

GENERIC LOCAL RINGS ON A SPECTRUM BETWEEN GOLOD AND GORENSTEIN

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To Lucho Avramov on the occasion of his 75th birthday

ABSTRACT. Artinian quotients R of the local ring $Q = k[[x, y, z]]$ are classified by multiplicative structures on $A = \mathrm{Tor}_*^Q(R, k)$; in particular, R is Gorenstein if and only if A is a Poincaré duality algebra while R is Golod if and only if all products in $A_{\geq 1}$ are trivial. There is empirical evidence that generic quotient rings with small socle ranks fall on a spectrum between Golod and Gorenstein in a very precise sense: The algebra A breaks up as a direct sum of a Poincaré duality algebra P and a graded vector space V , on which $P_{\geq 1}$ acts trivially. That is, A is a trivial extension, $A = P \ltimes V$, and the extremes $A = (k \oplus \Sigma k) \ltimes V$ and $A = P$ correspond to R being Golod and Gorenstein, respectively.

We prove that this observed behavior is, indeed, the generic behavior for graded quotients R of socle rank 2, and we show that the rank of P is controlled by the difference between the order and the degree of the socle polynomial of R .

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INTRODUCTION

A commutative noetherian local ring is the abstract form of the ring of germs of regular functions at a point on an algebraic variety. Accordingly, textbooks order local rings in a hierarchy based on the character of their singularity with the nonsingular rings, also called regular rings, being the most exclusive:

regular \subset hypersurface \subset complete intersection \subset Gorenstein .

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Most local rings, however, fall outside this hierarchy, and the characteristics of rings within it tell us little about local rings in general. For example, consider quotients of a regular local ring by high powers of its maximal ideal. Such rings do not fall within the geometric hierarchy, but they do belong to a recognized class, that of Golod rings, which has minimal overlap with the hierarchy. Thus one may wonder,

What does a typical local ring look like?

Any meaningful answer would presumably be a partial one, subject to restrictions on certain ring invariants. Within such restrictions there must, at the very least, be a systematic way to talk about *all* rings: a classification. We proceed to identify a viable set of restrictions that does not render the question trivial.

A fundamental invariant of a local ring is the minimal number of generators of its maximal ideal, it is called the *embedding dimension*. In embedding dimension 1, every singular local ring is a hypersurface, in particular a complete intersection, and meets the criterion for being Golod. In embedding dimension 2 the two notions separate decisively: A singular ring is either complete intersection or Golod. In embedding dimension 3 the range widens significantly: A singular local ring can now be Gorenstein without being complete intersection or it may not belong to any of the classes in the geometric hierarchy. Crucially, though, there is a classification: It is based on multiplicative structures in homology, and we discuss it below.

In the artinian case, another fundamental invariant of a local ring is the rank of its socle—the annihilator of the maximal ideal—which is called the *type*. All rings of type 1 are Gorenstein, but in the setting of artinian local rings of embedding dimension 3 and type 2 the question above is nontrivial and a terminology is available to express an answer. Before we discuss our answer, we make a further restriction to graded artinian quotients R of the trivariate power series algebra Q over a field k . Our answer is stated in the terminology of a classification of quotient rings R in terms of graded-commutative algebra structures on $\mathrm{Tor}_*^Q(R, k)$; it is due to Weyman [28] and to Avramov, Kustin, and Miller [4]. We recall the relevant details of the classification in Section 6; for now it suffices to say that it incorporates instances of two classic results of Golod: (1) Trivial multiplication on $\mathrm{Tor}_{\geq 1}^Q(R, k)$ characterizes Golod rings [18]; this uses that Q has embedding dimension 3. (2) Poincaré duality on $\mathrm{Tor}_*^Q(R, k)$ characterizes Gorenstein rings; this follows from Avramov and Golod [3] as $\mathrm{Tor}_*^Q(R, k)$ is isomorphic to the Koszul homology algebra of R .

Let R be a graded quotient of Q of type 2 with socle generators in degrees $s_1 \leq s_2$. One says that R is *compressed* if the length of R is as large as possible given s_1 and s_2 . Artinian k -algebras of type 2 can be parametrized in such a way that there is a nonempty Zariski open subset of the parameter space whose points correspond to compressed algebras. In this sense, assuming that k is large, a generic quotient ring R is compressed. This point is discussed in further detail in Section 8; till then we focus on compressed rings, as that notion is a more operational than “generic.”

Assume that R is compressed and that its defining ideal is generated by forms of degree 2 and higher. As R has type 2, this ideal has an irreducible decomposition $I_1 \cap I_2$, where I_1 and I_2 define artinian Gorenstein rings. Assume that also the rings Q/I_1 and Q/I_2 are compressed—also generic artinian Gorenstein rings are compressed. Our main result is Theorem 7.1; via Remark 6.8 it describes the k -algebra $\mathrm{Tor}_*^Q(R, k)$ as a trivial extension $P \ltimes V$ where P is a Poincaré duality algebra and $V \neq 0$ a graded vector space. The ring R is Golod if and only if P is *trivial*,

i.e. $P = k \oplus \Sigma k$. The key conclusions of the main theorem depend on the parity of s_2 and are summarized in Corollaries 7.3–7.5. They can be further condensed as:

If $s_2 \geq 5$, then $\text{Tor}_^Q(R, k)$ is a trivial extension $P \ltimes V$ of a Poincaré duality k -algebra P by a graded k -vector space V . Moreover,*

- *if s_2 is odd there is a number N , depending on s_2 , such that P is nontrivial for $s_1 < N$ and trivial for $s_1 \geq N$.*
- *if s_2 is even there are numbers $N_1 < N_2$, both depending on s_2 , such that P is nontrivial for $s_1 < N_1$ and trivial for $s_1 \geq N_2$.*

Our pursuit of the main theorem was spurred by data collected with the *Macaulay2* implementation [10] of the classification algorithm [11]. When the data first came in we were intrigued, because Avramov [2] had conjectured that this kind of nontrivial trivial extensions, i.e. $\text{Tor}_*^Q(R, k) = P \ltimes V$ with nontrivial P and $V \neq 0$, would not exist at all. Our attempts to prove this conjecture first led to the discovery of sporadic counterexamples [12], and with Weyman [14] we later developed a construction of rings with this kind of Tor-algebras and P of any size, but we still thought they were rare. This trajectory shows how our perspective on these rings changed with accrual of experimental data to eventually make a full 180° turn.

* * *

A brief synopsis of the paper is in place: Let I be a homogeneous ideal in the trivariate polynomial algebra Q over a field k and assume that I defines an artinian ring of type 2. An irreducible decomposition $I = I_1 \cap I_2$ yields two graded Gorenstein rings, Q/I_1 and Q/I_2 , and associated to this data is a Mayer–Vietoris sequence

$$(b) \quad 0 \longrightarrow Q/I \longrightarrow Q/I_1 \oplus Q/I_2 \longrightarrow Q/(I_1 + I_2) \longrightarrow 0.$$

The foundations of the proof of the main theorem are laid in Sections 1–4 with an analysis of this sequence. Particular attention is paid to the relations imposed by (b) on numerical invariants of Q/I , the Gorenstein rings, and $Q/(I_1 + I_2)$ as they relate to compressedness. Some of the broader conclusions of this analysis remain valid without the assumption that I is homogeneous and, indeed, Sections 1–3 deal with general artinian local rings. The next steps are comparative analyses of the minimal graded free resolutions (Section 5) and the multiplicative structures on the Tor algebras (Section 6) of the rings in (b). Central to both analyses is Construction 5.5, which harnesses an identification, as graded Q -modules, of the kernel of the homomorphism $Q/I \rightarrow Q/I_2$ with a power of the maximal ideal of the Gorenstein ring Q/I_1 and the canonical module of a quotient of Q by a power of its maximal ideal. The actual proof of the main theorem, which takes up most of Section 7, also draws on various nonhomological techniques to deal with issues not covered by the general homological analysis in Section 6. In the final Section 8 we describe the experiments that inspired this project and discuss how the main theorem and its underpinnings explain the generic behavior they reveal.

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1. ARTINIAN LOCAL RINGS OF TYPE 2

As is standard, we abbreviate the statement that a ring is local with unique maximal ideal to, say, “ (Q, \mathfrak{q}) is a local ring”, and when we need the notation \mathbf{k} for the residue field Q/\mathfrak{q} we say that $(Q, \mathfrak{q}, \mathbf{k})$ is local.

For \mathfrak{q} -primary Gorenstein ideals I_1 and I_2 in a regular local ring (Q, \mathfrak{q}) we identify necessary and sufficient conditions for the ideal $I_1 \cap I_2$ to define a ring of type 2.

1.1 Definitions. Let $(R, \mathfrak{m}, \mathbf{k})$ be an artinian local ring. For every element $x \neq 0$ in R the *valuation*,

$$v_R(x) = \max\{i \mid x \in \mathfrak{m}^i\},$$

is finite. The *socle* of R is the annihilator of the maximal ideal; it is a \mathbf{k} -vector space whose rank is called the *type* of R . That is,

$$\text{Soc } R = (0 :_R \mathfrak{m}) \quad \text{and} \quad \text{type } R = \text{rank}_{\mathbf{k}}(\text{Soc } R).$$

The ring R is Gorenstein if and only if $\text{type } R = 1$ holds. The *socle degree* of R is the integer s with $\mathfrak{m}^s \neq 0 = \mathfrak{m}^{s+1}$. Evidently, one has $\mathfrak{m}^s \subseteq \text{Soc } R$; if equality holds, then R is called *level*. Following Kustin, Şega, and Vraciu [23, 2.3(d)], the *socle polynomial* of R is defined as

$$\sum_{i=0}^s \text{rank}_{\mathbf{k}} \left(\frac{\mathfrak{m}^i \cap \text{Soc } R}{\mathfrak{m}^{i+1} \cap \text{Soc } R} \right) \chi^i.$$

Notice that R is level if and only if its socle polynomial is a monomial.

Let $R \cong Q/I$ be a minimal Cohen presentation of R , see [8, Thm. A.21], where $(Q, \mathfrak{q}, \mathbf{k})$ is a regular local ring and I is an ideal of Q contained in \mathfrak{q}^2 . One has

$$\mathfrak{q}^{s+1} \subseteq I \quad \text{and} \quad \mathfrak{q}^s \not\subseteq I.$$

The invariant t defined by

$$I \subseteq \mathfrak{q}^t \quad \text{and} \quad I \not\subseteq \mathfrak{q}^{t+1}$$

is, with a slight abuse of terminology, called the *initial degree* of I . In case I is homogeneous, it truly is the initial degree. Notice the inequalities

$$(1.1.1) \quad 2 \leq t \leq s + 1.$$

1.2. Recall, for example from Avramov [1, Sect. 5.2], that a local ring $(R, \mathfrak{m}, \mathbf{k})$ is called *Golod* if the ranks of the modules in the minimal free resolution of \mathbf{k} over R attain an upper bound established by Serre. If R is artinian, then it follows from an observation by Löfwall [24, Thm. 2.4] that R is Golod if the inequality

$$\left\lceil \frac{s+1}{2} \right\rceil < t$$

holds. Rossi and Şega give a different argument in the proof of [26, Prop. 6.3].

1.3 Lemma. *Let $(R, \mathfrak{m}, \mathbf{k})$ be an artinian local ring of type 2 and socle degree s . If one has $\text{rank}_{\mathbf{k}} \mathfrak{m}^s = 1$, then there exists an integer $v < s$ such that every nonzero element in $\text{Soc } R$ has valuation v or s .*

Proof. Let $x \neq 0$ be an element in \mathfrak{m}^s , and choose an element y such that $\{x, y\}$ is a basis for the \mathfrak{k} -vector space $\text{Soc } R$. Notice that $v = v_R(y)$ satisfies $v < s$, as every nonzero element of R has valuation at most s and $y \notin \mathfrak{m}^s$. Consider an element $z \in \text{Soc } R$ with $v_R(z) < s$; it is a linear combination of x and y , so $v_R(z) \geq \min\{v, s\} = v$ holds. Suppose one has $v_R(z) > v$ and choose $\alpha, \beta \in R \setminus \mathfrak{m}$ such that $z = \alpha x + \beta y$ holds. As α and β are nonzero, it follows that $y = \beta^{-1}(z - \alpha x)$ has valuation at least $v_R(z)$; a contradiction. Thus, every nonzero element of $\text{Soc } R$ has valuation v or s . \square

1.4. Let R be an artinian local ring of type 2; as in 1.1 consider a minimal Cohen presentation $R \cong Q/I$. The zero ideal of R has two irreducible components, so in Q one has $I = I_1 \cap I_2$, where I_1 and I_2 are irreducible ideals; see e.g. Gröbner [20, §6, Satz 3]. As I is \mathfrak{q} -primary, so are I_1 and I_2 . It follows that Q/I_1 and Q/I_2 are artinian Gorenstein rings. This leads us to consider the following situation.

1.5 Setup. Let $(Q, \mathfrak{q}, \mathfrak{k})$ be a regular local ring and I_1 and I_2 be \mathfrak{q} -primary ideals contained in \mathfrak{q}^2 . Set

$$I = I_1 \cap I_2 \quad \text{and} \quad I' = I_1 + I_2;$$

these are also \mathfrak{q} -primary ideals, and we adopt the following notation

Ideal	Quotient of Q	Initial degree	Socle degree
(1.5.1) I_1	R_1	t_1	s_1
I_2	R_2	t_2	s_2
I	R	t	s
I'	R'	t'	s'

Denote by e the common embedding dimension of Q, R_1, R_2, R , and R' . Without loss of generality, assume that $s_1 \leq s_2$ holds and set

$$(1.5.2) \quad a = \min\{i \geq 0 \mid \mathfrak{q}^i I_2 \subseteq I_1\} \quad \text{and} \quad b = \min\{i \geq 1 \mid \mathfrak{q}^{i+1} \cap I_2 \subseteq I_1\}.$$

The numbers a and b introduced in (1.5.2) capture crucial relations between the ideals I_1 and I_2 . Once we pass to the setting of compressed rings, b merges with s_1 , see Theorem 3.5, and with further restriction to graded rings, a is determined by the invariants from (1.5.1); see Theorem 4.4. The notation a remains convenient and is part of the statement of the main result, Theorem 7.1.

The first step in the analysis of Setup 1.5 is to record some elementary relations between the invariants from (1.5.1) and (1.5.2).

1.6 Proposition. *Adopt the setup in 1.5. One has:*

$$(a) \quad s = \max\{s_1, s_2\} = s_2 \quad \text{and} \quad t \geq \max\{t_1, t_2\}.$$

$$(b) \quad s' \leq \min\{s_1, s_2\} = s_1 \quad \text{and} \quad t' = \min\{t_1, t_2\}.$$

In particular, if the inequalities $\lceil \frac{s_1+1}{2} \rceil \leq t_1$ and $\lceil \frac{s_2+1}{2} \rceil \leq t_2$ hold, then one has

$$(c) \quad \left\lceil \frac{s+1}{2} \right\rceil \leq t \quad \text{and} \quad \left\lceil \frac{s'+1}{2} \right\rceil \leq t'.$$

Moreover, there are inequalities

$$(d) \quad 1 \leq b \leq s_1,$$

and if $I_2 \not\subseteq I_1$ holds, then one has

$$(e) \quad t_2 \leq a + t_2 - 1 \leq b \leq s_1.$$

Proof. (a): By the assumptions one has $\mathfrak{q}^{s_2+1} \subseteq \mathfrak{q}^{s_1+1} \subseteq I_1$ and $\mathfrak{q}^{s_2+1} \subseteq I_2$, so $\mathfrak{q}^{s_2+1} \subseteq I$ holds, while one has $\mathfrak{q}^{s_2} \not\subseteq I$ as $\mathfrak{q}^{s_2} \not\subseteq I_2$. The inequality for the initial degree t of I follows as one has $I \subseteq \mathfrak{q}^{t_1} \cap \mathfrak{q}^{t_2} = \mathfrak{q}^{\max\{t_1, t_2\}}$.

(b): The inequality for s' follows as one has $\mathfrak{q}^{s_1+1} \subseteq \mathfrak{q}^{s_1+1} + \mathfrak{q}^{s_2+1} \subseteq I_1 + I_2 = I'$. Finally, one has $I' \subseteq \mathfrak{q}^{t_1} + \mathfrak{q}^{t_2} = \mathfrak{q}^{\min\{t_1, t_2\}}$ while $I' \not\subseteq \mathfrak{q}^{t_1+1}$ and $I' \not\subseteq \mathfrak{q}^{t_2+1}$.

(c): Under the assumptions, these inequalities are immediate from (a) and (b).

(d): The definition of s_1 yields the nontrivial inequality.

(e): As I_2 is not contained in I_1 , one has $a \geq 1$, and that explains the first inequality. As $I_2 = \mathfrak{q}^{t_2} \cap I_2$ is not contained in I_1 one has $t_2 \leq b$. Now the second inequality follows, as the chains

$$\mathfrak{q}^{b-t_2+1}I_2 \subseteq I_2 \quad \text{and} \quad \mathfrak{q}^{b-t_2+1}I_2 \subseteq \mathfrak{q}^{b-t_2+1}\mathfrak{q}^{t_2} = \mathfrak{q}^{b+1}$$

yield $\mathfrak{q}^{b-t_2+1}I_2 \subseteq \mathfrak{q}^{b+1} \cap I_2 \subseteq I_1$. The third inequality holds by part (d). \square

1.7 Lemma. *Adopt the setup in 1.5. The following assertions hold.*

(a) *For every $x \in \mathfrak{q}^s \setminus I_2$ the element $x + I$ belongs to $\text{Soc } R$ and $v_R(x + I) = s$.*

(b) *For every $y \in (\mathfrak{q}^b \cap I_2) \setminus I_1$ the element $y + I$ belongs to $\text{Soc } R$ and $v_R(y + I) = b$.*

Proof. (a) For $x \in \mathfrak{q}^s \setminus I_2$, the element $x + I$ in R is nonzero, so it has valuation s and is evidently a socle element.

(b): Let $y \in (\mathfrak{q}^b \cap I_2) \setminus I_1$. The element $y + I$ in R is nonzero. By the definition of b one has $\mathfrak{q}y \subseteq I_1$, so $y + I$ is a socle element in R . Moreover, one has $v_R(y + I) \geq v_Q(y) \geq b$. To prove that equality holds, assume that one has $y = y' + y''$ with $y' \in \mathfrak{q}^{b+1}$ and $y'' \in I$. It follows that y' is in I_2 and, therefore, by the definition of b in I_1 . Now it follows that y is in I_1 , a contradiction. \square

The blanket assumption $s_1 \leq s_2$ informs the definitions in (1.5.2) and is responsible for the asymmetry in the next statement.

1.8 Theorem. *Adopt the setup in 1.5, let \mathfrak{m} denote the maximal ideal \mathfrak{q}/I of R , and assume that R_1 and R_2 are Gorenstein. The following conditions are equivalent.*

(i) *The ring R is of type 2.*

(ii) *One has $I_2 \not\subseteq I_1$.*

(iii) *One has $I_1 \not\subseteq I_2 \not\subseteq I_1$.*

Moreover, if R is of type 2, then one has $e \geq 2$ and the next assertions hold.

(a) *One has $\text{rank}_{\mathfrak{k}} \mathfrak{m}^s = 1$ if and only if $b < s$ holds, in which case one has*

$$v_R(z) = b \quad \text{for every } z \in (\text{Soc } R) \setminus \mathfrak{m}^s.$$

(b) *One has $\text{rank}_{\mathfrak{k}} \mathfrak{m}^s = 2$, i.e. R is level, if and only if $b = s$ holds, in which case also $s_1 = s$ holds.*

In particular, if R is of type 2, then its socle polynomial is $\chi^b + \chi^s$ and $a \geq 1$ holds.

Proof. Condition (iii) trivially implies (ii).

(i) \implies (iii): If type $R = 2$ holds, then R is not Gorenstein, so $R_1 \neq R \neq R_2$ and, therefore, $I_1 \not\subseteq I_2 \not\subseteq I_1$ hold. In particular, one has $a \geq 1$.

(ii) \implies (i): As $s = s_2$ holds, see Proposition 1.6(a), one can choose $x \in \mathfrak{q}^s \setminus I_2$. It follows from Lemma 1.7(a) that $x + I$ is a socle element of R of valuation s . The assumption $I_2 \not\subseteq I_1$ implies that the set $(\mathfrak{q}^b \cap I_2) \setminus I_1$ is not empty, so one can choose an element y in this set. It follows from Lemma 1.7(b) that $y + I$ is a socle element

of R of valuation b . Now, assume towards a contradiction that the elements $x + I$ and $y + I$ in $\text{Soc } R$ are linearly dependent. There exists then a unit α in Q , such that $y - \alpha x$ belongs to $I \subseteq I_2$. As y is in I_2 , this implies $x \in I_2$, which contradicts the choice of x . Thus, the elements $x + I$ and $y + I$ are linearly independent, whence type $R \geq 2$ holds. As the rings R_1 and R_2 are artinian and Gorenstein, the ideals I_1 and I_2 are irreducible. Thus one has type $R \leq 2$; see [20, §6, Satz 3].

If $e = 1$ holds, then Q is a DVR, so the ideals in Q are linearly ordered, whence one has $I = I_1$ or $I = I_2$. Thus, the assumption type $R = 2$ implies $e \geq 2$.

(a): The first assertion is immediate; indeed it was proved above that the basis elements $x + I$ and $y + I$ of $\text{Soc } R$ have valuations s and b . The second assertion follows from Lemma 1.3.

(b): By hypothesis and Proposition 1.6(a,d) one has $b \leq s_1 \leq s$. If R is level, then (a) yields $s \leq b$, whence $s_1 = b = s$ holds. Conversely, if $b = s$ then one has $s_1 = s$ and $\text{rank}_k \mathfrak{m}^s = 2$ as $x + I$ and $y + I$ are linearly independent in $\text{Soc } R$. \square

1.9 Lemma. *Adopt the setup in 1.5 and assume that R_1 and R_2 are Gorenstein. If R has type 2, then the following assertions hold.*

(a) *There are inequalities $\lceil \frac{s_1+1}{2} \rceil \geq t_1$ and $\lceil \frac{s_2+1}{2} \rceil \geq t_2$.*

(b) *If $\lceil \frac{s_2+1}{2} \rceil = t_2$ holds, then one has $t_1 \leq t_2 \leq s_1 \leq s_2 < 2s_1$.*

Proof. (a): If the inequality $\lceil \frac{s_1+1}{2} \rceil < t_1$ holds, then the Gorenstein ring R_1 is Golod by 1.2 and hence a hypersurface; see [1, Remark after 5.2.5]. Since R_1 is artinian, this implies that the embedding dimension e is 1, which by Theorem 1.8 contradicts the assumption type $R = 2$. Thus, $\lceil \frac{s_1+1}{2} \rceil \geq t_1$ holds, and by symmetry so does the second inequality.

(b): The inequality $s_1 \leq s_2$ holds by assumption. In view of part (a), one now has $t_1 \leq \lceil \frac{s_1+1}{2} \rceil \leq \lceil \frac{s_2+1}{2} \rceil = t_2$. Next, if the inequality $t_2 \geq s_1 + 1$ holds, then one has $I_2 \subseteq \mathfrak{q}^{t_2} \subseteq \mathfrak{q}^{s_1+1} \subseteq I_1$, which by Theorem 1.8 contradicts the assumption type $R = 2$. Thus, $t_2 \leq s_1$ holds. Finally, if the inequality $s_2 \geq 2s_1$ holds, then one has $t_2 = \lceil \frac{s_2+1}{2} \rceil \geq \lceil \frac{2s_1+1}{2} \rceil = s_1 + 1$, and as proved right above this contradicts the assumption type $R = 2$. \square

In the context of Theorem 1.8, the next example shows that even if R_1 and R_2 are Gorenstein of the same socle degree, the ring R may not be level.

1.10 Example. Let k be a field. In the regular local ring $Q = k[[x, y, z]]$ with maximal ideal $\mathfrak{q} = (x, y, z)$ consider the complete intersection ideals

$$I_1 = (x^2, xy + z^2, y^2) \quad \text{and} \quad I_2 = (x^2, xy + z^2, y^2 + z^3).$$

It is straightforward to check that one has:

$$\begin{aligned} \mathfrak{q}^4 &\subseteq I_1 \quad \text{and} \quad (I_1 : \mathfrak{q}) = (z^3) + I_1; \\ \mathfrak{q}^4 &\subseteq I_2 \quad \text{and} \quad (I_2 : \mathfrak{q}) = (z^3) + I_2 = (y^2) + I_2. \end{aligned}$$

The intersection of the two ideals is

$$I = I_1 \cap I_2 = (x^2, xy + z^2, y^3, y^2z, yz^2).$$

By Lemma 1.7 the elements $z^3 + I$ and $y^2 + z^3 + I$ belong to the socle of $R = Q/I$. That is, one has

$$(I : \mathfrak{q}) = (y^2, z^3) + I.$$

As I is homogeneous, this shows that R has socle polynomial $\chi^2 + \chi^3$; in particular R is not level.

For our purposes the next result completes the analysis of Setup 1.5 for $e = 2$.

1.11 Proposition. *Adopt the setup in 1.5 and assume that R_1 and R_2 are Gorenstein. If $e = 2$ and R is of type 2, then R and R' are Golod.*

Proof. By [1, Prop. (5.3.4)] each of the rings R_1 , R_2 , R , and R' is either a codimension 2 complete intersection or Golod; in particular R is Golod as it is of type 2. By Theorem 1.8 one has $I_2 \not\subseteq I_1$, so the ring $R' = Q/(I_1 + I_2) \cong R_1/((I_1 + I_2)/I_1)$ is a proper quotient of the artinian complete intersection R_1 and hence not a complete intersection. \square

2. COMPRESSED GORENSTEIN LOCAL RINGS

For a local ring $(R, \mathfrak{m}, \mathfrak{k})$ the Hilbert function, h_R , is defined by

$$h_R(i) = \text{rank}_{\mathfrak{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \text{ for } i \geq 0.$$

Let e denote the embedding dimension of R . Recall that if R is regular, then its Hilbert function is given by

$$(2.0.1) \quad h_R(i) = \binom{e-1+i}{e-1} \text{ for } i \geq 0.$$

One has $h_R(0) = 1$ and $h_R(1) = e$; notice the equality $\text{length}(R) = \sum_{i=0}^{\infty} h_R(i)$. If R has finite length, then we refer to the finite sequence $(h_R(0), h_R(1), \dots)$ of nonzero values of the Hilbert function as the *h -vector* of R .

In the balance of this section we adopt the following setup.

2.1 Setup. Let $(Q, \mathfrak{q}, \mathfrak{k})$ be a regular local ring of embedding dimension e , and $(R, \mathfrak{m}, \mathfrak{k})$ an artinian local ring with minimal Cohen presentation $R \cong Q/I$; in particular, R also has embedding dimension e . As in 1.1 let t denote the initial degree of I and s the socle degree of R ; both invariants can be detected from the Hilbert function:

$$(2.1.1) \quad s = \max\{i \mid h_R(i) \neq 0\} \quad \text{and} \quad t = \min\{i \mid h_R(i) \neq h_Q(i)\}.$$

We proceed to recall the notion of a compressed artinian Gorenstein ring. Compressedness of algebras was introduced by Iarrobino [22]; here we refer to the more recent treatment of compressed local Gorenstein rings in [26].

2.2. Assume that R is Gorenstein. For every $i \geq 0$ there is an inequality

$$h_R(i) \leq \min\{h_Q(i), h_Q(s-i)\} = \min\left\{\binom{e-1+i}{e-1}, \binom{e-1+s-i}{e-1}\right\}.$$

If equality holds for every i , then R is called *compressed*, see [26, Prop. 4.2]. Notice that if R is compressed, then its h -vector is symmetric and unimodal, and one has

$$(2.2.1) \quad \begin{aligned} t &= \min\{i \mid h_Q(i) > h_Q(s-i)\} \\ &= \min\left\{i \mid \binom{e-1+i}{e-1} > \binom{e-1+s-i}{e-1}\right\} = \left\lceil \frac{s+1}{2} \right\rceil. \end{aligned}$$

2.3 Remark. If $e = 1$, then R is a compressed Gorenstein ring. If R is Gorenstein of socle degree at most 2, then R is compressed: The h -vectors are $(1, 1)$ and $(1, e, 1)$.

For use in later sections, we record three technical statements about compressed Gorenstein rings.

2.4 Lemma. *Assume that R is compressed Gorenstein.*

(a) *If $e = 2$, then one has*

$$h_Q(t) - h_Q(s-t) = \begin{cases} 1 & \text{if } s \text{ is odd} \\ 2 & \text{if } s \text{ is even.} \end{cases}$$

(b) *If $e \geq 3$, then one has*

$$h_Q(t) - h_Q(s-t) = \binom{e-2+t}{e-2} + \begin{cases} 0 & \text{if } s \text{ is odd} \\ \binom{e-3+t}{e-2} & \text{if } s \text{ is even.} \end{cases}$$

Proof. From (2.0.1) one gets

$$h_Q(t) - h_Q(s-t) = \binom{e-1+t}{e-1} - \binom{e-1+s-t}{e-1}.$$

For s odd one has $s-t = t-1$ by (2.2.1), so the right-hand side becomes $\binom{e-2+t}{e-2}$; for $e = 1$ this equals 1. For s even one has $s-t = t-2$. For $e = 2$ the right-hand side now becomes $t+1 - (s-t+1) = 2t-s = 2$, and for $e \geq 3$ one gets

$$\binom{e-1+t}{e-1} - \binom{e-3+t}{e-1} = \binom{e-1+t}{e-1} - \binom{e-2+t}{e-1} + \binom{e-2+t}{e-1} - \binom{e-3+t}{e-1} = \binom{e-2+t}{e-2} + \binom{e-3+t}{e-2}. \quad \square$$

2.5 Proposition. *If R is compressed Gorenstein with $s \geq 2$, then the following assertions hold for all integers i with $2 \leq i \leq s$.*

(a) *The ring R/\mathfrak{m}^i is level of socle degree $i-1$.*

(b) *If $e \leq 2$ or $(i, s) \neq (3, 3)$, then the ring R/\mathfrak{m}^i is Golod.*

Proof. If $e = 1$, then R/\mathfrak{m}^i is trivially level, and it is Golod by [1, Prop. 5.2.5]. Now assume that $e \geq 2$ holds.

(a): Two applications of [26, Prop. 4.2(b)] yield

$$\begin{aligned} (0 :_{R/\mathfrak{m}^i} \mathfrak{m}/\mathfrak{m}^i) &= (\mathfrak{m}^i :_R \mathfrak{m})/\mathfrak{m}^i \\ &= ((0 :_R \mathfrak{m}^{s+1-i}) :_R \mathfrak{m})/\mathfrak{m}^i \\ &= (0 :_R \mathfrak{m}^{s+2-i})/\mathfrak{m}^i \\ &= \mathfrak{m}^{i-1}/\mathfrak{m}^i. \end{aligned}$$

(b): If $e = 2$ holds, then R/\mathfrak{m}^i is Golod by [13, Thm. (2.2)]. If $s \neq 3$ holds, then R/\mathfrak{m}^i is Golod by [26, Prop. 6.3]. Finally, without assumptions on s the ring R/\mathfrak{m}^2 is Golod; see [1, Prop. 5.2.4(1)]. \square

In [13, Prop. (3.3)] we give examples of compressed Gorenstein local rings (R, \mathfrak{m}) of socle degree 3 with R/\mathfrak{m}^3 not Golod, cf. Proposition 2.5(b).

The final result of this section recognizes high powers of the maximal ideal of a compressed Gorenstein local ring as dualizing modules of Golod rings; through Construction 5.5 it play a central role in Sections 5 and 6.

2.6 Proposition. *If R is compressed Gorenstein, then for every integer $i \geq t$ there is an isomorphism of Q -modules,*

$$\mathfrak{m}^i \cong \text{Ext}_Q^e(Q/\mathfrak{q}^{s+1-i}, Q).$$

Proof. The equality in the next display holds by [26, Prop. 4.2(b)],

$$\mathfrak{m}^i = (0 :_R \mathfrak{m}^{s+1-i}) \cong \text{Hom}_R(R/\mathfrak{m}^{s+1-i}, R) \cong \text{Hom}_R(Q/\mathfrak{q}^{s+1-i}, R).$$

The first isomorphism is standard and the second holds as I is contained in \mathfrak{q}^t , and one has $s+1-i \leq s+1-t \leq t$ by (2.2.1). Finally, since R is Gorenstein there is an isomorphism $\text{Hom}_R(Q/\mathfrak{q}^{s+1-i}, R) \cong \text{Ext}_Q^e(Q/\mathfrak{q}^{s+1-i}, Q)$; see [8, Thm. 3.3.7]. \square

3. COMPRESSED LOCAL RINGS OF TYPE 2

In [23] the notion of compressedness is extended beyond Gorenstein local rings.

3.1. Adopt Setup 2.1 and assume that R has type 2; by 1.4 and Theorem 1.8 one has $e \geq 2$. Let b be as in (1.5.2); the socle polynomial of R is

$$\chi^b + \chi^s$$

with $b = s$ if R is level and $b < s$ otherwise. If the equality

$$(3.1.1) \quad \begin{aligned} h_R(i) &= \min\{h_Q(i), h_Q(b-i) + h_Q(s-i)\} \\ &= \min\left\{\binom{e-1+i}{e-1}, \binom{e-1+b-i}{e-1} + \binom{e-1+s-i}{e-1}\right\} \end{aligned}$$

holds for every $i \geq 0$, then R is called *compressed*; see [23, Def. 2.5]. Notice that if R is compressed, then one has

$$\begin{aligned} t &= \min\{i \mid h_Q(i) > h_Q(b-i) + h_Q(s-i)\} \\ &= \min\left\{i \mid \binom{e-1+i}{e-1} > \binom{e-1+b-i}{e-1} + \binom{e-1+s-i}{e-1}\right\}. \end{aligned}$$

Moreover, the next inequality holds by [23, Thm. 4.4(c)],

$$(3.1.2) \quad \left\lceil \frac{s+1}{2} \right\rceil \leq t.$$

It follows that the h -vector of R is unimodal. Indeed, the function $h_Q(i)$ is increasing, and for $i \geq t$ the functions $h_Q(b-i)$ and $h_Q(s-i)$ are decreasing.

Strict inequality in (3.1.2) implies that R is Golod, see 1.2. As Golod rings sit at one extreme of the spectrum we are interested in, the focus of our attention is on rings with equality $\lceil \frac{s+1}{2} \rceil = t$; they may still be Golod.

3.2 Example. The complete intersections Q/I_1 and Q/I_2 from Example 1.10 have h -vectors $(1, 3, 3, 1)$; indeed I_1 and I_2 are both minimally generated by 3 elements in \mathfrak{q}^2 . Thus, both rings are compressed Gorenstein rings. The ideal I has two generators in $\mathfrak{q}^2 \setminus \mathfrak{q}^3$, and the socle polynomial of Q/I is $\chi^2 + \chi^3$. It follows that Q/I has h -vector $(1, 3, 4, 1)$ and is a compressed artinian ring of type 2.

In general, compressedness of the Gorenstein rings Q/I_1 and Q/I_2 does not guarantee compressedness of Q/I . On the other hand, the ring Q/I may be compressed though one of the Gorenstein rings is not.

3.3 Example. Let k be a field. In the regular local ring $Q = k[[x, y, z]]$ with maximal ideal $\mathfrak{q} = (x, y, z)$ consider the homogeneous complete intersection ideals

$$\begin{aligned} I_1 &= (x^2, y^2, z^2 + xy + yz), \\ I_2 &= (x^2, y^2, z^2), \quad \text{and} \\ I_3 &= (yz, x^2 + xy, y^3 - z^3). \end{aligned}$$

The corresponding quotient rings have h -vectors

$$h_{Q/I_1} = (1, 3, 3, 1) = h_{Q/I_2} \quad \text{and} \quad h_{Q/I_3} = (1, 3, 4, 3, 1),$$

so Q/I_1 and Q/I_2 are compressed but Q/I_3 is not. For the intersection ideals

$$\begin{aligned} I &= I_1 \cap I_2 = (x^2, y^2, xz^2 - z^3, yz^2) \quad \text{and} \\ J &= I_2 \cap I_3 = (yz^2, y^2z, x^2z, y^3 - z^3, x^2y + xy^2, x^3 - xy^2) \end{aligned}$$

the h -vectors are

$$h_{Q/I} = (1, 3, 4, 2) \quad \text{and} \quad h_{Q/J} = (1, 3, 6, 4, 1),$$

so Q/J is compressed but Q/I is not compressed.

As we explain in Remark 7.2, the ideal J in Example 3.3 can actually not be obtained as an intersection of ideals that define compressed Gorenstein rings. We complement this example with:

3.4 Example. In the regular ring $Q = \mathbb{Z}_2[[x, y, z]]$ consider the homogeneous ideals

$$\begin{aligned} I_1 &= (xz, xy + yz, x^3 + y^3 + y^2z + z^3), \\ I_2 &= (y^2z, x^2z + z^3, y^3 + xz^2, x^3, x^2y^2), \quad \text{and} \\ I_3 &= (z^3, y^2z + xz^2 + yz^2, x^2z + xyz, y^3 + xyz, xy^2, x^2y, x^3 + yz^2). \end{aligned}$$

Per *Macaulay2* [19] these are Gorenstein ideals with $I_1 \cap I_2 = I_2 \cap I_3$, and this common intersection defines a compressed ring with h -vector $(1, 3, 6, 9, 4, 1)$. The Gorenstein rings have h -vectors

$$h_{Q/I_1} = (1, 3, 4, 3, 1), \quad h_{Q/I_2} = (1, 3, 6, 6, 3, 1), \quad \text{and} \quad h_{Q/I_3} = (1, 3, 6, 3, 1)$$

so I_2 and I_3 define compressed rings, but I_1 does not.

The next theorem concludes our analysis of Setup 1.5 vis-à-vis compressedness. In the graded case stronger statements are available, see Theorem 4.4.

3.5 Theorem. *Adopt the setup in 1.5. Assume that R_1 and R_2 are compressed Gorenstein rings and that R has type 2. There are inequalities,*

$$(3.5.1) \quad \text{length}(R) \leq \sum_{i=0}^s \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}$$

and

$$(3.5.2) \quad 2 \leq t_1 \leq t_2 \leq s_1 \leq s < 2s_1.$$

If equality holds in (3.5.1), then R is compressed and the next assertions hold.

- (a) $h_R(i) = \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}$ for every $i \geq 0$.
- (b) $t = \min\{i \mid h_Q(i) > h_{R_1}(i) + h_{R_2}(i)\}$.
- (c) The socle polynomial of R is $\chi^{s_1} + \chi^s$.

Proof. The first inequality in (3.5.2) holds as I_1 is contained in \mathfrak{q}^2 . The remaining inequalities follow per (2.2.1) from Proposition 1.6(a) and Lemma 1.9(b).

By Theorem 1.8 the socle polynomial of R is $\chi^b + \chi^s$. For every $i \geq 0$ one has

$$(*) \quad \begin{aligned} \min\{h_Q(i), h_Q(b-i) + h_Q(s-i)\} &\leq \min\{h_Q(i), h_Q(s_1-i) + h_Q(s_2-i)\} \\ &\leq \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}. \end{aligned}$$

Indeed, the first inequality holds as $b \leq s_1$ and $s = s_2$ hold by Proposition 1.6(a,d). For the second inequality notice first that since R_2 is compressed one has $h_{R_2}(i) = h_Q(i) = \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}$ for $i < t_2$. Further, $t_1 \leq t_2$ holds by (3.5.2), so for $i \geq t_2$ one has $h_{R_1}(i) + h_{R_2}(i) = h_Q(s_1 - i) + h_Q(s_2 - i)$ as also R_1 is compressed. In view of (*) one now has

$$\begin{aligned}
 (**) \quad \text{length}(R) &\leq \sum_{i=0}^s \min\{h_Q(i), h_Q(b-i) + h_Q(s-i)\} \\
 &\leq \sum_{i=0}^s \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}
 \end{aligned}$$

where the first inequality holds by [23, Thm. 4.4(a)].

Further, it follows from [23, Thm. 4.4(b)] and (**) that R is compressed if the equality $\text{length}(R) = \sum_{i=0}^s \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}$ holds, and in that case equalities hold in (*) for every $i \geq 0$. This establishes (a), and (b) follows per (2.1.1). By Theorem 1.8 one has $I_2 \not\subseteq I_1$, so to prove (c) it suffices by Proposition 1.6(d) to establish the inequality $b \geq s_1$. Assume first that $t > s_1$ holds. As R and R_2 are compressed one has

$$\begin{aligned}
 (\dagger) \quad h_Q(s_1) &\leq h_Q(b - s_1) + h_Q(s_2 - s_1) \\
 &< h_Q(b - s_1) + h_Q(2s_1 - s_1) \\
 &= h_Q(b - s_1) + h_Q(s_1)
 \end{aligned}$$

where the first inequality holds by part (b), and the strict inequality comes from (3.5.2). Thus $h_Q(b - s_1)$ is positive, which forces $b \geq s_1$. Assume next that $t \leq s_1$ holds. Equalities hold in (*) for every $i \geq 0$, so for $i = s_1$ one has $h_Q(b - s_1) = h_Q(s_1 - s_1) = 1$ in view of part (b), and that forces $b \geq s_1$. \square

Notice from Example 3.2 that the equality in (3.5.1) need not hold for a compressed ring of type 2.

3.6 Proposition. *Adopt the setup in 1.5. Assume that R_1 and R_2 are compressed Gorenstein and R is compressed of type 2. The following assertions hold.*

- (a) *If $e \geq 3$, then one has $t \leq s_1$.*
- (b) *If $t > s_1$, then one has $e = 2$ and $s_2 = 2s_1 - 1$.*

Proof. If $t > s_1$ holds then, as in the proof of Theorem 3.5, one has $b = s_1$, and the inequalities (\dagger) in that proof yield $h_Q(s_1) = 1 + h_Q(s_2 - s_1)$. That is,

$$\binom{e-1+s_1}{e-1} = 1 + \binom{e-1+s_2-s_1}{e-1}.$$

As $s_2 - s_1 < s_1$ holds, the displayed equality can only hold if one has $e = 2$ and $s_2 = 2s_1 - 1$. In particular, the inequality $t \leq s_1$ holds for $e \geq 3$. \square

The next example illustrates Proposition 3.6(b).

3.7 Example. Let k be a field. In the regular local ring $k[[x, y]]$ with maximal ideal $\mathfrak{q} = (x, y)$ consider the homogeneous complete intersection ideals

$$I_1 = (xy, x^2 + y^2) \quad \text{and} \quad I_2 = (x^2, y^3).$$

It is straightforward to check that one has

$$\begin{aligned} \mathfrak{q}^3 &\subseteq I_1 \quad \text{and} \quad (I_1 : \mathfrak{q}) = (x^2) + I_1 . \\ \mathfrak{q}^4 &\subseteq I_2 \quad \text{and} \quad (I_2 : \mathfrak{q}) = (xy^2) + I_2 . \end{aligned}$$

The intersection of the two ideals is

$$I = I_1 \cap I_2 = (x^3, x^2y, y^3) .$$

Thus, in the notation from 1.5 one has $t_1 = t_2 = s_1 = 2 < 3 = t = s_2 = 2s_1 - 1$.

3.8 Remark. Adopt the setup in 1.5 and assume that R_2 is compressed Gorenstein. If the initial degree of I is larger than the initial degree of I_2 , then R is Golod. Indeed, Proposition 1.6(a) and (2.2.1) yield $\lceil \frac{s+1}{2} \rceil = \lceil \frac{s_2+1}{2} \rceil = t_2 \leq t$, so it follows from 1.2 that R is Golod if the strict inequality $t_2 < t$ holds. Even if $t = t_2$ holds, R may still be Golod, see Theorem 7.1(g).

3.9 Lemma. *Adopt the setup in 1.5. Assume that $e \geq 3$ holds, R_1 and R_2 are compressed Gorenstein, and R is compressed of type 2. The equality $\lceil \frac{s+1}{2} \rceil = t$ holds if and only if one has*

$$\binom{e-1+s_1-t_2}{e-1} < \binom{e-2+t_2}{e-2} + \begin{cases} 0 & \text{if } s \text{ is odd} \\ \binom{e-3+t_2}{e-2} & \text{if } s \text{ is even.} \end{cases}$$

Proof. As in Remark 3.8 one has $\lceil \frac{s+1}{2} \rceil = \lceil \frac{s_2+1}{2} \rceil = t_2 \leq t$. Theorem 3.5(b) yields $t = \min\{i \mid h_Q(i) > h_{R_1}(i) + h_{R_2}(i)\}$, so $t \leq t_2$ holds in view of 2.2 if and only if one has $h_Q(t_2) > h_Q(s_1-t_2) + h_Q(s_2-t_2)$, equivalently $h_Q(s_1-t_2) < h_Q(t_2) - h_Q(s_2-t_2)$. By Lemma 2.4(b) applied to R_2 this is the asserted inequality. \square

Compressed level rings of large socle degree are Golod.

3.10 Theorem. *Adopt the setup in 1.5. Assume that $e \geq 3$ holds, R_1 and R_2 are compressed Gorenstein, and R is compressed of type 2. If R is level, then it is Golod provided that one has*

$$s \geq 2e - 3 + \begin{cases} 0 & \text{if } s \text{ is odd} \\ \sqrt{8(e-1)^2 + 1} & \text{if } s \text{ is even.} \end{cases}$$

Proof. As R is level, $s_1 = s$ holds by Theorem 3.5(c). First assume that s is odd. It follows from 1.2 and Lemma 3.9 that R is Golod if the inequality $\binom{e-1+s-t_2}{e-1} \geq \binom{e-2+t_2}{e-2}$ holds. By Proposition 1.6(a) and (2.2.1) one has $t_2 = \frac{s+1}{2} = \frac{s_1+1}{2}$, and substituting this expression for t_2 into the inequality it reads

$$\binom{e-1+\frac{s-1}{2}}{e-1} \geq \binom{e-1+\frac{s-1}{2}}{e-2} .$$

Clearing common factors reduces this inequality to

$$\frac{1}{e-1} \geq \frac{1}{1+\frac{s-1}{2}} \quad \text{equivalently} \quad s \geq 2e - 3 .$$

Now assume that s is even. In this case one has $t_2 = \frac{s}{2} + 1$ and it follows as above that R is Golod if one has

$$\binom{e-2+\frac{s}{2}}{e-1} \geq \binom{e-1+\frac{s}{2}}{e-2} + \binom{e-2+\frac{s}{2}}{e-2} .$$

Clearing common factors reduces this inequality to

$$\frac{1}{e-1} \geq \frac{e-1+\frac{s}{2}}{\left(\frac{s}{2}+1\right)\frac{s}{2}} + \frac{1}{\frac{s}{2}} \quad \text{equivalently} \quad s^2 - 2(2e-3)s - 4e(e-1) \geq 0.$$

The quadratic polynomial in s has one positive root: $2e-3 + \sqrt{8(e-1)^2+1}$. \square

4. THE SUM AND INTERSECTION OF TWO GRADED GORENSTEIN IDEALS

From here on we work in a restricted version of Setup 1.5: We only consider homogeneous quotients of a ring of power series in three or more variables.

4.1 Setup. Let k be a field and (Q, \mathfrak{q}) the local k -algebra of power series in $e \geq 3$ variables with coefficients in k . Let I_1 and I_2 be homogeneous \mathfrak{q} -primary Gorenstein ideals contained in \mathfrak{q}^2 . Set $I = I_1 \cap I_2$ and $I' = I_1 + I_2$ and adopt the notation (1.5.1), the assumption $s_1 \leq s_2$, and the notation (1.5.2). Finally, assume that the Gorenstein rings R_1 and R_2 are compressed and that R has type 2; by Theorem 1.8, (2.2.1), and Proposition 1.6(a) one then has

$$(4.1.1) \quad I_2 \not\subseteq I_1 \not\subseteq I_2 \quad \text{and} \quad \left\lfloor \frac{s+1}{2} \right\rfloor = t_2.$$

Our interest is in the rings R and R' . The assumption $e \geq 3$ has been made part of the setup as the situation is trivial in lower embedding dimensions. Indeed, the assumption that R has type 2 rules out the possibility $e = 1$, and if $e = 2$, then R and R' are Golod, see Proposition 1.11. We start by noticing that in higher embedding dimension, compressed Gorenstein rings are rarely complete intersections.

4.2 Proposition. *Let (Q, \mathfrak{q}) be as in Setup 4.1 and $J \subseteq \mathfrak{q}^2$ be a homogeneous \mathfrak{q} -primary complete intersection ideal. If the ring Q/J is compressed, then one has $e = 3$ and $h_{Q/J} = (1, 3, 3, 1)$.*

Proof. Let \tilde{s} be the socle degree of Q/J ; by (2.2.1) the initial degree of J is $\tilde{t} = \left\lceil \frac{\tilde{s}+1}{2} \right\rceil$. As Q/J is complete intersection, J is minimally generated by e elements; in particular, one has $e \geq h_Q(\tilde{t}) - h_{Q/J}(\tilde{t}) = h_Q(\tilde{t}) - h_Q(\tilde{s} - \tilde{t})$; see 2.2. From Lemma 2.4(b) one now gets

$$e \geq \binom{e-2+\tilde{t}}{e-2} \geq \binom{e}{e-2} = \frac{e(e-1)}{2},$$

where the second inequality holds as one has $\tilde{t} \geq 2$ by the assumption $J \subseteq \mathfrak{q}^2$. It follows that one has $e \leq 3$, and the opposite inequality holds by assumption. Now one has $e = \frac{e(e-1)}{2}$, so $\tilde{t} = 2$ holds and it follows from Lemma 2.4(b) that \tilde{s} is odd, whence $\tilde{s} = 3$. By 2.2 the h -vector of Q/J is thus $(1, 3, 3, 1)$. \square

4.3 Remark. From the Mayer–Vietoris sequence $0 \rightarrow R \rightarrow R_1 \oplus R_2 \rightarrow R' \rightarrow 0$ one gets the equalities

$$(4.3.1) \quad h_R(i) + h_{R'}(i) = h_{R_1}(i) + h_{R_2}(i) \quad \text{for all } i \geq 0.$$

Theorem 3.5 can by way of (4.3.1) be strengthened as follows:

4.4 Theorem. *Adopt the setup in 4.1. There is an inequality,*

$$(4.4.1) \quad \text{length}(R) \leq \sum_{i=0}^s \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\};$$

and equality holds in if and only if R is compressed.

Moreover, if R is compressed, then there inequalities,

$$(4.4.2) \quad 2 \leq t_1 \leq t_2 \leq t \leq s_1 \leq s < 2s_1,$$

and the following assertions hold.

(a) $h_R(i) = \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}$ for $i \geq 0$.

(b) $t = \min\{i \mid h_Q(i) > h_{R_1}(i) + h_{R_2}(i)\}$.

(b') If the equality $h_Q(i) = h_{R_1}(i) + h_{R_2}(i)$ holds, then one has $t = i + 1$.

(c) The socle polynomial of R is $\chi^{s_1} + \chi^s$.

(d) $a = s_1 - t_2 + 1$.

Proof. Setup 4.1 is a special case of Setup 1.5, so the inequality (4.4.1) is a special case of (3.5.1). If equality holds in (4.4.1), then it follows from Theorem 3.5 that R is compressed and (a), (b), and (c) hold. Further, Propositions 1.6(b) and 3.6(a) yield $t_2 \leq t \leq s_1$; together with (3.5.2) this establishes the inequalities in (4.4.2). To complete the argument it suffices to prove two claims: (1) If equality holds in (4.4.1) then (b') holds. (2) If R is compressed, then (d) and equality in (4.4.1) hold.

(1): Assume that $h_Q(i) = h_{R_1}(i) + h_{R_2}(i)$ holds. By (b) and (4.4.2) one has $i < t \leq s_1 \leq s_2$ and, therefore, $0 < h_{R_2}(i) < h_Q(i)$. It follows that $t_2 \leq i$ holds; see (2.1.1). As $t_1 \leq t_2$ holds per (4.4.2), one now has

$$h_Q(i+1) > h_Q(i) = h_{R_1}(i) + h_{R_2}(i) > h_{R_1}(i+1) + h_{R_2}(i+1),$$

whence $i+1 = t$ holds by (b).

(2): Now assume that R is compressed. By Theorem 1.8 the socle polynomial of R is $\chi^b + \chi^s$ and $a \geq 1$ holds. To prove (d), choose a homogeneous element g of I_2 with $\mathfrak{q}^{a-1}g \not\subseteq I_1$ and set $d = v_Q(g)$. Choose $f \in \mathfrak{q}^{a-1} \setminus \mathfrak{q}^a$ such that $fg \notin I_1$ and notice that the degree $v_Q(f)$ is exactly $a-1$. As $\mathfrak{q}(fg)$ is in I_1 , the coset $fg + I_1$ is a nonzero socle element in R_1 . Thus one has

$$s_1 = v_{R_1}(fg + I_1) = v_Q(fg) = v_Q(f) + v_Q(g) = a - 1 + d.$$

As g is in I_2 one has $d \geq t_2$, and it suffices to prove that equality holds. One has $a = s_1 - d + 1$ and hence $\mathfrak{q}^{s_1-d+1}(I_2)_{t_2} \subseteq I_1$. In particular, one has

$$(I_2)_{t_2} \subseteq (I_1 : \mathfrak{q}^{s_1+1-d}) = \mathfrak{q}^d + I_1$$

where the equality holds by [26, Prop. 4.2] as R_1 is compressed. Assume towards a contradiction that $d > t_2$ holds. One then has $(I_2)_{t_2} \subseteq (\mathfrak{q}^d + I_1)_{t_2} = (I_1)_{t_2}$ and hence $(I_2)_{t_2} = I_{t_2}$. In particular, $t \leq t_2$ holds. The opposite inequality $t_2 \leq t$, holds by Proposition 1.6(a), so equality holds. As R_2 and R are compressed, one now has

$$h_Q(s-t_2) = h_{R_2}(t_2) = h_R(t_2) = h_Q(b-t_2) + h_Q(s-t_2)$$

where the second equality holds as one has $(I_2)_{t_2} = I_{t_2}$. The displayed equalities yield $h_Q(b-t_2) = 0$; that is, $b \leq t_2 - 1$, which contradicts Proposition 1.6(e).

To prove that equality holds in (4.4.1), notice first that it follows from (d) and Proposition 1.6(e) that $b = s_1$ holds, i.e. the socle polynomial of R is $\chi^{s_1} + \chi^s$. Next, recall from [23, Thm. 4.4] that since R is compressed one has

$$\text{length}(R) = \sum_{i=0}^s \min\{h_Q(i), h_Q(s_1-i) + h_Q(s-i)\}.$$

It thus suffices to prove that the next equality holds for all $i \geq 0$,

$$\min\{h_Q(i), h_Q(s_1 - i) + h_Q(s - i)\} = \min\{h_Q(i), h_{R_1}(i) + h_{R_2}(i)\}.$$

The inequality $t_1 \leq t_2$ holds by (4.4.2), so for $i \geq t_2$ one has $h_{R_1}(i) = h_Q(s_1 - i)$ and $h_{R_2}(i) = h_Q(s - i)$. For $i < t_2$ one has $h_{R_2}(i) = h_Q(i) \leq h_Q(s - i)$, where the inequality holds as $s = s_2$ by Proposition 1.6(a). Thus, both minima are $h_Q(i)$. \square

Tracking the properties of R is easier when one keeps an eye on R' .

4.5 Theorem. *Adopt the setup in 4.1 and assume that R is compressed. There are inequalities*

$$(4.5.1) \quad 2 \leq t' \leq s' + 1 \leq t \leq s' + 2.$$

Moreover, one has:

- (a) $h_{R'}(i) = \max\{0, h_{R_1}(i) + h_{R_2}(i) - h_Q(i)\}$ for $i \geq 0$.
- (b) $s' = \max\{i \mid h_Q(i) < h_{R_1}(i) + h_{R_2}(i)\}$.
- (c) If $t = t_2$ holds, then one has $s' + 1 = t$.
- (d) If $t' = s' + 1$ holds, then one has $t' = t_2$.

Proof. Part (a) follows immediately from (4.3.1) and Theorem 4.4(a).

(b): The first equality below holds by the definition of s' and the second follows from part (a):

$$s' = \max\{i \mid h_{R'}(i) \neq 0\} = \max\{i \mid h_Q(i) < h_{R_1}(i) + h_{R_2}(i)\}.$$

We can now prove the inequalities in (4.5.1). The first one holds as I' is contained in \mathfrak{q}^2 , and the second inequality holds by the definitions of s' and t' ; see (1.1.1). By Theorem 4.4(b) one has $t = \min\{i \mid h_Q(i) > h_{R_1}(i) + h_{R_2}(i)\}$, so the third inequality follows from part (b). For $i \leq t - 1$ one has $h_Q(i) \leq h_{R_1}(i) + h_{R_2}(i)$ by Theorem 4.4(b) and per 4.4(b') equality can only hold for $i = t - 1$. The inequality $t \leq s' + 2$ now follows from part (b).

(c): The inequality $s' + 1 \leq t$ holds by (4.5.1). Assume that $t = t_2$ holds, as R_2 is compressed one then has $h_{R_2}(t - 1) = h_Q(t - 1)$. By (4.4.2) one has $t \leq s_1$, so $h_{R_1}(t - 1)$ is positive, whence $h_{R_1}(t - 1) + h_{R_2}(t - 1) > h_Q(t - 1)$ holds. From part (b) one now gets $s' \geq t - 1$.

(d): By (4.4.2) one has $t_1 \leq t_2$, so $t' = t_1$ holds by Proposition 1.6(b). Thus, if $t' = s' + 1$ holds, then (b) yields $h_Q(t_1) \geq h_{R_1}(t_1) + h_{R_2}(t_1)$. As $h_{R_1}(t_1)$ is positive, this implies that $h_Q(t_1) > h_{R_2}(t_1)$ holds, so one has $t_1 \geq t_2$ as R_2 is compressed. \square

4.6 Proposition. *Adopt the setup in 4.1 and assume that R is compressed. The next assertions hold.*

- (a) If $s_1 = 2$ or $s_1 = 3 = s_2$ holds, then R' is Golod of type e and level of socle degree 1. That is, one has $R' = Q/\mathfrak{q}^2$.
- (b) If $s_1 = 3 < s_2$ holds, then R' is of type e and level of socle degree 2, and one has $R' = Q/(I_1 + \mathfrak{q}^3)$.
- (c) If $s_1 \geq 4$ holds, then R' is Golod. Moreover, if $t = t_2$ holds, then R' is of type $h_{R_1}(t - 1)$ and level of socle degree $t - 1$, and one has $R' = Q/(I_1 + \mathfrak{q}^t)$.

Proof. (a): It suffices to show that R' has h -vector $(1, e)$. Indeed, this implies that R' is Q/\mathfrak{q}^2 , so R' is Golod, see [1, Prop. 5.2.4], and evidently of type e and level of socle degree 1. Assume first that $s_1 = 2$ holds. By (4.4.2) the possible values of s_2

are 2 and 3. For $s_2 = 2$, both R_1 and R_2 have h -vector $(1, e, 1)$, so Theorem 4.5(a) yields $h_{R'} = (1, e)$. In case $s_2 = 3$, the h -vector of R_2 is $(1, e, e, 1)$, so 4.5(a) yields $h_{R'} = (1, e)$. Finally, if one has $s_1 = 3 = s_2$, then R_1 and R_2 both have h -vector $(1, e, e, 1)$. As $e \geq 3$, one has $2e \leq \binom{e+1}{2} = h_Q(2)$, so 4.5(a) yields $h_{R'} = (1, e)$.

(b): The h -vector of R_1 is $(1, e, e, 1)$. By (4.4.2) the possible values of s_2 are 4 and 5, so the h -vector of R_2 is

$$(1, e, \binom{e+1}{2}, e, 1) \quad \text{or} \quad (1, e, \binom{e+1}{2}, \binom{e+1}{2}, e, 1) .$$

In either case it follows from Theorem 4.5(a) that R' has h -vector $(1, e, e)$; in particular, it has type e . Further, $t_2 = 3$ holds in either case, see (2.2.1), so the ideal $I' = I_1 + I_2$ is contained in $I_1 + \mathfrak{q}^3$. As the quotients R' and $Q/(I_1 + \mathfrak{q}^3)$ have the same h -vector, they are equal as claimed. With $\mathfrak{m}_1 = \mathfrak{q}/I_1$ one can rewrite the equality of rings as an isomorphism $R' \cong R_1/\mathfrak{m}_1^3$. It now follows from Proposition 2.5(a) that R' is level of socle degree 2.

(c): Proposition 1.6(a) yields $t \geq t_2$. First we assume that equality holds and argue that one has

$$(*) \quad h_{R'}(i) = \begin{cases} h_{R_1}(i) & \text{for } i \leq t-1 \\ 0 & \text{for } i \geq t. \end{cases}$$

Indeed, Theorem 4.5(c) yields $s' = t-1$, so $h_{R'}(i) = 0$ holds for $i \geq t$. For $i \leq t-1$ the assumption $t = t_2$ implies the equalities $h_R(i) = h_Q(i) = h_{R_2}(i)$, so $h_{R'}(i) = h_{R_1}(i)$ holds by (4.3.1).

The ideal $I_1 + I_2$ is contained in $I_1 + \mathfrak{q}^t$, still by the assumption $t = t_2$, and (*) shows that the quotients R' and $Q/(I_1 + \mathfrak{q}^t)$ have the same h -vector, so they are equal as claimed. As above, one now has $R' \cong R_1/\mathfrak{m}_1^t$. It follows that R' is of type $h_{R_1}(t-1)$, and by Proposition 2.5(a) it is level of socle degree $t-1$. Finally, as $s_1 \geq 4$ holds by assumption, R' is Golod by Proposition 2.5(b).

Assuming now that $t > t_2$ holds, (4.4.2) yields $t_2 \leq s_1 - 1$. As the h -vectors of R_1 and R_2 are symmetric and unimodal, see 2.2, this explains the first inequality in the next display. The sharp inequality holds by the assumptions $e \geq 3$ and $s_1 \geq 4$, and the final equality holds in view of (2.0.1).

$$\begin{aligned} h_{R_1}(s_1 - 1) + h_{R_2}(s_1 - 1) &\leq h_{R_1}(1) + h_{R_2}(s_1 - 2) \\ &\leq e + h_Q(s_1 - 2) \\ &< \binom{e+s_1-3}{e-2} + h_Q(s_1 - 2) \\ &= h_Q(s_1 - 1) . \end{aligned}$$

This inequality, $h_{R_1}(s_1 - 1) + h_{R_2}(s_1 - 1) < h_Q(s_1 - 1)$, implies that t is at most $s_1 - 1$; see Theorem 4.4(b). Combining this with inequalities from (4.5.1), (2.2.1), (4.4.2), and Proposition 1.6(b) one gets

$$s' + 1 \leq t \leq s_1 - 1 \leq 2t_1 - 2 = 2t' - 2 ,$$

whence R' is Golod by 1.2. \square

4.7 Remark. Under the assumptions in Proposition 4.6(b) the ring R' may not be Golod. In the notation from the proof one has $R' \cong R_1/\mathfrak{m}_1^3$, and in [13, Thm. (4.2)] we identify conditions under which this quotient is not Golod.

With the next theorem, which conveniently describes R' as a truncation of the Gorenstein ring R_1 , we transition to the setup of Sections 5–8 where the focus is

exclusively on rings of embedding dimension 3. The theorem is important to the central Construction 5.5.

4.8 Theorem. *Adopt the setup in 4.1. Assume that $e = 3$ and R is compressed. If $\lceil \frac{s+1}{2} \rceil = t$ holds, then one has $R' = Q/(I_1 + \mathfrak{q}^t)$.*

Proof. By Proposition 1.6(a), the assumption on t , and (2.2.1) one has $t_2 = t$. For $s_1 \geq 4$ the assertion now follows from Proposition 4.6(c). For $s_1 = 3$ one has $3 \leq s \leq 5$, see (4.4.2). For $s = 3$ the h -vector of R is $(1, 3, 6, 2)$, see Table 4.9.1, so one has $\lceil \frac{s+1}{2} \rceil = 2 < 3 = t$. For $s = 4$ and $s = 5$ one has $R' = Q/(I_1 + \mathfrak{q}^3)$ by Proposition 4.6(b). For $s_1 = 2$ one has $2 \leq s \leq 3$ per (4.4.2). For $s = 2$ and $s = 3$ one has, respectively, $h_R = (1, 3, 2)$ and $h_R = (1, 3, 4, 1)$, see Table 4.9.1, so $\lceil \frac{s+1}{2} \rceil = 2 = t$ holds. As I_1 is contained in \mathfrak{q}^2 one has $I_1 + \mathfrak{q}^t = \mathfrak{q}^2$, so Proposition 4.6(a) yields $R' = Q/(I_1 + \mathfrak{q}^t)$. \square

4.9. Adopt the setup in 4.1; assume that $e = 3$ and R is compressed. For convenience we tabulate the h -vectors of R for frequently referenced combinations of s and s_1 ; the formula for the Hilbert function comes from (3.1.1).

(s_1, s)	$h_R(i)$	h_R
(2, 2)	$\min \left\{ \binom{2+i}{2}, 2\binom{4-i}{2} \right\}$	(1, 3, 2)
(2, 3)	$\min \left\{ \binom{2+i}{2}, \binom{4-i}{2} + \binom{5-i}{2} \right\}$	(1, 3, 4, 1)
(3, 3)	$\min \left\{ \binom{2+i}{2}, 2\binom{5-i}{2} \right\}$	(1, 3, 6, 2)
(4, 4)	$\min \left\{ \binom{2+i}{2}, 2\binom{6-i}{2} \right\}$	(1, 3, 6, 6, 2)
(4, 5)	$\min \left\{ \binom{2+i}{2}, \binom{6-i}{2} + \binom{7-i}{2} \right\}$	(1, 3, 6, 9, 4, 1)
(6, 6)	$\min \left\{ \binom{2+i}{2}, 2\binom{8-i}{2} \right\}$	(1, 3, 6, 10, 12, 6, 2)
(6, 7)	$\min \left\{ \binom{2+i}{2}, \binom{8-i}{2} + \binom{9-i}{2} \right\}$	(1, 3, 6, 10, 15, 9, 4, 1)
(7, 9)	$\min \left\{ \binom{2+i}{2}, \binom{9-i}{2} + \binom{11-i}{2} \right\}$	(1, 3, 6, 10, 15, 21, 13, 7, 3, 1)

TABLE 4.9.1. The h -vectors of select compressed rings R .

5. MINIMAL GRADED FREE RESOLUTIONS

Let k be a field and (Q, \mathfrak{q}) the local k -algebra of power series in three variables with coefficients in k . Let $J \subseteq \mathfrak{q}^2$ be a \mathfrak{q} -primary homogeneous ideal in Q and set $S = Q/J$. The minimal graded free resolution of S over Q has the form

$$Q \longleftarrow \bigoplus_{j \geq 1} Q^{\beta_{1j}(S)}(-j) \longleftarrow \bigoplus_{j \geq 1} Q^{\beta_{2j}(S)}(-j) \longleftarrow \bigoplus_{j \geq 1} Q^{\beta_{3j}(S)}(-j) \longleftarrow 0.$$

As the defining ideal J is contained in \mathfrak{q}^2 and the resolution is minimal, the graded Betti numbers $\beta_{ij}(S)$ vanish for $j \leq i$. The Hilbert series $H_S(\chi) = \sum_{j \geq 0} h_S(j) \chi^j$ is a rational function, see [8, Lem. 4.1.13],

$$(5.0.1) \quad H_S(\chi) = \frac{B_S(\chi)}{(1 - \chi)^3},$$

where the polynomial $B_S(\chi) = \sum_{j \geq 0} b_S(j) \chi^j$ has coefficients

$$(5.0.2) \quad b_S(0) = 1 \quad \text{and} \quad b_S(j) = -\beta_{1j}(S) + \beta_{2j}(S) - \beta_{3j}(S) \quad \text{for } j \geq 1.$$

The next result is not new—it follows from work of Boij [6, Prop. 3.3]—but included to match Theorem 5.4. Our proof uses the existence of algebra structures on free resolutions over Q , which we recall briefly in 5.2 and in further detail in 6.1.

5.1 Proposition. *Let $J \subseteq \mathfrak{q}^2$ be a \mathfrak{q} -primary homogeneous ideal in Q such that the quotient $S = Q/J$ is compressed Gorenstein of socle degree s and initial degree t .*

If s is odd, then the minimal graded free resolution of S over Q has the form

$$Q \leftarrow \begin{array}{ccc} Q^{t+1}(-t) & & Q^\beta(-t-1) \\ \oplus & \leftarrow & \oplus \\ Q^\beta(-t-1) & & Q^{t+1}(-t-2) \end{array} \leftarrow Q(-s-3) \leftarrow 0,$$

for some integer $\beta \geq 0$.

If s is even, then the minimal graded free resolution of S over Q has the form

$$Q \leftarrow Q^{2t+1}(-t) \leftarrow Q^{2t+1}(-t-1) \leftarrow Q(-s-3) \leftarrow 0.$$

The odd socle degree case in 5.1 also follows from recent work of Vandeboert [27, Prop. 3.3], who additionally shows that the integer β is at most t .

5.2. By a result of Buchsbaum and Eisenbud [9] the minimal free resolution of S over Q has a structure of a commutative differential graded algebra. This structure is not unique, but the induced graded-commutative algebra structure on $\text{Tor}_*^Q(S, \mathfrak{k})$ is unique. If S is Gorenstein, then $\text{Tor}_*^Q(S, \mathfrak{k})$ is a Poincaré duality algebra; this is due to Avramov and Golod, see for example [2, 1.4.2] or the original [3].

Proof of 5.1. By the definition of t one has $\beta_{1j}(S) = 0$ for $j \leq t-1$ and hence $\beta_{2j}(S) = 0$ for $j \leq t$ and $\beta_{3j}(S) = 0$ for $j \leq t+1$. Per (5.0.2) one thus has

$$(1) \quad b_S(t) = -\beta_{1t}(S) \quad \text{and} \quad b_S(t+1) = -\beta_{1t+1}(S) + \beta_{2t+1}(S).$$

The assumption $J \subseteq \mathfrak{q}^2$ implies that t and hence s is at least 2. By 2.2 one has $h_S(i) = h_Q(i)$ for $i \leq t-1$ and, therefore,

$$\begin{aligned} (1-\chi)^3 H_S(\chi) &= 1 + \sum_{i=t}^s (h_S(i) - 3h_S(i-1) + 3h_S(i-2) - h_S(i-3)) \chi^i \\ &\quad + (-3h_S(s) + 3h_S(s-1) - h_S(s-2)) \chi^{s+1} \\ &\quad + (3h_S(s) - h_S(s-1)) \chi^{s+2} \\ &\quad - h_S(s) \chi^{s+3}. \end{aligned}$$

Per (2.2.1) the equality $t = \lceil \frac{s+1}{2} \rceil$ holds; together with (5.0.1) and the formula for $h_S(i)$ from 2.2 it yields

$$\begin{aligned} b_S(t) &= h_Q(s-t) - 3h_Q(t-1) + 3h_Q(t-2) - h_Q(t-3) \\ &= \binom{s-t+2}{2} - 3\binom{t+1}{2} + 3\binom{t}{2} - \binom{t-1}{2} \\ (2) \quad &= \binom{s-t+2}{2} - 3t - \binom{t-1}{2} \\ &= - \begin{cases} t+1 & \text{if } s \text{ is odd} \\ 2t+1 & \text{if } s \text{ is even.} \end{cases} \end{aligned}$$

For $s \geq 3$ one has $t + 1 \leq s$, as the equality $t + 1 = \lceil \frac{s+3}{2} \rceil$ holds. In view of 2.2 and (5.0.1) one now gets

$$(3) \quad \begin{aligned} b_S(t+1) &= h_Q(s-t-1) - 3h_Q(s-t) + 3h_Q(t-1) - h_Q(t-2) \\ &= \binom{s-t+1}{2} - 3\binom{s-t+2}{2} + 3\binom{t+1}{2} - \binom{t}{2} \\ &= \begin{cases} 0 & \text{if } s \text{ is odd} \\ 2t+1 & \text{if } s \geq 4 \text{ is even.} \end{cases} \end{aligned}$$

For $s = 2$ one has $t = 2$; a direct computation now yields

$$(4) \quad b_S(t+1) = 5 = 2t+1 \quad \text{if } s = 2.$$

As S is Gorenstein, $\text{Tor}_*^Q(S, \mathbf{k})$ is a Poincaré duality algebra; see 5.2. In particular, for every nonzero homogeneous element $x \in \text{Tor}_1^Q(S, \mathbf{k})$ there is a homogeneous element $y \in \text{Tor}_2^Q(S, \mathbf{k})$ with $xy \neq 0$, so one has

$$(5) \quad |x| \geq t, \quad |y| \geq t+1, \quad \text{and} \quad s+3 = |xy| = |x| + |y|.$$

If s is odd, then $s+3 = 2t+2$ holds, so by (5) one has $|x| = t$ and $|y| = t+2$ or $|x| = t+1 = |y|$. It follows from (3) and (1) that $\beta_{1t+1}(S) = \beta_{2t+1}(S)$ holds, and with the abbreviated notation β for this number it now follows from (1) and (2) that the graded minimal free resolution of S has the asserted format.

If s is even, then $s+3 = 2t+1$ holds, and (5) yields $|x| = t$ and $|y| = t+1$. The desired conclusion now follows from (1)–(4). \square

In the balance of this section we adopt Setup 4.1 with $e = 3$ and further assume that also R is compressed.

5.3 Lemma. *If $\lceil \frac{s+1}{2} \rceil = t$ holds, then one has $a = s_1 - t + 1$ and there are equalities*

$$\beta_{1t}(R) = -\binom{a+1}{2} + \begin{cases} t+1 & \text{if } s \text{ is odd} \\ 2t+1 & \text{if } s \text{ is even} \end{cases}$$

and

$$\beta_{1t+1}(R) - \beta_{2t+1}(R) = a(a+2) - \begin{cases} 0 & \text{if } s \text{ is odd} \\ 2t+1 & \text{if } s \text{ is even.} \end{cases}$$

Proof. By the assumption on t and Theorem 4.4(d) one has $a = s_1 - t + 1$. By (3.1.1) one has $h_R(i) = h_Q(i)$ for $i \leq t-1$ and, therefore,

$$\begin{aligned} (1-\chi)^3 H_R(\chi) &= 1 + \sum_{i=t}^s (h_R(i) - 3h_R(i-1) + 3h_R(i-2) - h_R(i-3))\chi^i \\ &\quad + (-3h_R(s) + 3h_R(s-1) - h_R(s-2))\chi^{s+1} \\ &\quad + (3h_R(s) - h_R(s-1))\chi^{s+2} \\ &\quad - h_R(s)\chi^{s+3}. \end{aligned}$$

By Theorem 4.4(c) and (3.1.1) one has $h_R(i) = h_Q(s_1 - i) + h_Q(s - i)$ for $i \geq t$. In view of (5.0.1) this yields

$$\begin{aligned}
 b_R(t) &= h_Q(s_1 - t) + h_Q(s - t) - 3h_Q(t - 1) + 3h_Q(t - 2) - h_Q(t - 3) \\
 &= \binom{s_1 - t + 2}{2} + \binom{s - t + 2}{2} - 3\binom{t + 1}{2} + 3\binom{t}{2} - \binom{t - 1}{2} \\
 (1) \quad &= \binom{a + 1}{2} + \binom{s - t + 2}{2} - 3t - \binom{t - 1}{2} \\
 &= \binom{a + 1}{2} - \begin{cases} t + 1 & \text{if } s \text{ is odd} \\ 2t + 1 & \text{if } s \text{ is even.} \end{cases}
 \end{aligned}$$

For $s \geq 3$ one has $t + 1 \leq s$, as $t + 1 = \lceil \frac{s+3}{2} \rceil$ holds by assumption. In view of (3.1.1) and (5.0.1) one now gets

$$\begin{aligned}
 b_R(t + 1) &= h_Q(s_1 - t - 1) + h_Q(s - t - 1) - 3(h_Q(s_1 - t) + h_Q(s - t)) \\
 &\quad + 3h_Q(t - 1) - h_Q(t - 2) \\
 &= \binom{s_1 - t + 1}{2} + \binom{s - t + 1}{2} - 3\left(\binom{s_1 - t + 2}{2} + \binom{s - t + 2}{2}\right) + 3\binom{t + 1}{2} - \binom{t}{2} \\
 (2) \quad &= \binom{a}{2} + \binom{s - t + 1}{2} - 3\left(\binom{a + 1}{2} + \binom{s - t + 2}{2}\right) + 3\binom{t + 1}{2} - \binom{t}{2} \\
 &= -a(a + 2) + \binom{s - t + 1}{2} - 3\binom{s - t + 2}{2} + 3\binom{t + 1}{2} - \binom{t}{2} \\
 &= -a(a + 2) + \begin{cases} 0 & \text{if } s \text{ is odd} \\ 2t + 1 & \text{if } s \geq 4 \text{ is even.} \end{cases}
 \end{aligned}$$

For $s = 2$ one has $s_1 = 2 = t$ and $a = 1$; a direct computation yields

$$(3) \quad b_R(t + 1) = 2 = -a(a + 2) + 2t + 1 \quad \text{if } s = 2.$$

As shown in the first lines of the proof of Proposition 5.1 one has $b_R(t) = -\beta_{1t}(R)$ and $b_R(t + 1) = -\beta_{1t+1}(R) + \beta_{2t+1}(R)$, so (1)–(3) yield the asserted equalities. \square

We can now give a detailed description of the minimal graded free resolution of R over Q in the case of interest: $\lceil \frac{s+1}{2} \rceil = t$.

5.4 Theorem. *Assume that $\lceil \frac{s+1}{2} \rceil = t$ holds and set*

$$f_0 = \binom{a+1}{2}, \quad f_1 = a(a + 2), \quad \text{and} \quad f_2 = \binom{a+2}{2}.$$

If s is odd, then the minimal graded free resolution of R over Q has the form

$$\begin{array}{ccccccc}
 & & Q^{t+1-f_0}(-t) & & Q^\beta(-t-1) & & Q(-s_1-3) \\
 Q \longleftarrow & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & 0 \\
 & & Q^{f_1+\beta}(-t-1) & & Q^{t+1+f_2}(-t-2) & & Q(-s-3) & &
 \end{array}$$

for some integer $\beta \geq 0$.

If s is even, then the minimal graded free resolution of R over Q has the form

$$\begin{array}{ccccccc}
 & & Q^{2t+1-f_0}(-t) & & Q^{2t+1-f_1+\beta}(-t-1) & & Q(-s_1-3) \\
 Q \longleftarrow & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \longleftarrow & 0 \\
 & & Q^\beta(-t-1) & & Q^{f_2}(-t-2) & & Q(-s-3) & &
 \end{array}$$

for some integer $\beta \geq \max\{0, f_1 - 2t - 1\}$.

We prepare for the proof with a construction that is reused in the next section.

5.5 Construction. Adopt the assumptions and notation from Theorem 5.4. The kernel of the surjection $R \rightarrow R_2$ is the ideal $I_2/(I_1 \cap I_2)$, which as a Q -module is isomorphic to $(I_1 + I_2)/I_1 = (I_1 + \mathfrak{q}^t)/I_1$; see Theorem 4.8. This is the t^{th} power of the maximal ideal of the compressed Gorenstein ring R_1 . One has $t \geq t_1$ by Proposition 1.6(a), and $a = s_1 - t + 1$ holds by Lemma 5.3, so as a Q -module, $(I_1 + \mathfrak{q}^t)/I_1$ is by Proposition 2.6 isomorphic to $D = \text{Ext}_Q^3(Q/\mathfrak{q}^a, Q)$.

The module D is the dualizing module of the Cohen–Macaulay ring Q/\mathfrak{q}^a . The socle degree of Q/\mathfrak{q}^a is $a - 1$, so as a graded module D is concentrated in degree $a + 2$. As a homomorphism of graded Q -modules, the surjection $R \rightarrow R_2$ has kernel concentrated in degree t . Thus one has an exact sequence of graded Q -modules, $0 \rightarrow D(-t - a - 2) \rightarrow R \rightarrow R_2 \rightarrow 0$, and the Horseshoe Lemma yields an exact sequence of graded free resolutions,

$$(5.5.1) \quad 0 \rightarrow F' \rightarrow \tilde{F} \rightarrow F'' \rightarrow 0,$$

where F' and F'' are minimal. The resolution F'' of R_2 is described in Proposition 5.1, and the resolution F' of $D(-t - a - 2)$ can be described in similar detail. Indeed, one has $t + a + 2 = s_1 + 3$; the resolution F' of $D(-s_1 - 3)$ is obtained from the Q -dual of the minimal graded free resolution of Q/\mathfrak{q}^a . Thus, F' has the form

$$(5.5.2) \quad Q^{f_0}(-t) \leftarrow Q^{f_1}(-t-1) \leftarrow Q^{f_2}(-t-2) \leftarrow Q(-s_1-3) \leftarrow 0.$$

Indeed, in the resolution of Q/\mathfrak{q}^a the free modules in degrees 0 and 1 have ranks 1 and $h_Q(a) = f_2$, and the rank of the free module in degree 3 is the rank of the socle of Q/\mathfrak{q}^a , i.e. $h_Q(a-1) = f_0$. Finally, one has $f_0 + f_2 - 1 = f_1$.

Proof of 5.4. The minimal graded free resolution F of R is a direct summand of the complex \tilde{F} in the diagram (5.5.1), where F' is described in (5.5.2) and F'' in Proposition 5.1. By Theorem 4.4(c) the socle polynomial of R is $\chi^{s_1} + \chi^s$, so F_3 is a free module of rank 2 with generators in degrees $s_1 + 3$ and $s + 3$. (Notice that this means that one has $F_3 = \tilde{F}_3$.)

Assume first that s is odd. It follows from (5.5.1) and the descriptions of F' and F'' that $\tilde{F}_1 = F'_1 \oplus F''_1$ and, therefore, F_1 has generators in degrees t and $t + 1$ only. By Lemma 5.3 there are $t + 1 - f_0$ generators in degree t , and with $\beta = \beta_{2t+1}(R)$ there are $f_1 + \beta$ generators in degree $t + 1$. Similarly it follows that F_2 is generated in degrees $t + 1$ and $t + 2$. There are β generators in degree $t + 1$; that also determines the number of generators in degree $t + 2$, as the total rank of the free module F_2 is $\text{rank}_R F_1 + \text{rank}_R F_3 - \text{rank}_R F_0 = t + 1 - f_0 + f_1 + \beta + 2 - 1 = t + 1 + f_2 + \beta$.

Assume now that s is even. As in the odd case, F_1 has generators in degrees t and $t + 1$ only. By Lemma 5.3 there are $2t + 1 - f_0$ generators in degree t ; set $\beta = \beta_{1t+1}(R)$. As in the odd case, F_2 is generated in degrees $t + 1$ and $t + 2$ with $2t + 1 - f_1 + \beta$ generators in degree $t + 1$, and that also determines that there are f_2 generators in degree $t + 2$. \square

6. PARAMETERS OF MULTIPLICATION ON THE TOR-ALGEBRA

Throughout this section we adopt Setup 4.1; we further assume that $e = 3$ holds and that R , like R_1 and R_2 , is compressed. Recall from Theorem 4.4(d) the equality

$$(6.0.1) \quad a = s_1 - t_2 + 1.$$

As in Theorem 5.4 set

$$(6.0.2) \quad f_0 = \binom{a+1}{2}, \quad f_1 = a(a+2), \quad \text{and} \quad f_2 = \binom{a+2}{2},$$

and recall from (5.5.2) the relation

$$(6.0.3) \quad f_0 - f_1 + f_2 - 1 = 0.$$

6.1. The algebra $A = \text{Tor}_*^Q(R, k)$, see 5.2, is bigraded; we refer to its homogeneous components with double indices: $A_{i,j} = (\mathbf{A}_i)_j = \text{Tor}_i^Q(R, k)_j$. The multiplicative structure on A can be described in terms of three parameters

$$p = \text{rank}_k(\mathbf{A}_1 \cdot \mathbf{A}_1), \quad q = \text{rank}_k(\mathbf{A}_1 \cdot \mathbf{A}_2), \quad \text{and} \quad r = \text{rank}_k \delta$$

where $\delta: \mathbf{A}_2 \rightarrow \text{Hom}_k(\mathbf{A}_1, \mathbf{A}_3)$ is defined by $\delta(y)(x) = xy$ for $x \in \mathbf{A}_1$ and $y \in \mathbf{A}_2$. By [4, Thm. 2.1] and [2, 3.4.2 and 3.4.3] there exist bases

$$\mathbf{e}_1, \dots, \mathbf{e}_m \text{ for } \mathbf{A}_1, \quad \mathbf{f}_1, \dots, \mathbf{f}_{m+1} \text{ for } \mathbf{A}_2, \quad \text{and} \quad \mathbf{g}_1, \mathbf{g}_2 \text{ for } \mathbf{A}_3$$

such that the multiplicative structure on A is one of following:

$$(6.1.1) \quad \begin{array}{ll} \mathbf{B} : & \mathbf{e}_1 \mathbf{e}_2 = \mathbf{f}_3 & \mathbf{e}_i \mathbf{f}_i = \mathbf{g}_1 \text{ for } 1 \leq i \leq 2 \\ \mathbf{G}(r) : & & \mathbf{e}_i \mathbf{f}_i = \mathbf{g}_1 \text{ for } 1 \leq i \leq r \\ \mathbf{H}(p, q) : & \mathbf{e}_{p+1} \mathbf{e}_i = \mathbf{f}_i \text{ for } 1 \leq i \leq p & \mathbf{e}_{p+1} \mathbf{f}_{p+j} = \mathbf{g}_j \text{ for } 1 \leq j \leq q. \end{array}$$

Here it is understood that all products that are not listed, and not determined by those listed and the rules of graded commutativity, are zero.

We say that R is of class \mathbf{B} if the multiplicative structure on A is given by \mathbf{B} in (6.1.1) etc. Notice that the classes $\mathbf{G}(1)$ and $\mathbf{H}(0, 1)$ coincide as do $\mathbf{G}(0)$ and $\mathbf{H}(0, 0)$. This overlap is usually avoided by only using $\mathbf{G}(r)$ with $r \geq 2$, see [2, 1.3], but here it is convenient to refer to the first class as $\mathbf{G}(1)$. For the class $\mathbf{G}(0) = \mathbf{H}(0, 0)$ we only use the latter symbol; the rings of this class are precisely the Golod rings, and they are mostly referred to as such.

The relationship between the parameters p , q , and r and the classes is simple:

$$(6.1.2) \quad \begin{array}{c|ccc} \text{Class of } R & p & q & r \\ \hline \mathbf{B} & 1 & 1 & 2 \\ \mathbf{G}(r) [r \geq 1] & 0 & 1 & r \\ \mathbf{H}(p, q) [q \leq 2] & p & q & q \end{array}$$

6.2 Remark. The multiplicative structures described in 6.1 is the subset of those found in [4, Thm. 2.1] that can be realized by quotient rings Q/J of type 2, where J is \mathfrak{q} -primary and contained in \mathfrak{q}^2 ; see also Remark 7.6. A quotient ring Q/J of type 1, such as R_1 or R_2 , is of class $\mathbf{C}(3)$ if it is complete intersection and otherwise of class $\mathbf{G}(r)$ with r equal to the minimal number of generators of J . A quotient ring of type 3 or higher— R' is typically such a ring, see Proposition 4.6—may in addition to the classes in 6.1 be of what is known as class \mathbf{T} , and that is exactly what happens in [13, Thm. (4.2)]—the case discussed in Remark 4.7.

6.3 Lemma. *The equality $\lceil \frac{s+1}{2} \rceil = t$ holds if and only if one has*

$$f_0 = \binom{2 + s_1 - \lceil \frac{s+1}{2} \rceil}{2} < \begin{cases} \lceil \frac{s+1}{2} \rceil + 1 & \text{if } s \text{ is odd} \\ 2 \lceil \frac{s+1}{2} \rceil + 1 & \text{if } s \text{ is even.} \end{cases}$$

Proof. As one has $a = s_1 - t_2 + 1$ per (6.0.1) and the equality $\lceil \frac{s+1}{2} \rceil = t_2$ is part of the setup, the assertion follows from Lemma 3.9. \square

Per 1.2 the next result shows that R is Golod if the difference $s - s_1$ is not too big; this is recorded in detail in Corollaries 7.4 and 7.5.

6.4 Proposition. *One has $\lceil \frac{s+1}{2} \rceil < t$ if and only if the next inequality holds,*

$$s - s_1 \leq \begin{cases} \frac{s+2-\sqrt{4s+13}}{2} & \text{if } s \text{ is odd} \\ \frac{s+1-\sqrt{8s+25}}{2} & \text{if } s \text{ is even.} \end{cases}$$

Proof. The inequality $\lceil \frac{s+1}{2} \rceil \leq t$ holds by (3.1.2). Thus it follows from Lemma 6.3 that the sharp inequality holds if and only if one has

$$\left(\frac{2 + s_1 - \lceil \frac{s+1}{2} \rceil}{2} \right) \geq \begin{cases} \lceil \frac{s+1}{2} \rceil + 1 & \text{if } s \text{ is odd} \\ 2 \lceil \frac{s+1}{2} \rceil + 1 & \text{if } s \text{ is even.} \end{cases}$$

This inequality simplifies to

$$(4 + 2s_1 - 2 \lceil \frac{s+1}{2} \rceil) (2 + 2s_1 - 2 \lceil \frac{s+1}{2} \rceil) \geq \begin{cases} 4(2 + 2 \lceil \frac{s+1}{2} \rceil) & \text{if } s \text{ is odd} \\ 8(1 + 2 \lceil \frac{s+1}{2} \rceil) & \text{if } s \text{ is even.} \end{cases}$$

If s is odd, then one has $2 \lceil \frac{s+1}{2} \rceil = s + 1$, and the inequality simplifies to a quadratic inequality in $(s - s_1)$:

$$(*) \quad 4(s - s_1)^2 - 4(s + 2)(s - s_1) + s^2 - 9 \geq 0.$$

If s is even, then one has $2 \lceil \frac{s+1}{2} \rceil = s + 2$, and the inequality simplifies to:

$$(**) \quad 4(s - s_1)^2 - 4(s + 1)(s - s_1) + s^2 - 6s - 24 \geq 0.$$

The asserted bounds come from the meaningful (smaller) roots of the quadratic polynomials corresponding to $(*)$ and $(**)$. \square

The central results of this section are Propositions 6.7 and 6.9 and Corollary 6.12. Between them they show that R , with exception for a few special cases, is Golod or of class $\mathbf{G}(r)$. The proofs rely on the next two lemmas.

6.5 Lemma. *If $3 \leq s_1$ holds, then one has $\mathrm{Tor}_1^Q(R, \mathbf{k}) \cdot \mathrm{Tor}_1^Q(R, \mathbf{k}) = 0$.*

Proof. By (3.1.2) there is an inequality $\lceil \frac{s+1}{2} \rceil \leq t$. If strict inequality holds, then R is Golod by 1.2, i.e. of class $\mathbf{H}(0, 0)$; in particular, one has $p = 0$, cf. (6.1.2). We now assume that $\lceil \frac{s+1}{2} \rceil = t$ holds. Together with the assumption $3 \leq s_1$ and Proposition 6.4 this implies that s is at least 4 and, therefore, $3 \leq t$. The minimal graded free resolution of R over Q is given by Theorem 5.4. In the bigraded \mathbf{k} -algebra $\mathbf{A} = \mathrm{Tor}_*^Q(R, \mathbf{k})$, the internal degree of a product of nonzero homogeneous elements from \mathbf{A}_1 is at least $2t$. Since \mathbf{A}_2 is concentrated in internal degrees $t + 1$ and $t + 2$ and the inequality $2t > t + 2$ holds, one has $\mathbf{A}_1 \cdot \mathbf{A}_1 = 0$. \square

6.6 Lemma. *If one has $s_1 = 4$ and $s = 5$, then the internal degree of a nonzero element in the subspace $\mathrm{Tor}_1^Q(R, \mathbf{k}) \cdot \mathrm{Tor}_2^Q(R, \mathbf{k})$ of $\mathrm{Tor}_3^Q(R, \mathbf{k})$ is $8 = s + 3$.*

Proof. The h -vector of R is $(1, 3, 6, 9, 4, 1)$, see Table 4.9.1. In particular, one has $\lceil \frac{s+1}{2} \rceil = 3 = t$, so (6.0.1) yields $a = 2$, and the minimal graded free resolution of R over Q is given by Theorem 5.4 with $f_0 = 3$, $f_1 = 8$, and $f_2 = 6$; see (6.0.2). As one has $t + 1 - f_0 = 1$, the ideal I has a single generator of degree 3, and the remaining generators have degree 4. The single generator in degree 3 generates a subspace of rank 3 in I_4 , so since one has $h_R(4) = 4$ compared to $h_Q(4) = 15$, there are $(15 - 3) - 4 = 8$ generators of degree 4. In particular, one has $\beta = 8 - f_1 = 0$. Thus the minimal graded free resolution of R over Q has the form

$$Q \leftarrow Q(-3) \oplus Q^8(-4) \leftarrow Q^{10}(-5) \leftarrow Q(-7) \oplus Q(-8) \leftarrow 0.$$

In the bigraded \mathbf{k} -algebra $\mathbf{A} = \mathrm{Tor}_*^Q(R, \mathbf{k})$, the internal degree of a product of nonzero homogeneous elements $x \in \mathbf{A}_1$ and $y \in \mathbf{A}_2$ is $|xy| = |x| + |y| \geq 3 + 5 = 8$, so if xy is nonzero, then equality must hold as \mathbf{A}_3 is concentrated in degrees 7 and 8. \square

6.7 Proposition. *If the inequalities $3 \leq s_1 < s$ hold, then the internal degree of a nonzero element in the subspace $\mathrm{Tor}_1^Q(R, \mathbf{k}) \cdot \mathrm{Tor}_2^Q(R, \mathbf{k})$ of $\mathrm{Tor}_3^Q(R, \mathbf{k})$ is $s + 3$; in particular, the subspace has rank at most 1. Moreover, R is Golod or of class $\mathbf{G}(r)$.*

Proof. Set $\mathbf{A} = \mathrm{Tor}_*^Q(R, \mathbf{k})$. It $\lceil \frac{s+1}{2} \rceil < t$ holds, then R is Golod by 1.2, so there are no nonzero products in $\mathbf{A}_{\geq 1}$; see 6.1. We may thus assume that $\lceil \frac{s+1}{2} \rceil = t$ holds. The minimal graded free resolution of R over Q is given by Theorem 5.4; it shows that the bigraded \mathbf{k} algebra \mathbf{A} decomposes as follows:

$$\mathbf{A} = \mathbf{A}_{00} \oplus (\mathbf{A}_{1t} \oplus \mathbf{A}_{1t+1}) \oplus (\mathbf{A}_{2t+1} \oplus \mathbf{A}_{2t+2}) \oplus (\mathbf{A}_{3s_1+3} \oplus \mathbf{A}_{3s+3}).$$

The internal degree of a product of nonzero homogeneous elements from \mathbf{A}_1 and \mathbf{A}_2 is at least $2t + 1$. If s is even, then one has $2t + 1 = s + 3 > s_1 + 3$, so a nonzero product in $\mathbf{A}_1 \cdot \mathbf{A}_2$ has degree $s + 3$; in particular $q \leq 1$ holds. If s is odd, then one has $2t + 1 = s + 2$, and the assumptions imply that s is at least 5. If $s \geq s_1 + 2$ holds, then one has $2t + 1 > s_1 + 3$, so a nonzero product in $\mathbf{A}_1 \cdot \mathbf{A}_2$ has degree $s + 3$; in particular $q \leq 1$ holds. If $s = s_1 + 1$ holds, then the assumption $\lceil \frac{s+1}{2} \rceil = t$, via Proposition 6.4, yields $s = 5$; now invoke Lemma 6.6.

By Lemma 6.5 one has $p = 0$, and as argued above q is at most 1, so R is Golod or of class $\mathbf{G}(r)$; see 6.1. \square

6.8 Remark. In case R is of class $\mathbf{G}(r)$ with $r \geq 2$, the algebra $\mathbf{A} = \mathrm{Tor}_*^Q(R, \mathbf{k})$ is described in [2, 1.3] as a trivial extension of a Poincaré duality algebra \mathbf{P} , of total rank $2(r + 1)$, by a graded \mathbf{k} -vector space \mathbf{V} with the almost trivial \mathbf{P} -module structure: $\mathbf{P}_{\geq 1}\mathbf{V} = 0$. For R of class $\mathbf{H}(0, 1) = \mathbf{G}(1)$, see 6.1, \mathbf{A} is described similarly with $\mathbf{P} = (\mathbf{k} \times \Sigma^2 \mathbf{k}) \otimes_{\mathbf{k}} (\mathbf{k} \times \Sigma \mathbf{k})$ of total rank 4, and for R of class $\mathbf{H}(0, 0)$ with $\mathbf{P} = \mathbf{k} \times \Sigma \mathbf{k}$ of rank 2.

Remark 6.2 explains why we use r_2 below to denote the minimal number of generators of the defining ideal I_2 of the Gorenstein ring R_2 .

6.9 Proposition. *Assume that $\lceil \frac{s+1}{2} \rceil = t$ holds and let m and r_2 denote the minimal number of generators of the ideals I and I_2 . The next inequality holds:*

(a) $r \geq m - f_1$.

Moreover, if one has $s_1 < s$, then the following (in)equalities hold:

(b) $r \leq r_2 - f_0$.

(c) $r = m - f_1$ if s is odd.

- (d) $r \leq m - f_1 + f_0$ if s is even.
(e) $r = m - f_1 = m - 3$ if s is even and $s_1 = \frac{s}{2} + 1$.

Proof. As in Construction 5.5 let D be the canonical module of Q/\mathfrak{q}^a and set

$$A' = \mathrm{Tor}_*^Q(D(-s_1 - 3), \mathfrak{k}), \quad A = \mathrm{Tor}_*^Q(R, \mathfrak{k}), \quad \text{and} \quad A'' = \mathrm{Tor}_*^Q(R_2, \mathfrak{k}).$$

The ranks of the \mathfrak{k} -vector spaces A'_i , A_i , and A''_i are given by (5.5.2), Theorem 5.4, and Proposition 5.1; in particular one has

$$\mathrm{rank}_{\mathfrak{k}} A_1 = m = \mathrm{rank}_{\mathfrak{k}} A_2 - 1 \quad \text{and} \quad \mathrm{rank}_{\mathfrak{k}} A_1'' = r_2 = \mathrm{rank}_{\mathfrak{k}} A_2''.$$

The exact sequence $0 \rightarrow D(-t - a - 2) \rightarrow R \rightarrow R_2 \rightarrow 0$ induces an exact sequence of \mathfrak{k} -vector spaces

$$\begin{array}{ccccccc}
0 & \longrightarrow & A'_3 & \xrightarrow[\text{(1)}]{\phi_3} & A_3 & \xrightarrow[\text{(1)}]{\psi_3} & A''_3 & \longrightarrow & 0 \\
& & & & & & & \searrow & \\
& & & & & & & & \text{(0)} \\
& & & & & & & \searrow & \\
& & & & & & & & A'_2 & \xrightarrow[\text{(f}_2\text{)}]{\phi_2} & A_2 & \xrightarrow[\text{(m+1-f}_2\text{)}]{\psi_2} & A''_2 & \longrightarrow & 0 \\
& & & & & & & \searrow & \\
& & & & & & & & \text{(r}_2\text{-m-1+f}_2\text{)} \\
& & & & & & & \searrow & \\
& & & & & & & & A'_1 & \xrightarrow[\text{(m-r}_2\text{+f}_0\text{)}]{\phi_1} & A_1 & \xrightarrow[\text{(r}_2\text{-f}_0\text{)}]{\psi_1} & A''_1 & \longrightarrow & 0 \\
& & & & & & & \searrow & \\
& & & & & & & & \text{(f}_0\text{)} \\
& & & & & & & \searrow & \\
& & & & & & & & A'_0 & \xrightarrow[\text{(0)}]{\phi_0} & A_0 & \xrightarrow[\text{(1)}]{\psi_0} & A''_0 & \longrightarrow & 0
\end{array}$$

(†)

where the numbers under the arrows indicate the ranks of the maps. These ranks are computed by way of (6.0.3). The minimal free resolutions F and F'' are differential graded algebras, see 5.2, and the map $R \rightarrow R_2$ lifts to a morphism

$$(*) \quad F \longrightarrow F''$$

of such algebras. Therefore, $\psi = (\psi_i: A_i \rightarrow A''_i)_{0 \leq i \leq 3}$ is a morphism of bigraded \mathfrak{k} -algebras.

Recall the map $\delta: A_2 \rightarrow \mathrm{Hom}_{\mathfrak{k}}(A_1, A_3)$ from 6.1 and consider the map

$$\tilde{\delta}: \psi_2(A_2) \longrightarrow \mathrm{Hom}_{\mathfrak{k}}(\psi_1(A_1), A''_3)$$

given by

$$\tilde{\delta}(\psi_2(\mathbf{y}))(\psi_1(\mathbf{x})) = \psi_1(\mathbf{x})\psi_2(\mathbf{y}) = \psi_3(\mathbf{xy}).$$

The rank \tilde{r} of $\tilde{\delta}$ is a lower bound for r , the rank of δ : To see this, notice that if $\tilde{\delta}(\psi_2(\mathbf{y}_1)), \dots, \tilde{\delta}(\psi_2(\mathbf{y}_{\tilde{r}}))$ are linearly independent maps in $\mathrm{Hom}_{\mathfrak{k}}(\psi_1(A_1), A''_3)$, then $\delta(\mathbf{y}_1), \dots, \delta(\mathbf{y}_{\tilde{r}})$ are linearly independent in $\mathrm{Hom}_{\mathfrak{k}}(A_1, A_3)$.

Part (a): Since A'' is a Poincaré duality algebra, see 5.2, the subspace $\psi_1(A_1)$ determines a subspace U of A''_2 of the same rank, such that for every $\mathbf{x} \in \psi_1(A_1)$ one has $\mathbf{x}u \neq 0$ for some $u \in U$. Thus, a lower bound for \tilde{r} is the rank of the intersection $\psi_2(A_2) \cap U$, which via (6.0.3) is at least

$$\mathrm{rank}_{\mathfrak{k}} \psi_1 + \mathrm{rank}_{\mathfrak{k}} \psi_2 - r_2 = m + 1 - f_2 - f_0 = m - f_1.$$

Part (a) facilitates the proof an auxiliary result akin to Lemma 6.6.

Sublemma. *If one has $s_1 = 2$ and $s = 3$, then the internal degree of a nonzero element in the subspace $A_1 \cdot A_2$ of A_3 is $6 = s + 3$.*

Proof. The h -vector of R is $(1, 3, 4, 1)$, see Table 4.9.1, so $\lceil \frac{s+1}{2} \rceil = 2 = t$ holds. Now (6.0.1) yields $a = 1$, and the ranks of the nonzero components of the bigraded algebra A are given by Theorem 5.4 with $f_0 = 1$, $f_1 = 3$, and $f_2 = 3$; see (6.0.2).

$$\begin{aligned} \operatorname{rank}_k A_{35} &= 1 & \text{and} & & \operatorname{rank}_k A_{36} &= 1 \\ \operatorname{rank}_k A_{23} &= \beta & \text{and} & & \operatorname{rank}_k A_{24} &= 6 \\ \operatorname{rank}_k A_{12} &= 2 & \text{and} & & \operatorname{rank}_k A_{13} &= 3 + \beta. \end{aligned}$$

If $\beta = 0$ holds, then a homogeneous product in $A_1 \cdot A_2 \subseteq A_3$ has internal degree at least $2 + 4 = 6 = s + 3$, so a nonzero product has degree $s + 3$. If β is positive, then one has $r \geq 3$ as part (a) yields $r \geq m - f_1 = 2 + \beta$. One now has $q \leq 2 < r$, which means that R is of class $\mathbf{G}(r)$, see (6.1.2); in particular $q = 1$ holds, so all nonzero products of homogeneous elements from A_1 and A_2 have the same internal degree. If this degree were 5, then the parameter r would be limited by the rank of A_{12} , which is 2; a contradiction. Thus nonzero products have degree $6 = s + 3$. \square

Assume now that $s_1 < s$ holds. For $3 \leq s_1$ it follows from Proposition 6.7 that a nonzero product of homogeneous elements from A_1 and A_2 has internal degree $s + 3$; in particular, q from 6.1 is at most 1. The Sublemma above shows that the same is true for $s_1 = 2$, as the assumption $s_1 < s$ and (4.4.2) forces $s = 3$. It follows that A has a Poincaré duality subalgebra P of total rank $2r + 2$, which captures all nontrivial products of elements from A_1 and A_2 ; see (6.1.1). For homogeneous elements $x \in P_1$ and $y \in P_2$ with $xy \neq 0$ in A_3 one has $|xy| = s + 3$. In particular, xy is not in the image of ϕ_3 whence $\psi_3(xy) \neq 0$. As ψ is a morphism of graded k -algebras, it follows that the restriction of ψ to P is injective.

Parts (b) and (d): The ranks of the graded components of ψ are computed in (\dagger), so one has

$$\begin{aligned} r &= \operatorname{rank}_k P_1 \leq \operatorname{rank}_k \psi_1 = r_2 - f_0 & \text{and} \\ r &= \operatorname{rank}_k P_2 \leq \operatorname{rank}_k \psi_2 = m + 1 - f_2. \end{aligned}$$

In view of (6.0.3) these are the asserted bounds on r .

Part (e): As the restriction of ψ to P is injective, the rank \tilde{r} of $\tilde{\delta}$ equals r . Set $W = \{w \in A_2'' \mid \psi_1(A_1) \cdot w = 0\}$, and notice that because A'' is a Poincaré duality algebra, one has

$$\tilde{r} \leq \operatorname{rank}_k \psi_2 - \operatorname{rank}_k(\psi_2(A_2) \cap W).$$

Assume that s is even and $s_1 = \frac{s}{2} + 1 < s$ holds. By (2.2.1) one has $t_2 = s_1$, which per (6.0.1) implies $a = 1$. Now (\dagger) and (6.0.2) yield $\operatorname{rank}_k \psi_2 = m + 1 - f_2 = m - 2$, while the lower bound on r from part (a) is $m - 3$. Thus one has

$$m - 3 \leq r = \tilde{r} \leq m - 2 - \operatorname{rank}_k(\psi_2(A_2) \cap W),$$

so it suffices to show that the intersection $\psi_2(A_2) \cap W$ is nonzero.

By Theorem 5.4 the ideal I has $2t$ minimal generators, say, x_1, \dots, x_{2t} of degree t , and without loss of generality one can assume that they are the first $2t$ of the $2t + 1$ minimal generators of I_2 ; see Proposition 5.1. By [9, Thm. 2.1] the ideal I_2 can be generated by the submaximal Pfaffians of a $(2t + 1) \times (2t + 1)$ skew-symmetric matrix T , and R_2 has a minimal free resolution G over Q with $\partial_2^G = T$; see also [8, Thm. 3.4.1]. A standard change of basis argument shows one can assume that $\partial_2^{F''}$

6.10 Proposition. *If one has $s = k(k+1) - 1$ and $s_1 = \frac{1}{2}k(k+3) - 1$ for some integer $k \geq 2$, then $t = \frac{1}{2}k(k+1)$ holds and I is generated by one form of degree t and $k(k+2)$ forms of degree $t+1$. Moreover, R is of class $\mathbf{G}(1)$.*

Proof. With s and s_1 as given, one has $a = \frac{1}{2}k(k+3) - \frac{1}{2}k(k+1) = k$. In the notation from Theorem 5.4 one has $f_0 = \frac{1}{2}k(k+1) = \lceil \frac{s+1}{2} \rceil$, whence Lemma 6.3 yields $\lceil \frac{s+1}{2} \rceil = t$, and we notice that $t = f_0$ holds. From Theorem 5.4 it now follows that I has exactly one generator of degree t . In turn, this implies that the first syzygy of I has no generators of degree $t+1$, i.e. $\beta = 0$ in 5.4, and it follows that the number of generators in degree $t+1$ is $f_1 = k(k+2)$. Finally, one has $4 \leq s_1 < s$, so Proposition 6.9(c) yields $r = m - f_1 = 1$, whence R belongs to class $\mathbf{G}(1)$ by Proposition 6.7 and (6.1.2). \square

6.11 Proposition. *If s is even and one has $s = \frac{1}{2}k(k+1) - 2$ and $s_1 = \frac{1}{4}k(k+5) - 1$ for some integer $k \geq 4$, then $t = \frac{1}{4}k(k+1)$ holds and I is generated by one form of degree t and $\frac{1}{2}k(k+3) - 1$ forms of degree $t+1$. Moreover, R is Golod.*

Proof. With s and s_1 as given, one has $a = \frac{1}{4}k(k+5) - \frac{1}{4}k(k+1) = k$. In the notation from Theorem 5.4 one has $f_0 = \frac{1}{2}k(k+1) = 2 \lceil \frac{s+1}{2} \rceil$, whence Lemma 6.3 yields $\lceil \frac{s+1}{2} \rceil = t$, and we notice that $2t = f_0$ holds. From Theorem 5.4 it now follows that I has exactly one generator of degree t . In turn, this implies that the first syzygy of I has no generators of degree $t+1$, i.e. $\beta - f_1 + 2t + 1 = 0$ in 5.4. Thus, the number of generators in degree $t+1$ is

$$\beta = f_1 - 2t - 1 = k(k+2) - \frac{1}{2}k(k+1) - 1 = \frac{1}{2}k(k+3) - 1.$$

Finally, one has $8 \leq s_1 < s$, so Proposition 6.9(d) yields

$$r \leq m - f_1 + f_0 = \beta + 1 - f_1 + f_0 = -2t + f_0 = 0.$$

By Proposition 6.7 and (6.1.2) it now follows that R is Golod. \square

The next statement is folded in to Theorem 7.1 but worth recording separately.

6.12 Corollary. *Assume that R is level. If $s \geq 8$ or s is odd, then R is Golod.*

Proof. For $s \geq 10$ and odd $s \geq 3$ it follows from Theorem 3.10 that R is Golod, and $s \geq 2$ holds by (4.4.2). For $s = 8$ apply Proposition 6.11 with $k = 4$. \square

The final result of this section does not per se deal with the setup in 4.1, it only invokes the local ring (Q, \mathfrak{q}) , but it does come in handy in the proof of the main theorem. We have adapted the proof and notation from Herzog, Reiner, and Welker's result [21, Thm. 4] on Golodness of componentwise linear ideals.

6.13 Lemma. *Let $J \subseteq \mathfrak{q}^2$ be a homogeneous \mathfrak{q} -primary ideal in Q and set $\mathbf{A} = \mathrm{Tor}_*^Q(Q/J, \mathfrak{k})$. Let u be the initial degree of J and $J_{\langle u \rangle}$ the ideal generated by J_u ; set $\mathbf{B} = \mathrm{Tor}_*^Q(Q/J_{\langle u \rangle}, \mathfrak{k})$. For integers $j, k \leq u$ and $\ell \leq u+1$ there are inequalities*

$$(a) \quad \mathrm{rank}_{\mathfrak{k}}(\mathbf{A}_{1j} \cdot \mathbf{A}_{1k}) \leq \mathrm{rank}_{\mathfrak{k}}(\mathbf{B}_{1j} \cdot \mathbf{B}_{1k})$$

$$(b) \quad \mathrm{rank}_{\mathfrak{k}}(\mathbf{A}_{1j} \cdot \mathbf{A}_{2\ell}) \leq \mathrm{rank}_{\mathfrak{k}}(\mathbf{B}_{1j} \cdot \mathbf{B}_{2\ell})$$

$$(c) \quad \mathrm{rank}_{\mathfrak{k}}(\mathbf{A}_{2\ell} \xrightarrow{\delta'} \mathrm{Hom}_{\mathfrak{k}}(\mathbf{A}_{1j}, \mathbf{A}_{3j+\ell})) \leq \mathrm{rank}_{\mathfrak{k}}(\mathbf{B}_{2\ell} \xrightarrow{\delta''} \mathrm{Hom}_{\mathfrak{k}}(\mathbf{B}_{1j}, \mathbf{B}_{3j+\ell}))$$

with δ' and δ'' defined as in 6.1. In particular, if $Q/J_{\langle u \rangle}$ is Golod, then one has

$$(\mathbf{A}_1)_{\leq u} \cdot (\mathbf{A}_1)_{\leq u} = 0 = (\mathbf{A}_1)_{\leq u} \cdot (\mathbf{A}_2)_{\leq u+1}.$$

Proof. Let K be the Koszul complex on a minimal set of generators for \mathfrak{q} . The surjective homomorphism $\pi: Q/J_{\langle u \rangle} \rightarrow Q/J$ of graded rings is an isomorphism in degrees from 0 to u . It induces a surjective morphism of differential graded algebras,

$$\pi \otimes_Q K: Q/J_{\langle u \rangle} \otimes_Q K \longrightarrow Q/J \otimes_Q K.$$

There are isomorphisms of graded-commutative k -algebras $A \cong H(Q/J \otimes_Q K)$ and $B \cong H(Q/J_{\langle u \rangle} \otimes_Q K)$; see [2, (1.2.1)]. Thus $H(\pi \otimes_Q K)$ induces a morphism, $\tilde{\pi}: B \rightarrow A$, of graded-commutative algebras. Since the differential on K is linear and $\pi_{\leq u}$ is an isomorphism, it follows that

$$\tilde{\pi}_{i,j}: B_{i,j} \longrightarrow A_{i,j}$$

is an isomorphism for all i and $j \leq i + u - 1$.

(a): For integers $j, k \leq u$ and elements $x \in A_{1,j}$ and $x' \in A_{1,k}$ one has

$$xx' = \tilde{\pi}_{1,j} \tilde{\pi}_{1,j}^{-1}(x) \cdot \tilde{\pi}_{1,k} \tilde{\pi}_{1,k}^{-1}(x') = \tilde{\pi}_{2,j+k}(\tilde{\pi}_{1,j}^{-1}(x) \cdot \tilde{\pi}_{1,k}^{-1}(x')).$$

Thus, if the product xx' is non-zero, then so is $\tilde{\pi}^{-1}(x) \cdot \tilde{\pi}^{-1}(x')$ in $B_{1,j} \cdot B_{1,k}$. It follows that every nonzero element in $A_{1,j} \cdot A_{1,k}$ lifts to a nonzero element in $B_{1,j} \cdot B_{1,k}$ and, further, that linearly independent elements in $A_{1,j} \cdot A_{1,k}$ lift to linearly independent elements in $B_{1,j} \cdot B_{1,k}$.

Parts (b) and (c) follow from parallel arguments, and the final assertion follows as $B_1 \cdot B_1 = 0 = B_1 \cdot B_2$ holds if $J_{\langle u \rangle}$ is Golod. \square

7. THE MAIN THEOREM

We have hitherto focused on a setup where we are given compressed artinian Gorenstein rings defined by ideals I_1 and I_2 , and we have analyzed the situation where the ideal $I_1 \cap I_2$ defines a compressed ring of type 2. In the statement of our main theorem below, the point of view is slightly shifted: The focus is now on a compressed ring of type 2 whose defining ideal is assumed to be obtainable as an intersection of ideals that define compressed Gorenstein ring; see also Remark 7.2.

7.1 Theorem. *Let k be a field, set $Q = k[[x, y, z]]$ and $\mathfrak{q} = (x, y, z)$. Let $I \subseteq \mathfrak{q}^2$ be a homogeneous ideal such that $R = Q/I$ is compressed artinian of type 2. Assume that I is the intersection of homogeneous ideals I_1 and I_2 that define compressed Gorenstein rings. Denote by m and t the minimal number of generators and the initial degree of I and set $a = \min\{i \geq 0 \mid \mathfrak{q}^i I_2 \subseteq I_1\}$; let $\chi^{s_1} + \chi^s$ be the socle polynomial of R with $s_1 \leq s$.*

If s is odd, then the following assertions hold.

- (a) *If $\frac{s+1}{2} < t$, then R is of class $\mathbf{H}(0, 0)$, i.e. Golod.*
- (b) *If $\frac{s+1}{2} = t$ and $s \geq 5$, then $s_1 \neq s$ and R is of class $\mathbf{G}(r)$ with*

$$\begin{aligned} r &= m - a(a+2) \\ &\geq \frac{1}{2}(s+3 - a(a+1)) \geq 1 \quad \text{where } a = s_1 - \frac{s-1}{2}. \end{aligned}$$

- (c) *If $\frac{s+1}{2} = t$ and $s = 3$, then $s_1 = 2$ and R is of one of the following classes*

$$\begin{cases} \mathbf{B} & \text{with } m = 5 \\ \mathbf{G}(3) & \text{with } m = 6. \end{cases}$$

If s is even, then the following assertions hold.

- (d) *If $\frac{s}{2} + 1 < t$, then R is of class $\mathbf{H}(0, 0)$, i.e. Golod.*

- (e) If $\frac{s}{2} + 1 = s_1 \neq s$, then $\frac{s}{2} + 1 = t$ and R is of class $\mathbf{G}(m - 3)$.
 (f) If $\frac{s}{2} + 1 = t$ and $s_1 \neq s$, then R is of class $\mathbf{H}(0, 0)$ or of class $\mathbf{G}(r)$ with

$$\begin{aligned} m - \frac{1}{2}a(a + 3) &\geq r \geq m - a(a + 2) \\ &\geq s + 3 - \frac{3}{2}a(a + \frac{5}{3}) \quad \text{where } a = s_1 - \frac{s}{2}. \end{aligned}$$

- (g) If $\frac{s}{2} + 1 = t$ and $s_1 = s$, then $s \leq 8$ and the following assertions hold.

If $s = 2$, then R is of one of the following classes

$$\begin{cases} \mathbf{H}(3, 2) & \text{with } m = 4 \\ \mathbf{B} & \text{with } m = 5. \end{cases}$$

If $s = 4$, then R is of one of the following classes

$$\begin{cases} \mathbf{H}(0, 0) & \text{with } 5 \leq m \leq 8 \\ \mathbf{G}(r) & \text{with } 6 \leq m \leq 7 \text{ and } r \leq m - 5 \\ \mathbf{H}(0, 2) & \text{with } m = 7. \end{cases}$$

If $s = 6$, then R is of one of the following classes

$$\begin{cases} \mathbf{H}(0, 0) & \text{with } 9 \leq m \leq 11 \\ \mathbf{G}(1) & \text{with } m = 10. \end{cases}$$

If $s = 8$, then R is of class $\mathbf{H}(0, 0)$ with $m = 14$.

7.2 Remark. For an ideal I as in Theorem 7.1 one can always find homogeneous Gorenstein ideals I_1 and I_2 with $I_1 \cap I_2 = I$, but the assumption that they define compressed rings is crucial. Indeed, for the ideals I_2 and I_3 from Example 3.3 the intersection $J = I_2 \cap I_3$ is a six-generated ideal of initial degree 3, and it defines a compressed ring with socle polynomial $\chi^3 + \chi^4$. With the *Macaulay2* package [10] one can verify that Q/J is of class $\mathbf{G}(1)$. Had Q/I_3 been compressed—or had it in any way been possible to obtain J as the intersection of two homogeneous ideals that define compressed Gorenstein rings—then Q/J would by 7.1(e) have been of class $\mathbf{G}(3)$. Thus, it follows that this ideal J can not be obtained as an intersection of ideals that define compressed Gorenstein rings. It can happen, though, that an ideal I as in 7.1 can be obtained as intersections of Gorenstein ideals, $I_1 \cap I_2 = I = I_2 \cap I_3$, in such a way that the rings Q/I_2 and Q/I_3 are compressed but Q/I_1 is not, see Example 3.4.

Proof of 7.1. Let I_1 and I_2 be homogeneous Gorenstein ideals with $I = I_1 \cap I_2$ and such that Q/I_1 and Q/I_2 are compressed; this means that I_1 and I_2 fit Setup 4.1 with $e = 3$. Further, the assumption that R is compressed means that the results in Section 6 apply. They take care of most of the proof, but certain special cases elude them. To deal with those cases, we enlist a result of Bigatti, Geramita, and Migliore [5] on growth of Hilbert functions as well as our joint work with Weyman [15], which is based on linkage theory.

Parts (a) and (d): By (3.1.2) one has $\lceil \frac{s+1}{2} \rceil \leq t$, and if strict inequality holds, then R is Golod by 1.2, which precisely means that R is of class $\mathbf{H}(0, 0)$; see 6.1.

Part (b): Let s be odd and assume that $\frac{s+1}{2} = t$ and $s \geq 5$ hold. By Proposition 6.4 one has $s_1 < s$ and (4.4.2) yields $s_1 \geq 3$, so Proposition 6.9(c), Theorem 5.4,

and Lemma 6.3 yield

$$r = m - f_1 \geq t + 1 - f_0 > 0.$$

Now it follows from Proposition 6.7 and (6.1.2) that R is of class $\mathbf{G}(r)$. By Lemma 5.3 one has $a = s_1 - \frac{s-1}{2}$, and direct computations based on (6.0.2) yield

$$m - f_1 = m - a(a + 2) \quad \text{and} \quad t + 1 - f_0 = \frac{1}{2}(s + 3 - a(a + 1)).$$

Part (c): Assume that $s = 3$ and $t = 2$ hold. By Proposition 6.4 and (4.4.2) one has $s_1 = 2$, so Lemma 5.3 yields $a = 1$. Per (6.0.2) one now has $f_0 = 1$ and $f_1 = 3$. It follows from Theorem 5.4 that the ideal I is minimally generated by two quadratic and $\beta + 3$ cubic forms. Our first step is to prove that β is at most 1.

The h -vector of R is $(1, 3, 4, 1)$, see Table 4.9.1. Let $I_{(2)}$ be the ideal generated by the two quadratic forms. As

$$h_{Q/I_{(2)}}(2) = 4 = \binom{3}{2} + \binom{1}{1} \quad \text{one has} \quad h_{Q/I_{(2)}}(3) \leq 4^{(2)} = \binom{4}{3} + \binom{2}{2} = 5$$

by Macaulay's theorem, see [8, Thm. 4.2.10]. Thus $\beta + 3 \leq 5 - 1$ and hence $\beta \leq 1$.

If $\beta = 0$ holds, then m is 5 and Proposition 6.9(c) yields $r = 2$. If R were of class $\mathbf{H}(p, q)$, then one would have $q \leq 1$ by [15, Thm. 1.1], which is impossible as $r = q$ holds by (6.1.2). Since the type of R is 2, it now follows from [15, Thm. 4.5(b)] that R is of class \mathbf{B} .

If $\beta = 1$ holds, then m is 6 and Proposition 6.9(c) yields $r = 3$; since the type of R is 2, it is of class $\mathbf{G}(3)$; see (6.1.2).

Part (e): Let s be even and assume that one has $\frac{s}{2} + 1 = s_1 \neq s$. Lemmas 6.3 and 5.3 yield $\frac{s}{2} + 1 = t$, $a = 1$, and $m \geq 2t = s + 2$. By Proposition 6.9(e) one has $r = m - 3$, and as $s \geq 4$ holds, r is at least 3, whence R is of class $\mathbf{G}(r)$, see (6.1.2).

Part (f): Let s be even and assume that $\frac{s}{2} + 1 = t$ and $s_1 < s$ hold. It follows that s is at least 4, so s_1 is at least 3 by (4.4.2). From Propositions 6.7 and 6.9(a,d) it follows that R is Golod or of class $\mathbf{G}(r)$ with

$$m - f_1 + f_0 \geq r \geq m - f_1.$$

By Lemma 5.3 one has $a = s_1 - \frac{s}{2}$ and direct computations based on (6.0.2) yield

$$m - f_1 + f_0 = m - f_2 + 1 = m - \frac{1}{2}(a + 2)(a + 1) + 1 = m - \frac{1}{2}a(a + 3)$$

and

$$m - f_1 = m - a(a + 2).$$

By Theorem 5.4 one has $m - f_1 \geq 2t + 1 - f_0 - f_1$, and another computation yields

$$2t + 1 - f_0 - f_1 = s + 3 - \frac{1}{2}a(a + 1) - a(a + 2) = s + 3 - \frac{3}{2}a(a + \frac{5}{3}).$$

Part (g): Let s be even and assume that $\frac{s}{2} + 1 = t$ and $s_1 = s$ hold. It follows from Proposition 6.4 that s is at most 8, and by Lemma 5.3 one has $a = \frac{s}{2}$. In the balance of the proof, let \mathbf{A} be the bigraded \mathbf{k} -algebra $\text{Tor}_*^Q(R, \mathbf{k})$.

The case $s = 8$ is covered by Corollary 6.12. We address the remaining cases in descending order.

Case $s = 6$

One has $t = 4$, $a = 3$, $f_0 = 6$, $f_1 = 15$, and $f_2 = 10$; see (6.0.1) and (6.0.2). Recall from Theorem 5.4 that for some $\beta \geq 15 - 8 - 1 = 6$ the minimal graded free resolution of R over Q has the form

$$(*) \quad Q \longleftarrow \begin{array}{c} Q^3(-4) \\ \oplus \\ Q^\beta(-5) \end{array} \longleftarrow \begin{array}{c} Q^{\beta-6}(-5) \\ \oplus \\ Q^{10}(-6) \end{array} \longleftarrow Q^2(-9) \longleftarrow 0.$$

Thus the ideal I is minimally generated by three quartic forms and β quintics. Our first step is to prove that β is at most 8.

The h -vector of R is $(1, 3, 6, 10, 12, 6, 2)$, see Table 4.9.1. Let $I_{\langle 4 \rangle}$ be the ideal generated by the three quartic forms. As

$$h_{Q/I_{\langle 4 \rangle}}(4) = 12 = \binom{5}{4} + \binom{4}{3} + \binom{3}{2} \quad \text{one has} \quad h_{Q/I_{\langle 4 \rangle}}(5) \leq 12^{\langle 4 \rangle} = \binom{6}{5} + \binom{5}{4} + \binom{4}{3} = 15$$

by Macaulay's theorem; thus $\beta \leq 15 - 6 = 9$ holds. Assume towards a contradiction that β is 9, that is, $h_{Q/I_{\langle 4 \rangle}}(5) = 15$. This assumption implies that the Hilbert function of $Q/I_{\langle 4 \rangle}$ has maximal growth in degree 4. It follows from [5, Prop. 2.7] that the generators of $I_{\langle 4 \rangle}$ have a common cubic factor f , that is, $I_{\langle 4 \rangle} = \mathfrak{q}f$. Since $I_{\langle 4 \rangle}$ is contained in I , the element $f + I$ is a socle element in R . As $f + I$ has degree 3 this contradicts the assumption that R is level of socle degree 6. Thus one has $6 \leq \beta \leq 8$ and, therefore, $9 \leq m \leq 11$.

By Lemma 6.5 one has $p = 0$, so R is of class $\mathbf{H}(0, 0)$, $\mathbf{H}(0, 2)$, or $\mathbf{G}(r)$; see 6.1. Consider the bigraded \mathbf{k} -algebra \mathbf{A} . For homogeneous elements $x \in \mathbf{A}_1$ and $y \in \mathbf{A}_2$ with $xy \neq 0$ in \mathbf{A}_3 it follows from (*) that the internal degrees are $|x| = 4$ and $|y| = 5$; that is, $\mathbf{A}_1 \cdot \mathbf{A}_2 = \mathbf{A}_{14} \cdot \mathbf{A}_{25}$. Per (*) one has $\text{rank}_{\mathbf{k}} \mathbf{A}_{14} = 3$ and $\text{rank}_{\mathbf{k}} \mathbf{A}_{25} = \beta - 6 = m - 9$. It follows that r is at most $m - 9$, so m is at least 10 for rings of class $\mathbf{G}(r)$. We proceed to prove that R is Golod for $m = 11$; this rules out the possibility $\mathbf{H}(0, 2)$, and it means that R can be of class $\mathbf{G}(r)$ only for $m = 10$ and $r = 1$.

Assume that $m = 11$ holds, i.e. $\beta = 8$. It suffices to show that the minimal free resolution of $Q/I_{\langle 4 \rangle}$ has the form $Q \longleftarrow Q^3(-4) \longleftarrow Q^2(-5) \longleftarrow 0$. It then follows that $Q/I_{\langle 4 \rangle}$ is a Golod ring, see for example [1, 5.3.4]. As established above, one has $\mathbf{A}_1 \cdot \mathbf{A}_2 = \mathbf{A}_{14} \cdot \mathbf{A}_{25}$, and Lemma 6.13 yields $\mathbf{A}_{14} \cdot \mathbf{A}_{25} = 0$. Thus, $q = 0$ holds and R is of class $\mathbf{H}(0, 0)$. To establish that $Q/I_{\langle 4 \rangle}$ has the asserted free resolution, notice first from (*) that the three quartic generators of $I_{\langle 4 \rangle}$ have $\beta - 6 = 2$ linear syzygies. It suffices to show that they have no further syzygies, and since the Koszul relations are of degree 8 this comes down to verifying that the minimal free resolution of $Q/I_{\langle 4 \rangle}$ has no $Q(-u)$ summand in degree 2 for $6 \leq u \leq 8$. As $\beta = 8$ holds, one has

$$h_{Q/I_{\langle 4 \rangle}}(5) = h_R(5) + \beta = 6 + 8 = 14 = \binom{6}{5} + \binom{5}{4} + \binom{3}{3} + \binom{2}{2} + \binom{1}{1},$$

so Macaulay's theorem yields

$$h_{Q/I_{\langle 4 \rangle}}(6) \leq \binom{7}{6} + \binom{6}{5} + \binom{4}{4} + \binom{3}{3} + \binom{2}{2} = 16.$$

A straightforward calculation based on (5.0.2) and (5.0.1) now yields

$$\beta_{26}(Q/I_{\langle 4 \rangle}) - \beta_{36}(Q/I_{\langle 4 \rangle}) = b_{Q/I_{\langle 4 \rangle}}(6) = h_{Q/I_{\langle 4 \rangle}}(6) - 16 \leq 0.$$

A linear relation between the two first syzygies of the three quartics would also show in (*), so one has $\beta_{36}(Q/I_{\langle 4 \rangle}) = \beta_{36}(R) = 0$, this forces $\beta_{26}(Q/I_{\langle 4 \rangle}) = 0$ and $h_{Q/I_{\langle 4 \rangle}}(6) = 16$. Further, this implies $\beta_{37}(Q/I_{\langle 4 \rangle}) = \beta_{37}(R) = 0$ as a quadratic

relation between the first syzygies of the three quartics would also show in (*). Repeating the procedure, one gets $h_{Q/I_{(4)}}(7) \leq 18$ and

$$\beta_{27}(Q/I_{(4)}) - \beta_{37}(Q/I_{(4)}) = b_{Q/I_{(4)}}(7) = h_{Q/I_{(4)}}(7) - 18 \leq 0.$$

As above this forces $\beta_{27}(Q/I_{(4)}) = 0$ and $h_{Q/I_{(4)}}(7) = 18$ and, in addition, one gets $\beta_{38}(Q/I_{(4)}) = \beta_{38}(R) = 0$ as a cubic relation between the first syzygies of the three quartics would also show in (*). Repeating the procedure a third time yields $h_{Q/I_{(4)}}(8) \leq 20$ and

$$\beta_{28}(Q/I_{(4)}) - \beta_{38}(Q/I_{(4)}) = b_{Q/I_{(4)}}(8) = h_{Q/I_{(4)}}(8) - 20 \leq 0,$$

which this forces $\beta_{28}(Q/I_{(4)}) = 0$ as desired.

Case $s = 4$

One has $t = 3$, $a = 2$, $f_0 = 3$, $f_1 = 8$, and $f_2 = 6$; see (6.0.1) and (6.0.2). Recall from Theorem 5.4 that for some $\beta \geq 8 - 6 - 1 = 1$ the minimal graded free resolution of R over Q has the form

$$(**) \quad \begin{array}{ccccccc} & Q^4(-3) & & Q^{\beta-1}(-4) & & & \\ & \oplus & \longleftarrow & \oplus & \longleftarrow & Q^2(-7) & \longleftarrow 0. \\ & Q^\beta(-4) & & Q^6(-5) & & & \end{array}$$

Thus the ideal I is minimally generated by four cubic forms and β quartics. Our first step is to show that β is at most 4.

The h -vector of R is $(1, 3, 6, 6, 2)$, see Table 4.9.1. Let $I_{(3)}$ denote the ideal generated by the four cubics. As

$$h_{Q/I_{(3)}}(3) = 6 = \binom{4}{3} + \binom{2}{2} + \binom{1}{1} \quad \text{one has} \quad h_{Q/I_{(3)}}(4) \leq 6^{\binom{3}{4}} = \binom{5}{4} + \binom{3}{3} + \binom{2}{2} = 7$$

by Macaulay's theorem; thus $\beta \leq 7 - 2 = 5$ holds. Assume towards a contradiction that β is 5, i.e. $h_{Q/I_{(3)}}(4) = 7$. The assumption $h_{Q/I_{(3)}}(4) = 7$ implies that the Hilbert function of $Q/I_{(3)}$ has maximal growth in degree 3. It follows from [5, Prop. 2.7] that the generators of $I_{(3)}$ have a common linear factor, i.e. the four cubics have the form lq_1, \dots, lq_4 for some linear form l and quadratic forms q_1, \dots, q_4 . The exact sequence of graded Q -modules,

$$0 \rightarrow Q(-1)/(I_2 : l) \rightarrow Q/I_2 \rightarrow Q/(I_2 + (l)) \rightarrow 0,$$

yields $h_{Q/I_2}(i) = h_{Q/(I_2:l)}(i-1) + h_{Q/(I_2+(l))}(i)$ for all i . As I is contained in I_2 , the quadratic forms q_1, \dots, q_4 belong to $(I_2 : l)$, so one has $h_{Q/(I_2:l)}(2) \leq 2$. The ideal $(I_2 : l)$ defines a Gorenstein ring of socle degree $s-1 = 3$, see [29, Ch. IV Thm. 35] and [25, Cor. I.2.4], so the possible h -vectors of this ring are $(1, 1, 1, 1)$ and $(1, 2, 2, 1)$. As one has $h_{Q/I_2} = (1, 3, 6, 3, 1)$, see 2.2, the h -vector of $Q/(I_2 + (l))$ would have to be $(1, 2, 5, 2)$ or $(1, 2, 4, 1)$, but by Macaulay's theorem neither is a possible h -vector; a contradiction. Thus one has $1 \leq \beta \leq 4$ and $5 \leq m \leq 8$.

By Lemma 6.5 one has $p = 0$, so R is of class $\mathbf{H}(0, 0)$, $\mathbf{H}(0, 2)$, or $\mathbf{G}(r)$, see 6.1. Now consider the bigraded k -algebra A . For homogeneous elements $x \in A_1$ and $y \in A_2$ with $xy \neq 0$ in A_3 it follows from (**) that the internal degrees are $|x| = 3$ and $|y| = 4$; that is, $A_1 \cdot A_2 = A_{1,3} \cdot A_{2,4}$. Per (**) one has $\text{rank}_k A_{1,3} = 4$ and $\text{rank}_k A_{2,4} = \beta - 1 = m - 5$. It follows that r is at most $m - 5$, so m is at least 6 for rings of class $\mathbf{G}(r)$ and at least 7 for rings of class $\mathbf{H}(0, 2)$. It remains to prove that R is Golod for $m = 8$.

Assume that $m = 8$ holds, i.e. $\beta = 4$. It suffices to show that the minimal free resolution of $Q/I_{(3)}$ has the form $Q \longleftarrow Q^4(-3) \longleftarrow Q^3(-4) \longleftarrow 0$. It then follows

that $Q/I_{\langle 3 \rangle}$ is a Golod ring, see for example [1, Prop. 5.3.4]. As established above, one has $A_1 \cdot A_2 = A_{13} \cdot A_{24}$, and Lemma 6.13 yields $A_{13} \cdot A_{24} = 0$. Thus, $q = 0$ holds and R is of class $\mathbf{H}(0, 0)$. To establish that $Q/I_{\langle 3 \rangle}$ has the asserted free resolution, notice first from (***) that the four cubic generators of $I_{\langle 3 \rangle}$ have $\beta - 1 = 3$ linear syzygies. It suffices to show that they have no further syzygies, which comes down to verifying that the minimal free resolution of $Q/I_{\langle 3 \rangle}$ has no $Q(-5)$ or $Q(-6)$ summand in degree 2. As $\beta = 4$ holds, one has

$$h_{Q/I_{\langle 3 \rangle}}(4) = h_R(4) + \beta = 2 + 4 = 6 = \binom{5}{4} + \binom{3}{3}$$

and, therefore, $h_{Q/I_{\langle 3 \rangle}}(5) \leq \binom{6}{5} + \binom{4}{4} = 7$ by Macaulay's theorem. A straightforward calculation based on (5.0.2) and (5.0.1) now yields

$$\beta_{25}(Q/I_{\langle 3 \rangle}) - \beta_{35}(Q/I_{\langle 3 \rangle}) = b_{Q/I_{\langle 3 \rangle}}(5) = h_{Q/I_{\langle 3 \rangle}}(5) - 6 \leq 1.$$

A linear relation between the first syzygies of the four cubics would also show in (**), so one has $\beta_{35}(Q/I_{\langle 3 \rangle}) = \beta_{35}(R) = 0$, which implies that $\beta_{25}(Q/I_{\langle 3 \rangle}) \leq 1$ holds. Equality would force $h_{Q/I_{\langle 3 \rangle}}(5) = 7$ and $h_{Q/I_{\langle 3 \rangle}}(6) \leq 8$ which would yield

$$\beta_{26}(Q/I_{\langle 3 \rangle}) - \beta_{36}(Q/I_{\langle 3 \rangle}) = b_{Q/I_{\langle 3 \rangle}}(6) = h_{Q/I_{\langle 3 \rangle}}(6) - 9 < 0;$$

this is absurd, as one has $\beta_{36}(Q/I_{\langle 3 \rangle}) = \beta_{36}(R) = 0$ since a relation in degree 6 between the first syzygies of the four cubics would also show in (**). Thus one has $\beta_{25}(Q/I_{\langle 3 \rangle}) = 0$ and $h_{Q/I_{\langle 3 \rangle}}(5) = 6 = \binom{6}{5}$. Macaulay's theorem now yields $h_{Q/I_{\langle 3 \rangle}}(6) \leq 6^{\binom{5}{5}} = 7$, and as above one gets

$$(\dagger) \quad \beta_{26}(Q/I_{\langle 3 \rangle}) - \beta_{36}(Q/I_{\langle 3 \rangle}) = b_{Q/I_{\langle 3 \rangle}}(6) = h_{Q/I_{\langle 3 \rangle}}(6) - 6 \leq 1.$$

Since $\beta_{36}(Q/I_{\langle 3 \rangle}) = 0$ this implies $\beta_{26}(Q/I_{\langle 3 \rangle}) \leq 1$. Assume towards a contradiction that equality holds. This implies $h_{Q/I_{\langle 3 \rangle}}(6) = 7$, i.e. the Hilbert function of $Q/I_{\langle 3 \rangle}$ has maximal growth in degree 5. It now follows from [5, Prop. 2.7] that the generators of $(I_{\langle 3 \rangle})_{\langle 5 \rangle}$ have a common linear factor l . That is, one has $\mathfrak{q}^2 I_{\langle 3 \rangle} \subseteq (l)$. As (l) is a prime ideal, one has $(l) : \mathfrak{q} = (l)$ and, therefore, $(l) : \mathfrak{q}^2 = ((l) : \mathfrak{q}) : \mathfrak{q} = (l)$. It follows that $I_{\langle 3 \rangle}$ is contained in (l) , so the four cubic generators have a common linear factor; as in the subcase $\beta = 4$ above this leads to a contradiction. Thus $h_{Q/I_{\langle 3 \rangle}}(6) \leq 6$ holds, whence (\dagger) yields $\beta_{26}(Q/I_{\langle 3 \rangle}) - \beta_{36}(Q/I_{\langle 3 \rangle}) \leq 0$ and as $\beta_{36}(Q/I_{\langle 3 \rangle})$ is 0 this implies $\beta_{26}(Q/I_{\langle 3 \rangle}) = 0$ as desired.

Case $s = 2$

One has $t = 2$, $a = 1 = f_0$, and $f_1 = 3 = f_2$; see (6.0.1) and (6.0.2). Recall from Theorem 5.4 that for some $\beta \geq 0$ the minimal graded free resolution of R over Q has the form

$$(***) \quad Q \longleftarrow \begin{array}{c} Q^4(-2) \\ \oplus \\ Q^\beta(-3) \end{array} \longleftarrow \begin{array}{c} Q^{2+\beta}(-3) \\ \oplus \\ Q^3(-4) \end{array} \longleftarrow Q^2(-5) \longleftarrow 0.$$

Thus the ideal I is minimally generated by four quadratic forms and β cubics. Proposition 6.9(a) yields $r \geq m - 3 = \beta + 1$. Our first step is to show that $\beta \leq 1$.

The h -vector of R is $(1, 3, 2)$, see Table 4.9.1. Let $I_{\langle 2 \rangle}$ denote the ideal generated by the four quadratics. As

$$h_{Q/I_{\langle 2 \rangle}}(2) = 2 = \binom{2}{2} + \binom{1}{1} \quad \text{one has} \quad h_{Q/I_{\langle 2 \rangle}}(3) \leq 2^{\binom{2}{2}} = \binom{3}{3} + \binom{2}{2} = 2$$

by Macaulay's theorem; thus $\beta \leq 2$ holds. Towards a contradiction assume $\beta = 2$. In (***) there are now 4 syzygies of degree 3. They must be linear syzygies

of the four quadratic generators, so they appear in the minimal free resolution of $Q/I_{\langle 2 \rangle}$ which, therefore, must have length 3. Notice also that $I_{\langle 2 \rangle}$ is not a Gorenstein ideal, since it is minimally generated by an even number of generators, see [9, Thm. 2.1]. Now consider the bigraded k -algebra $\mathbf{B} = \text{Tor}_*^Q(Q/I_{\langle 2 \rangle}, k)$. For homogeneous elements $x \in \mathbf{A}_1$ and $y \in \mathbf{A}_2$ with $xy \neq 0$ in \mathbf{A}_3 it follows from $(***)$ that their internal degrees are $|x| = 2$ and $|y| = 3$. Thus one has

$$\text{rank}_k(\mathbf{A}_{2,3} \rightarrow \text{Hom}_k(\mathbf{A}_{1,2}, \mathbf{A}_{3,5})) = r \geq 3,$$

so by Lemma 6.13(c) the map $\delta: \mathbf{B}_2 \rightarrow \text{Hom}_k(\mathbf{B}_1, \mathbf{B}_3)$, cf. 6.1, has rank at least 3. This contradicts [2, Thm. 3.1] which bounds the rank of δ above by 2.

If $\beta = 0$ holds, then m is 4, so R is an almost complete intersection and hence of class $\mathbf{H}(3, 2)$; see [2, 3.4.2] or [15, Thm. 4.1].

If $\beta = 1$ holds, then m is 5, and Proposition 6.9(c) yields $r = 2$. As R has type 2 it follows from [15, Thm. 4.5] that R is of class \mathbf{B} or $\mathbf{H}(p, q)$. The latter option is ruled out by [15, Thm. 1.1] and (6.1.2), which yield $q = 1$ and $q = r$. \square

The statement of Theorem 7.1 is organized according to: first the parity of s , second the relation of $\lceil \frac{s+1}{2} \rceil$ to t , cf. (3.1.2), and finally a comparison of m to r . In Corollaries 7.3–7.5 below the conclusions from 7.1 are summarized with an emphasis on the difference between the degrees of the socle generators of R .

7.3 Corollary. *Let R be as in Theorem 7.1 with $s \leq 4$. One has $2 \leq s_1 \leq 4$ and the next assertions hold.*

(a) *If $s_1 = 2$, then $s \leq 3$ and R is of class*

$$\begin{cases} \mathbf{H}(3, 2) & \text{with } s = 2 \\ \mathbf{B} & \text{with } 2 \leq s \leq 3 \\ \mathbf{G}(3) & \text{with } s = 3 \end{cases}$$

(b) *If $s_1 = 3$, then R is of class*

$$\begin{cases} \mathbf{H}(0, 0) & \text{with } s = 3 \\ \mathbf{G}(r) & \text{with } s = 4 \end{cases}$$

(c) *If $s_1 = 4$, then $s = 4$ and R is of class $\mathbf{H}(0, 0)$, $\mathbf{G}(r)$, or $\mathbf{H}(0, 2)$.*

Proof. Under the the assumptions in Theorem 7.1, Theorem 4.4 applies; in particular inequalities $2 \leq s_1 \leq 4$ hold by (4.4.2).

(a): If $s_1 = 2$, then (4.4.2) yields $s \leq 3$. If $s = 2$, then $h_R = (1, 3, 2)$ holds, and for $s = 3$ one has $h_R = (1, 3, 4, 1)$, see Table 4.9.1; in either case $t = 2 = \lceil \frac{s+1}{2} \rceil$ holds, so the assertions follow from 7.1(c,g).

(b): If $s_1 = 3$, then (4.4.2) yields $3 \leq s \leq 4$. If $s = 3$, then $h_R = (1, 3, 6, 2)$ holds, see Table 4.9.1, so $t = 3 > \lceil \frac{s+1}{2} \rceil$ holds; for $s = 4$ one gets $h_R = (1, 3, 6, 4, 1)$, so $t = 3 = \lceil \frac{s+1}{2} \rceil$ holds. The assertions now follow from 7.1(a,e).

(c): If $s_1 = 4$, then (4.4.2) yields $s = 4$, and the assertion summarizes the case $s = 4$ in 7.1(g). \square

Remark 6.8 translates Corollaries 7.4 and 7.5 into the summary given in the introduction.

7.4 Corollary. *Let R be as in Theorem 7.1 with s odd and $s \geq 5$. Set*

$$N(s) = \frac{s - 2 + \sqrt{4s + 13}}{2}.$$

There are inequalities

$$\frac{s+1}{2} \leq s_1 \leq s \quad \text{and} \quad \frac{s+1}{2} < N(s) < s,$$

and the next assertions hold.

- (a) *If one has $s_1 < N(s)$, then R is of class $\mathbf{G}(r)$.*
- (b) *If one has $N(s) \leq s_1$, then R is of class $\mathbf{H}(0, 0)$, i.e. Golod.*

Proof. The first set of inequalities comes from (4.4.2); as s is at least 5, the second set follows immediately from the definition of $N(s)$. It follows from Proposition 6.4 that the inequality $\frac{s+1}{2} < t$ holds if and only if one has $N(s) \leq s_1$, and the two assertions now follow immediately from Theorem 7.1(a,b). \square

The next corollary describes the situation for even socle degree $s \geq 6$. In the odd case, Corollary 7.4, a single bound on s_1 determines whether R is Golod or of class $\mathbf{G}(r)$. In Corollary 7.5 it takes two bounds, $N_1(s) < N_2(s)$, to definitively separate the classes Golod and $\mathbf{G}(r)$; in the intermediate interval both possibilities occur, see Table 8.2.1 but also 8.5(d).

7.5 Corollary. *Let R be as in Theorem 7.1 with s even and $s \geq 6$. Set*

$$N_1(s) = \frac{3s - 5 + \sqrt{24s + 97}}{6} \quad \text{and} \quad N_2(s) = \frac{s - 1 + \sqrt{8s + 25}}{2}.$$

There are inequalities

$$\frac{s}{2} + 1 \leq s_1 \leq s \quad \text{and} \quad N_1(s) < N_2(s),$$

and the next assertions hold.

- (a) *If one has $s_1 < N_1(s)$, then R is of class $\mathbf{G}(r)$.*
- (b) *If one has $N_2(s) \leq s_1$, then R is of class $\mathbf{H}(0, 0)$, i.e. Golod.*
- (c) *If one has $N_1(s) \leq s_1 < N_2(s)$, then R is of class $\mathbf{H}(0, 0)$ or $\mathbf{G}(r)$.*

Proof. The first set of inequalities comes from (4.4.2), and it is immediate from the definitions that $N_1(s) < N_2(s)$ holds. It follows from Proposition 6.4 that the inequality $\frac{s}{2} + 1 < t$ holds if and only if one has $N_2(s) \leq s_1$, so part (b) holds by Theorem 7.1(d). Further, $\frac{s}{2} + 1 = t$ holds for $s_1 < N_2(s)$, in particular for $s_1 < N_1(s)$, and in that case Theorem 7.1(f) yields $a = s_1 - \frac{s}{2}$ and

$$r \geq s + 3 - \frac{3}{2}a\left(a + \frac{5}{3}\right) = -\frac{3}{2}a^2 - \frac{5}{2}a + s + 3.$$

This quadratic expression is positive for $a < \frac{-5 + \sqrt{24s + 97}}{6}$, i.e. for $s_1 < N_1(s)$. This proves (a), and (c) follows from Theorem 7.1(f,g). \square

7.6 Remark. Let (Q, \mathfrak{q}) be as in Theorem 7.1 and $J \subseteq \mathfrak{q}^2$ be a homogeneous \mathfrak{q} -primary ideal such that $S = Q/J$ has type 2. With \tilde{m} denoting the minimal number

of generators of J , it is known—see for example [15, Thm. 1.1 and Sect. 4]—that S is of one of the following classes

$$\left\{ \begin{array}{ll} \mathbf{H}(3, 2) & \text{with } \tilde{m} = 4 \\ \mathbf{H}(0, 0) & \text{with } \tilde{m} \geq 5 \\ \mathbf{B} & \text{with } \tilde{m} \geq 5 \text{ and } \tilde{m} \text{ odd} \\ \mathbf{G}(r) & \text{with } \tilde{m} \geq 6 \text{ and } r \leq \tilde{m} - 3 \\ \mathbf{H}(1, 2) & \text{with } \tilde{m} \geq 6 \text{ and } \tilde{m} \text{ even} \\ \mathbf{H}(0, 2) & \text{with } \tilde{m} \geq 7. \end{array} \right.$$

Empirical evidence suggests that essentially all of these classes materialize, cf. [15, Conj. 7.4]. In comparison, the extra assumptions imposed in Theorem 7.1 restrict rings R as in 7.1 with $m \geq 8$ to the classes $\mathbf{H}(0, 0)$ and $\mathbf{G}(r)$.

We close this section with a result that provides further evidence for [15, Conjecture 7.4(a)] and suggests that the answer to [27, Question 9.7] is negative.

7.7 Proposition. *Let R be as in Theorem 7.1. If R is of class $\mathbf{G}(r)$, then $r \leq m - 3$.*

Proof. As R is of class $\mathbf{G}(r)$, it follows from (3.1.2) and 1.2 that $\lceil \frac{s+1}{2} \rceil = t$ holds. If s is odd, then it follows from Corollaries 7.3 and 7.4 that $s_1 < s$ holds. Thus one has $r = m - f_1$ by Proposition 6.9(c), and $f_1 \geq 3$ holds by (6.0.2) as a is positive by Proposition 1.6(e). Now assume that s is even. If $s_1 = s$ holds, then Theorem 7.1(g) yields $s \leq 6$ and shows that $r \leq m - 5$ holds. Assuming now that $s_1 < s$ holds, Proposition 6.9(d) yields $r \leq m - f_1 + f_0$. For $a \geq 2$ one has $f_1 - f_0 \geq 3$, and $a = 1$ implies $s_1 = \frac{s}{2} + 1$, so $r = m - 3$ holds by 7.1(e). \square

8. GENERIC BEHAVIOR

In this final section we first elaborate on the remarks made in the introduction about generic algebras being compressed. The statement of Theorem 7.1 was informed by experiments, and we share some of the collected data in Table 8.2.1. This data suggests a number of questions; we address a few of them. Finally, we discuss in which sense Theorem 7.1 explains the class of a randomly chosen graded artinian type 2 quotient of the trivariate power series algebra over a field.

8.1 Compressedness of generic artinian algebras of type 2. Let \mathbf{k} be a field and $e \geq 2$ an integer. Fröberg and Laksov [17, Sect. 7] prove that given a polynomial $P(\chi)$ that satisfies certain numerical conditions involving e —in [23, Rmk. 4.2] they are referred to as “legal socle polynomials”—there is a non-empty Zariski open set in affine space \mathbf{k}^d , where d depends on e and the coefficients of $P(\chi)$, whose points are in one-to-one correspondence with homogeneous ideals in $\mathbf{k}[x_1, \dots, x_e]$, equivalently in $\mathbf{k}[[x_1, \dots, x_e]]$, that define compressed artinian \mathbf{k} -algebras with socle polynomial $P(\chi)$. For $e \geq 3$ and numbers $s_1 \leq s < 2s_1$ the polynomial $P(\chi) = \chi^{s_1} + \chi^s$ satisfies the numerical conditions in [17, Prop. 5]: In the notation of [17] one has

$$r_{s_1} = \binom{s_1+e-2}{e-2} - 1 - \binom{s-s_1+e-1}{e-1} \geq \binom{s_1+e-1}{e-1} - 1 - \binom{s_1+e-2}{e-1} = \binom{s_1+e-2}{e-2} - 1 \geq 0,$$

so the discriminating value, which in [17, Prop. 5] is called b , is at most s_1 . A closer look at the proof of [17, Prop. 16] reveals that this b is actually t from (1.5.1), and the verification above amounts to the inequality $t \leq s_1$ from Proposition 3.6(a).

8.2 Random artinian algebras of type 2. In the sense discussed above in 8.1, a generic graded artinian type 2 quotient of the power series algebra $Q = \mathbb{k}[[x, y, z]]$ is compressed. It is even simpler to see that a generic, in the same sense, artinian Gorenstein quotient of Q is compressed; see also Boij and Laksov [7, Thm. 3.4] and [23, Thm. 4.1(d)]. Thus, with $\mathfrak{q} = (x, y, z)$, if one generates random \mathfrak{q} -primary Gorenstein ideals $I_1 \subseteq \mathfrak{q}^2$ and $I_2 \subseteq \mathfrak{q}^2$, then one expects the rings Q/I_1 , Q/I_2 , and $Q/(I_1 \cap I_2)$ to be compressed. In particular, the h -vector of $Q/(I_1 \cap I_2)$ should be determined by the socle degrees of Q/I_1 and Q/I_2 , and since the ideals are chosen randomly, one expects $I_1 \cap I_2$ to be minimally generated by the least possible number of elements, given the h -vector. Theorem 7.1 should, therefore, determine the class of $Q/(I_1 \cap I_2)$ based on the socle degrees of Q/I_1 and Q/I_2 , and indeed it does; we explain how in 8.5. In fact, the statement of Theorem 7.1 was informed by the outcomes of such experiments conducted with *Macaulay2* [19]:

TABLE 8.2.1. Let \mathbb{k} be a field and \mathfrak{q} the maximal ideal of the local ring $Q = \mathbb{k}[[x, y, z]]$. For fixed integers $2 \leq s_1 \leq s < 2s_1$, cf. (4.4.2), and various choices of \mathbb{k} we generated random \mathfrak{q} -primary Gorenstein ideals $I_1 \subseteq \mathfrak{q}^2$ and $I_2 \subseteq \mathfrak{q}^2$ with quotients Q/I_1 and Q/I_2 of socle degrees s_1 and s . Using [11] we classified the rings $Q/(I_1 \cap I_2)$ and recorded the generic, i.e. prevalent class. If $Q/(I_1 \cap I_2)$ was not of the generic class, we still recorded it if the rings Q/I_1 , Q/I_2 , and $Q/(I_1 \cap I_2)$ were all compressed, cf. Theorem 7.1. Here we reproduce the results for $s \leq 10$.

s_1	s	h -vector	t	Generic class	m	Other compressed classes
2	2	(1, 3, 2)	2	H (3, 2)	4	<i>Not possible, see 7.1(g)</i>
					5	B
2	3	(1, 3, 4, 1)	2	B	5	<i>Not possible, see 7.1(c)</i>
					6	G (3)
3	3	(1, 3, 6, 2)	3	H (0, 0)	8	<i>Not possible, see 7.1(a)</i>
3	4	(1, 3, 6, 4, 1)	3	G (3)	6	<i>Not possible, see 7.1(e)</i>
					7	G (4)
4	4	(1, 3, 6, 6, 2)	3	H (0, 0)	5	<i>Not possible, see 7.1(g)</i>
					6	H (0, 0), G (1)
					7	G (1), G (2), H (0, 2)
					8	H (0, 0)
3	5	(1, 3, 6, 7, 3, 1)	3	G (3)	6	<i>Not possible, see 8.5(c)</i>
					7	G (4)
					8	G (5)
4	5	(1, 3, 6, 9, 4, 1)	3	G (1)	9	<i>Not possible, see 6.10</i>
5	5	(1, 3, 6, 10, 6, 2)	4	H (0, 0)	9	<i>Not possible, see 7.1(a)</i>
					10	H (0, 0)
4	6	(1, 3, 6, 10, 7, 3, 1)	4	G (5)	8	<i>Not possible, see 7.1(e)</i>
					9	G (6)
5	6	(1, 3, 6, 10, 9, 4, 1)	4	G (1)	6	H (0, 0)
					7	G (1), G (2)
					8	H (0, 0), G (2), G (3)
					9	G (1), G (3), G (4)
					10	G (2)
6	6	(1, 3, 6, 10, 12, 6, 2)	4	H (0, 0)	9	<i>Not possible, see 7.1(g)</i>
					10	H (0, 0), G (1)
					11	H (0, 0)
4	7	(1, 3, 6, 10, 11, 6, 3, 1)	4	G (4)	7	<i>Not possible, see 8.5(c)</i>

					8	$\mathbf{G}(5)$
					9	$\mathbf{G}(6)$
					10	$\mathbf{G}(7)$
5	7	(1, 3, 6, 10, 13, 7, 3, 1)	4	$\mathbf{G}(2)$	10	<i>Not possible, see 8.5(c)</i>
					11	$\mathbf{G}(3)$
6	7	(1, ..., 15, 9, 4, 1)	5	$\mathbf{H}(0, 0)$	12	<i>Not possible, see 7.1(a)</i>
7	7	(1, ..., 15, 12, 6, 2)	5	$\mathbf{H}(0, 0)$	9	<i>Not possible, see 7.1(a)</i>
					10	$\mathbf{H}(0, 0)$
					11	$\mathbf{H}(0, 0)$
5	8	(1, ..., 15, 11, 6, 3, 1)	5	$\mathbf{G}(7)$	10	<i>Not possible, see 7.1(e)</i>
					11	$\mathbf{G}(8)$
6	8	(1, ..., 15, 13, 7, 3, 1)	5	$\mathbf{G}(3)$	8	$\mathbf{H}(0, 0), \mathbf{G}(2)$
					9	$\mathbf{G}(1), \mathbf{G}(3), \mathbf{G}(4)$
					10	$\mathbf{G}(2), \mathbf{G}(4), \mathbf{G}(5)$
					11	$\mathbf{G}(3), \mathbf{G}(5), \mathbf{G}(6)$
7	8	(1, ..., 15, 16, 9, 4, 1)	5	$\mathbf{H}(0, 0)$	9	
					10	$\mathbf{H}(0, 0), \mathbf{G}(1)$
					11	$\mathbf{H}(0, 0), \mathbf{G}(1), \mathbf{G}(2)$
					12	$\mathbf{H}(0, 0), \mathbf{G}(3)$
8	8	(1, ..., 15, 20, 12, 6, 2)	5	$\mathbf{H}(0, 0)$	14	<i>Not possible, see 7.1(g)</i>
5	9	(1, ..., 15, 16, 10, 6, 3, 1)	5	$\mathbf{G}(5)$	8	<i>Not possible, see 8.5(c)</i>
					9	$\mathbf{G}(6)$
					10	$\mathbf{G}(7)$
					11	$\mathbf{G}(8)$
6	9	(1, ..., 15, 18, 11, 6, 3, 1)	5	$\mathbf{G}(3)$	11	<i>Not possible, see 8.5(c)</i>
					12	$\mathbf{G}(4)$
7	9	(1, ..., 21, 13, 7, 3, 1)	6	$\mathbf{H}(0, 0)$	15	<i>Not possible, see 7.1(a)</i>
8	9	(1, ..., 21, 16, 9, 4, 1)	6	$\mathbf{H}(0, 0)$	12	<i>Not possible, see 7.1(a)</i>
					13	$\mathbf{H}(0, 0)$
					14	$\mathbf{H}(0, 0)$
9	9	(1, ..., 21, 20, 12, 6, 2)	6	$\mathbf{H}(0, 0)$	8	<i>Not possible, see 7.1(a)</i>
					9	$\mathbf{H}(0, 0)$
					10	$\mathbf{H}(0, 0)$
					11	$\mathbf{H}(0, 0)$
					12	$\mathbf{H}(0, 0)$
6	10	(1, ..., 21, 16, 10, 6, 3, 1)	6	$\mathbf{G}(9)$	12	<i>Not possible, see 7.1(e)</i>
					13	$\mathbf{G}(10)$
7	10	(1, ..., 21, 18, 11, 6, 3, 1)	6	$\mathbf{G}(5)$	10	$\mathbf{G}(2), \mathbf{G}(4)$
					11	$\mathbf{G}(3), \mathbf{G}(5), \mathbf{G}(6)$
					12	$\mathbf{G}(4), \mathbf{G}(6), \mathbf{G}(7)$
					13	$\mathbf{G}(8)$
8	10	(1, ..., 21, 21, 13, 7, 3, 1)	6	$\mathbf{H}(0, 0)$	9	
					10	$\mathbf{H}(0, 0), \mathbf{G}(1)$
					11	$\mathbf{H}(0, 0), \mathbf{G}(1), \mathbf{G}(2)$
					12	$\mathbf{H}(0, 0), \mathbf{G}(1), \mathbf{G}(2), \mathbf{G}(3)$
					13	$\mathbf{H}(0, 0)$
9	10	(1, ..., 21, 25, 16, 9, 4, 1)	6	$\mathbf{H}(0, 0)$	14	
					15	$\mathbf{H}(0, 0), \mathbf{G}(1)$
10	10	(1, ..., 28, 20, 12, 6, 2)	7	$\mathbf{H}(0, 0)$	16	<i>Not possible, see 7.1(d)</i>
					17	$\mathbf{H}(0, 0)$

Corollaries 7.4 and 7.5 show that R is of class $\mathbf{H}(0,0)$ or $\mathbf{G}(r)$ if the socle degree, s , is at least 5. The data shows that this is the generic behavior for $s \geq 4$; we elaborate on this in 8.5.

8.3 Socle polynomials $\chi^{s_1} + \chi^s$ that uniquely determine m . For R as in Theorem 7.1 and certain socle polynomials, the class of R as well as m is uniquely determined. Propositions 6.10 and 6.11 yield some such cases—among them those where R has socle polynomial $\chi^4 + \chi^5$ or $2\chi^8$. The data in Table 8.2.1 suggests that there are more cases of this behavior, and below we provide ad hoc arguments on Betti tables to account for those that fall within the parameters of the table. We use the *Macaulay2* convention for compact presentation of Betti tables.

- (a) If R is as in Theorem 7.1 with socle polynomial $2\chi^3$, then R is of class $\mathbf{H}(0,0)$ with $m = 8$. Indeed, one has $h_R = (1, 3, 6, 2)$, see Table 4.9.1; in particular, the initial degree of I is 3, so R is of class $\mathbf{H}(0,0)$ by 7.1(a). A direct computation, see (5.0.1), yields $B_R(\chi) = 1 - 8\chi^3 + 9\chi^4 - 2\chi^6$. As $b_R(5) = 0$ and R has type 2, one has $\beta_{2,5}(R) = 0 = \beta_{3,5}(R)$, so the Betti table of the minimal free resolution of R over Q is

	0	1	2	3
0	1	.	.	.
1
2	.	8	9	.
3	.	.	.	2.

- (b) If R is as in Theorem 7.1 with socle polynomial $\chi^7 + \chi^9$, then R is of class $\mathbf{H}(0,0)$ with $m = 15$. Indeed, one has $h_R = (1, 3, 6, 10, 15, 13, 7, 3, 1)$, see Table 4.9.1; in particular, the initial degree of I is 6, so R is of class $\mathbf{H}(0,0)$ by 7.1(a). A direct computation, see (5.0.1), yields $B_R(\chi) = 1 - 15\chi^6 + 16\chi^7 - \chi^{10} - \chi^{12}$. As $b_R(11) = 0$ and R has type 2, one has $\beta_{2,11}(R) = 0$; this forces $\beta_{1,10}(R) = 0$, and continuing this standard analysis one sees that the Betti table of the minimal free resolution of R over Q is

	0	1	2	3
0	1	.	.	.
:	:	:	:	:
5	.	15	16	.
6
7	.	.	.	1
8
9	.	.	.	1.

- (c) If R is as in Theorem 7.1 with socle polynomial $\chi^6 + \chi^7$, then R is of class $\mathbf{H}(0,0)$ with $m = 12$. Indeed, one has $h_R = (1, 3, 6, 10, 15, 9, 4, 1)$, see Table 4.9.1; in particular, the initial degree of I is 5, so R is of class $\mathbf{H}(0,0)$ by 7.1(a). A direct computation, see (5.0.1), yields $B_R(\chi) = 1 - 12\chi^5 + 12\chi^6 + \chi^7 - \chi^9 - \chi^{10}$. As above one argues that for some integer $\beta \geq 0$ the Betti table of the minimal free resolution of R over Q is

	0	1	2	3
0	1	.	.	.
:	:	:	:	:
4	.	12	$12 + \beta$.
5	.	β	1	.
6	.	.	.	1
7	.	.	.	1

We use Boij–Söderberg theory to show that β is zero. Assume towards a contradiction that β is positive. Performing the first two steps of the algorithm

provided by Eisenbud and Schreyer in [16, Sect. 1] one gets

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & . & . \\ . & . & . & . \\ 4 & . & 12 & 12 + \beta \\ 5 & . & \beta & 1 \\ 6 & . & . & . \\ 7 & . & . & 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & . & . \\ . & . & . & . \\ 4 & . & 27 & 30 \\ 5 & . & . & . \\ 6 & . & . & 5 \\ 7 & . & . & . \end{pmatrix} - \frac{11}{40} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & . & . \\ . & . & . & . \\ 4 & . & 24 & 35 \\ 5 & . & . & . \\ 6 & . & . & . \\ 7 & . & . & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & \frac{1}{20} & . & . \\ . & . & . & . \\ 4 & . & . & \beta - \frac{7}{8} \\ 5 & . & \beta & 1 \\ 6 & . & . & . \\ 7 & . & . & \frac{7}{40} \end{pmatrix}.$$

The degree sequence of the resulting table is invalid; a contradiction.

8.4 Unexplained patterns. Theorem 7.1 does not explain all the patterns one can glean from Table 8.2.1. One example is the absence of rings of class $\mathbf{H}(0, 0)$ with socle polynomial $2\chi^4$ and $m = 7$. A similar unexplained pattern is this one: Rings of class $\mathbf{G}(r)$ with socle polynomial $\chi^6 + \chi^8$ or $\chi^7 + \chi^{10}$ were observed with both the minimal and maximal values $r = m - 8$ and $r = m - 5$ determined by Theorem 7.1(f), but rings with $r = m - 7$ never materialized.

We close with an explanation of the generic behavior recorded in Table 8.2.1.

8.5 Explanation of observed generic behavior. Let R be as in Theorem 7.1. Implicit in some of the arguments below is an assumption that the field k is infinite, or at least large. The generic behavior is, nevertheless, also observed when the coefficient field is as small as \mathbb{Z}_2 . If R is random, then m is as small as possible given the h -vector of R , which is determined by the socle polynomial as R is compressed. In parts (a)–(c) below no further assumptions are made, so they are rigorous statements about the class of R when m is minimal. If $\lceil \frac{s+1}{2} \rceil = t$ holds, then this minimal m is determined by Theorem 5.4; for $s \leq 10$ some consequences of (a)–(c) are recorded in the right-most column of Table 8.2.1.

- (a) If $s = 2$, then R is of class $\mathbf{H}(3, 2)$ by (4.4.2) and Theorem 7.1(g).
- (b) If $s = 3$, then Theorem 7.1(c) and Corollary 7.3 show that R is of class

$$\begin{cases} \mathbf{B} & \text{if } s_1 = 2 \\ \mathbf{H}(0, 0) & \text{if } s_1 = 3. \end{cases}$$

- (c) If $s \geq 5$ is odd and one sets $N(s) = \frac{1}{2}(s - 2 + \sqrt{4s + 13})$, then R is of class

$$\begin{cases} \mathbf{G}(\frac{1}{2}(s + 3 - a(a + 1))) & \text{if } s_1 < N(s) \\ \mathbf{H}(0, 0) & \text{if } N(s) \leq s_1 \end{cases} \quad \text{with } a = s_1 - \frac{s-1}{2}.$$

Indeed, if $N(s) \leq s_1$ holds, then R is of class $\mathbf{H}(0, 0)$, and if $s_1 < N(s)$ holds then R is of class $\mathbf{G}(r)$ by Corollary 7.4. In the second case we compute r as follows: The equality $\frac{s+1}{2} = t$ holds, see e.g. 1.2, so $a = s_1 - \frac{s-1}{2}$ holds by Theorem 7.1(b) and Theorem 5.4 yields $m = t + 1 - f_0 + f_1$. Proposition 6.9(c) and (6.0.2) now yield

$$r = m - f_1 = t + 1 - f_0 = \frac{s+3}{2} - \frac{1}{2}a(a + 1) = \frac{1}{2}(s + 3 - a(a + 1)).$$

- (d) If $s \geq 4$ is even and one sets $N(s) = \frac{s}{2} - 1 + \sqrt{s + 4}$, then R is of class

$$\begin{cases} \mathbf{G}(s - 1) & \text{if } \frac{s}{2} + 1 = s_1 \\ \mathbf{G}(s + 3 - a(a + 2)) & \text{if } \frac{s}{2} + 1 < s_1 < N(s) \\ \mathbf{H}(0, 0) & \text{if } N(s) \leq s_1. \end{cases} \quad \text{with } a = s_1 - \frac{s}{2}.$$

Indeed, notice first that one has $\frac{s}{2} + 1 \leq t \leq s_1$ by (3.1.2) and (4.4.2). If equalities hold, then (6.0.1) yields $a = 1$, and it follows from Theorem 7.1(e), Theorem 5.4, and (6.0.2) that R is of class $\mathbf{G}(r)$ with

$$r = m - 3 = 2t + 1 - f_0 - 3 = s - 1.$$

If $N_2(s) \leq s_1$ holds, then R is of class $\mathbf{H}(0, 0)$ by Corollary 7.5. Assuming now that $s_1 < N_2(s)$ holds, Proposition 6.4 yields $\frac{s}{2} + 1 = t$, so Theorem 7.1(f) gives $a = s_1 - \frac{s}{2}$, and in view of (6.0.2) one has

$$(\#) \quad 2t + 1 - f_1 = s + 3 - a(a + 2).$$

From this equality it is straightforward to verify that one has

$$(\diamond) \quad 2t + 1 - f_1 \leq 0 \iff N(s) \leq s_1.$$

By Corollary 7.5 the ring R is of class $\mathbf{H}(0, 0)$ or $\mathbf{G}(r)$. Theorem 5.4 yields

$$(*) \quad m = \max\{2t + 1 - f_0, f_1 - f_0\}.$$

From Proposition 6.9(d) one now gets a bound on r ,

$$(\dagger) \quad r \leq m - f_1 + f_0 = \max\{2t + 1 - f_1, 0\}.$$

Thus, if $N(s) \leq s_1$ holds, then it follows from (\diamond) and (\dagger) that R is of class $\mathbf{H}(0, 0)$. Further it is straightforward to verify the inequality $N(s) < N_2(s)$, cf. Corollary 7.5.

Finally, for $\frac{s}{2} + 1 < s_1 < N(s)$ one has $2t + 1 - f_1 > 0$ by (\diamond) and, therefore, $\beta = 0$ and $m = 2t + 1 - f_0$; see Theorem 5.4. Per $(\#)$ the upper bound on r from (\dagger) is $s + 3 - a(a + 2) = 2t + 1 - f_1 > 0$. To see that, generically, this bound is achieved, and R hence of class $\mathbf{G}(s + 3 - a(a + 2))$, we reason along the lines of the proof of Proposition 6.9(e): Recall that, in the notation from that proof, the upper bound on r , see (\dagger) , is $\text{rank}_k \psi_2$. By assumption $s_1 > \frac{s}{2} + 1$ holds, whence one has $a \geq 2$ and, therefore, $f_0 \geq 3$ by (6.0.1) and (6.0.2). By Theorem 5.4 the ideal I is minimally generated by $m \leq 2t - 2$ elements, say, x_1, \dots, x_m of degree t , and without loss of generality one can assume that they are the first m of the $2t + 1$ minimal generators x_i of I_2 ; see Proposition 5.1. As in the proof of 6.9(e) one can assume that $\partial_2^{F''}$ is given by a $(2t + 1) \times (2t + 1)$ skew-symmetric matrix with linear entries, and that the only nonzero products of elements in A_1'' and A_2'' are $e_i f_i$ for $1 \leq i \leq 2t + 1$. As the entries in the skew-symmetric matrix are linear, every relation between the elements x_1, \dots, x_m is a combination of the minimal relations between the generators x_i of I_2 . Generically, the skew-symmetric matrix has nonzero entries everywhere off the diagonal, so the i^{th} minimal relation involves all of the $2t + 1$ generators of I_2 save one, namely x_i . It follows that writing a relation that involves at most $2t - 2$ of the generators of I_2 in terms of those that involve $2t$ of them will, generically, require all $2t + 1$ of those minimal relations. Thus, the homomorphism $F_2 \rightarrow F_2''$ from $(*)$ in the proof of 6.9(e) maps the basis elements in internal degree $t + 1$ to random k -linear combinations of the elements of the basis for F_2'' . While $\psi_1(A_1)$ is spanned by e_1, \dots, e_m , it follows that the image $\psi_2(A_2)$ is a random subspace of A_2'' . As A'' is a Poincaré duality algebra, each element of $\psi_2(A_2)$ has nonzero products with all of the basis vectors e_1, \dots, e_m . As one has

$$\text{rank}_k \psi_2 = m - f_1 + f_0 = m - \frac{3}{2}a(a + 1) < m$$

it follows that the rank \tilde{r} of the map $\tilde{\delta}$ from the proof of 6.9 equals $\text{rank}_k \psi_2$, and \tilde{r} is a lower bound for r , so one has $r = \text{rank}_k \psi_2$.

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