G-LEVELS OF PERFECT COMPLEXES

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Abstract. We prove that a commutative noetherian ring R is Gorenstein of dimension at most d if $d+1$ is an upper bound on the G-level of perfect R-complexes, in which case $d+1$ is an upper bound on the G-level of R-complexes with finitely generated homology.

1. Introduction

Throughout this paper, R denotes a commutative noetherian ring. Let C be a collection of objects in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$, the bounded derived category of complexes with degreewise finitely generated homology. The number of mapping cones needed, up to summands, finite direct sums, and shifts, to build N from C is known as the level of N with respect to C and denoted by $\operatorname{level}^{\mathsf{C}}_R N$. That is, levels stratify the derived category of R using its triangulated structure. Levels with respect to the class $\{R\}$, sensibly known as R-levels, have been studied extensively. It has been known since the first study of R-levels by Avramov, Buchweitz, Iyengar, and Miller [2] that a commutative noetherian ring R is regular of finite Krull dimension if and only if every complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ has R-level at most $\dim R + 1$.

In this paper, we are concerned with levels with respect to the collection $\mathsf{G}(R)$ of finitely generated Gorenstein projective R-modules. This study was initiated by Awadalla and Marley [3], who proved that R is Gorenstein of finite Krull dimension if and only if there is an upper bound on the $\mathsf{G}(R)$-levels of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. They also gave such a bound, namely $2(\dim R + 1)$ and showed that it was optimal for zero-dimensional Gorenstein rings.

Our first main result, Theorem 2.10, says that a Gorenstein ring can be recognized by an upper bound on the $\mathsf{G}(R)$-levels of perfect complexes: If the inequality $\operatorname{level}^{\mathsf{G}(R)}_R P \le d+1$ holds for every perfect R-complex P, then R is Gorenstein of Krull dimension at most d. This parallels a well-known characterization of regular rings; see for example Krause [8].

Our second main result, Theorem 3.10, improves the bound from [3] on $\mathsf{G}(R)$-levels over Gorenstein rings: If R is Gorenstein, then $\operatorname{level}^{\mathsf{G}(R)}_R M \le \max\{2, \dim R + 1\}$ holds for every complex M in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. The appearance of 2 reflects the fact that the $2(\dim R + 1)$ bound is optimal when $\dim R = 0$ holds, and the natural question of what happens when the upper bound on $\mathsf{G}(R)$-levels is 1 is answered in Proposition 3.12. The findings from the results discussed so far are distilled in Theorem 3.13 which, in part, says that R is Gorenstein of Krull dimension $d \ge 1$ if and only if one has

$$\sup\{\operatorname{level}^{\mathsf{G}(R)}_R M \mid M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)\} = d+1 = \sup\{\operatorname{level}^{\mathsf{G}(R)}_R P \mid P \text{ is a perfect } R\text{-complex}\}\,.$$

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2. G-Levels of perfect complexes

We start by introducing the most central notation; for any unexplained notation or terminology we refer the reader to Christensen, Holm, and Foxby [5].

For an R-complex

$$M \coloneqq \cdots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \cdots$$

and every integer i we set $\mathrm{Z}_i(M) \coloneqq \ker(\partial_i^M)$, $\mathrm{B}_i(M) \coloneqq \operatorname{im}(\partial_{i+1}^M)$, $\mathrm{C}_i(M) \coloneqq \operatorname{coker}(\partial_{i+1}^M)$, and $\mathrm{H}_i(M) \coloneqq \mathrm{Z}_i(M)/\,\mathrm{B}_i(M)$.

2.1. Let C be a collection of objects in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. Recall from [2], the C-level of a complex M in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$, denoted by $\operatorname{level}_R^{\mathsf{C}} M$, is given as follows:

(1) $\operatorname{level}_R^{\mathsf{C}} M = 0$ if M is 0 in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$.
(2) $\operatorname{level}_R^{\mathsf{C}} M = 1$ if M is nonzero, but can be built from objects in C using shifts, summands, and finite direct sums.
(3) For $n > 1$, $\operatorname{level}_R^{\mathsf{C}} M = n$ if n is the infimum of i such that there is an exact triangle

$$K \longrightarrow L \oplus M \longrightarrow N \longrightarrow$$

with $\operatorname{level}_R^{\mathsf{C}} K = 1$ and $\operatorname{level}_R^{\mathsf{C}} N = i - 1$.

2.2. Recall that when R is local and $M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ has finite Gorenstein dimension, we have the following well-known formula due to Auslander and Bridger, see [5, Theorem 19.4.25]:

$$\operatorname{Gdim}_R M = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

2.3. Let C be a collection of objects in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. A morphism $\alpha\colon M \to N$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ is C-ghost if the induced maps

$$\operatorname{Ext}_R^n(C,M) \longrightarrow \operatorname{Ext}_R^n(C,N)$$

are zero for all integers n and all objects C in C.

The next result was first proved by Kelly [7, Theorem 3].

Ghost Lemma 2.4. *Let C be a collection of objects from $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ and $\alpha_i\colon M_i \to M_{i+1}$ for $0 \le i \le n-1$ a sequence of morphisms in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ such that $\alpha_{n-1}\alpha_{n-2}\cdots\alpha_0$ is a nonzero morphism in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. If each α_i is C-ghost, then one has $\operatorname{level}_R^{\mathsf{C}} M_0 \ge n+1$.*

Let $\mathsf{G}(R)$ denote the collection of finitely generated Gorenstein projective R-modules.

Lemma 2.5. *Let M be a finitely generated R-module. If there exists an integer $b \ge 1$ such that $\operatorname{Ext}_R^n(G,M) = 0$ holds for all $n \ge b$ and all G in $\mathsf{G}(R)$, then $\operatorname{Ext}_R^n(G,M) = 0$ holds for all $n \ge 1$ and all G in $\mathsf{G}(R)$.*

Proof. Assume that $\operatorname{Ext}_R^n(G,M) = 0$ holds for all $n \ge b$ and all G in $\mathsf{G}(R)$, and assume towards a contradiction that one has $\operatorname{Ext}_R^n(G,M) \neq 0$ for some $n \ge 1$ and some module G from $\mathsf{G}(R)$. As G is a b^{th} syzygy of a module G' in $\mathsf{G}(R)$, one has

$$\operatorname{Ext}_R^{b+n}(G',M) \cong \operatorname{Ext}_R^n(G,M) \neq 0\,,$$

a contradiction. ∎

Ideas for the proof of the next result come from [3, Theorem 3.3] and [8, Proposition A.1.2].

Theorem 2.6. *Let R be a commutative noetherian local ring. If there exists an integer ℓ such that $\operatorname{level}_R^{\mathsf{G}(R)} P \le \ell$ holds for every perfect R-complex P, then R is Gorenstein.*

Proof. Without loss of generality one can assume that ℓ is at least 2. Let k be the residue field of R, let K be the Koszul complex on a minimal set of generators for the maximal ideal of R, and set $D = \operatorname{Hom}_R(K, \mathrm{E}_R(k))$. Then D is a bounded complex of injective R-modules and has finite length homology. To show that R is Gorenstein, it is by [5, Corollary 19.5.12] enough to show that D has finite projective dimension over R.

To this end, let $F \xrightarrow{\simeq} D$ be a semi-free resolution over R with F degreewise finitely generated. For $n \geq \sup \mathrm{H}(D)$, one has $\operatorname{pd}_R D \leq \operatorname{pd}_R \mathrm{C}_n(F) + n$, so it suffices to fix an integer $s \geq \sup \mathrm{H}(D)$ and show that $\mathrm{C}_s(F)$ has finite projective dimension; after a shift one can take $s = 0$. As the injective dimension of D over R is finite, it follows that for every R-module G one has $\operatorname{Ext}_R^m(G, D) = 0$ for all $m > \operatorname{id}_R D$. Thus, for every $n \geq 0$ and G in $\mathsf{G}(R)$, there is an integer b_n such that $\operatorname{Ext}_R^m(G, \mathrm{C}_n(F)) = \operatorname{Ext}_R^{m+n}(G, D) = 0$ for all $m \geq b_n$, see Christensen, Frankild, and Holm [6, Corollary 2.10], whence one has $\operatorname{Ext}_R^m(G, \mathrm{C}_n(F)) = 0$ for all $m \geq 1$ and all $G \in \mathsf{G}(R)$ by Lemma 2.5.

For $n \geq 0$, set

$$X^n = 0 \longrightarrow F_{\ell+n} \xrightarrow{\partial_{\ell+n}} \cdots \xrightarrow{\partial_{n+1}} F_n \longrightarrow 0\,.$$

Notice that we have an exact sequence of R-complexes

$$0 \longrightarrow \Sigma^{\ell+n}\,\mathrm{C}_{\ell+n+1}(F) \longrightarrow X^n \longrightarrow \Sigma^n\,\mathrm{C}_n(F) \longrightarrow 0\,,$$

and from the associated exact sequence of Ext modules one gets for every $G \in \mathsf{G}(R)$

$$(*) \qquad \operatorname{Ext}_R^m(G, X^n) = 0 \quad \text{for} \quad m \neq -(\ell+n) \text{ and } m \neq -n\,.$$

Now, consider the canonical morphisms $\alpha^n \colon X^n \to X^{n+1}$ for $n = 0, \ldots, \ell - 1$. It follows from $(*)$ that they are all $\mathsf{G}(R)$-ghost. Therefore, if the composite $\alpha = \alpha^{\ell-1}\alpha^{\ell-2}\cdots\alpha^0$ is a nonzero morphism in $\mathsf{D}_\mathsf{b}^\mathsf{f}(R)$, then Ghost Lemma 2.4 yields

$$\operatorname{level}_R^{\mathsf{G}(R)} X^0 \geq \ell + 1\,,$$

which contradicts the assumption. Thus, α is a zero morphism in $\mathsf{D}_\mathsf{b}^\mathsf{f}(R)$. Since α is a morphism of semi-projective R-complexes it is null-homotopic; in particular there exist homomorphisms

$$\sigma_\ell \colon F_\ell \to F_{\ell+1} \quad \text{and} \quad \sigma_{\ell-1} \colon F_{\ell-1} \to F_\ell$$

such that

$$1^{F_\ell} = \alpha_\ell = \sigma_{\ell-1}\partial_\ell + \partial_{\ell+1}\sigma_\ell\,.$$

Consider the composite

$$\mathrm{B}_\ell(F) \hookrightarrow F_\ell \xrightarrow{\sigma_\ell} F_{\ell+1} \xrightarrow{\partial_{\ell+1}} \mathrm{B}_\ell(F)\,.$$

For x in $\mathrm{B}_\ell(F)$, one has

$$x = 1^{F_\ell}(x) = \sigma_{\ell-1}\partial_\ell(x) + \partial_{\ell+1}\sigma_\ell(x) = \partial_{\ell+1}\sigma_\ell(x)\,,$$

so $\partial_{\ell+1}\sigma_\ell(x)$ is a left inverse to the inclusion $\mathrm{B}_\ell(F) \hookrightarrow F_\ell$, which means that $\mathrm{B}_\ell(F)$ is a summand of a free module, whence $\mathrm{C}_0(F)$ has projective dimension at most $\ell + 1$. □

Remark 2.7. Notice that the complexes X^n in the proof above have amplitude ℓ. Thus, for $\ell \geq 2$ one gets the desired conclusion that R is Gorenstein, as long as perfect R-complexes of amplitude ℓ have $\mathsf{G}(R)$-level at most ℓ.

The following fact will be used several times in the next couple of proofs.

2.8. Let $\mathfrak{p}$ be a prime ideal in R and P a perfect $R_\mathfrak{p}$-complex. Since every finitely generated free $R_\mathfrak{p}$-module is a localization of a finitely generated free R-module, Letz's proof of [9, Lemma 3.9] shows that there exists a bounded complex F of finitely generated free R-modules with $F_\mathfrak{p} \simeq P$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R_\mathfrak{p})$.

Lemma 2.9. *Let R be Cohen–Macaulay of finite Krull dimension. There is a bounded complex F of finitely generated free R-modules with $\operatorname{level}_R^{\mathsf{G}(R)} F \geq \dim R + 1$. Moreover, if R is local then one can take F to be the Koszul complex on a sequence of parameters for R, and in that case equality holds.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R with $\dim R_\mathfrak{m} = \dim R$. Let K be the Koszul complex on a sequence of parameters for $R_\mathfrak{m}$ and choose by 2.8 a bounded complex F of finitely generated free R-modules with $F_\mathfrak{m} \simeq K$. In view of [2, Lemma 2.4(6)] this explains the inequality in the display below. Per [3, Corollary 3.4] and 2.2 the equalities hold as K is the minimal free resolution of $\mathrm{H}_0(K)$,

$$\operatorname{level}_R^{\mathsf{G}(R)} F \geq \operatorname{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})} K = \operatorname{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})} \mathrm{H}_0(K) = \operatorname{Gdim}_{R_\mathfrak{m}} \mathrm{H}_0(K) + 1 = \dim R + 1\,. \qquad \square$$

For $d = 0$ the conclusion in the next result is far from optimal; see Proposition 3.12.

Theorem 2.10. *Let R be a commutative noetherian ring. If there exists an integer d such that $\operatorname{level}_R^{\mathsf{G}(R)} P \leq d + 1$ holds for every perfect R-complex P, then R is a Gorenstein of Krull dimension at most d.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R; it suffices to show that the local ring $R_\mathfrak{m}$ is Gorenstein of Krull dimension at most d. Let P be a perfect $R_\mathfrak{m}$-complex. Per 2.8 there exists a perfect R-complex F with $F_\mathfrak{m} \simeq P$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R_\mathfrak{m})$. Now, [2, Lemma 2.4(6)] yields the first inequality in the chain

$$\operatorname{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})} P = \operatorname{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})} F_\mathfrak{m} \leq \operatorname{level}_R^{\mathsf{G}(R)} F \leq d + 1\,,$$

which implies that $R_\mathfrak{m}$ is Gorenstein by Theorem 2.6.

Finally, one has $\dim R_\mathfrak{m} \leq d$ by Lemma 2.9. $\square$

Remark 2.11. Let M be a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ of finite Gorenstein projective dimension. It follows from [5, Proposition 9.1.27] that M fits in a triangle with a perfect R-complex and a module from $\mathsf{G}(R)$. This provides some measure of an *a posteriori* explanation of why it suffices to bound the $\mathsf{G}(R)$-level of perfect R-complexes to conclude that R is Gorenstein.

3. G-levels over Gorenstein rings

It is proved in [3, Theorem 3.11] that if R is Gorenstein, then the $\mathsf{G}(R)$-level of a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ is at most $2(\dim R + 1)$. For artinian rings this bound is optimal, but we prove in this section that $\dim R + 1$ is the optimal bound if R has positive Krull dimension.

Construction 3.1. Let M be an R-complex. There is a graded free R-module F and a surjective homomorphism $\bar{\pi}\colon F \twoheadrightarrow \mathrm{H}(M)$ of graded R-modules. By graded-projectivity of F, it lifts to a graded homomorphism $\pi\colon F \to \mathrm{Z}(M)$. Considering F as a complex with zero differential, π is a morphism $F \to M$ of complexes. Note that if M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$, then we can choose F to be a bounded complex of finitely generated free modules.

Set $\Omega_R^0(M) \coloneqq M$ and $\Omega_R^1(M) \coloneqq \Sigma^{-1}\operatorname{Cone}(\pi)$. Applying the construction above recursively, set $\Omega_R^{n+1}(M) \coloneqq \Omega_R^1(\Omega_R^n(M))$ for $n \geq 1$. Taken together the ensuing triangles

$$\Omega_R^{n+1}(M) \longrightarrow F^n \xrightarrow{\pi^n} \Omega_R^n(M) \longrightarrow$$

form an Adams resolution for M in the sense of Christensen [4]. They induce short exact sequences in homology

$$0 \longrightarrow \mathrm{H}(\Omega_R^{n+1}(M)) \longrightarrow F^n \longrightarrow \mathrm{H}(\Omega_R^n(M)) \longrightarrow 0\,.$$

Note that if M is a complex of free modules, then so are the complexes $\Omega_R^n(M)$ and, more generally, if M is a complex of modules of depth at least p for some $p \le \operatorname{depth} R$, then so is each complex $\Omega_R^n(M)$.

3.2. For an R-complex M, set $M^{\oplus} = \bigoplus_{n\in\mathbb{Z}} M_n$. This notation comes in handy as the equality $\operatorname{level}_R^{\mathsf{G}(R)} M = \operatorname{level}_R^{\mathsf{G}(R)} M^{\oplus}$ holds for a complex M with zero differential.

Lemma 3.3. *Let $n \ge 1$ be an integer and M a complex in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$. There is an inequality*

$$\operatorname{level}_R^{\mathsf{G}(R)} M \le \operatorname{level}_R^{\mathsf{G}(R)} \Omega_R^n(M) + n\,.$$

Proof. It suffices to show that $\operatorname{level}_R^{\mathsf{G}(R)} M \le \operatorname{level}_R^{\mathsf{G}(R)} \Omega_R^1(M)+1$ holds. To this end consider the triangle from Construction 3.1

$$\Omega_R^1(M) \longrightarrow F \xrightarrow{\pi} M \longrightarrow\ .$$

Since F is a complex with zero differential one has $\operatorname{level}_R^{\mathsf{G}(R)} F = \operatorname{level}_R^{\mathsf{G}(R)} F^{\oplus} \le 1$, and the asserted inequality holds by [2, Lemma 2.4(2)]. ☐

Lemma 3.4. *Let M be an R-complex. For every integer $n \ge 1$, we have an exact sequence of R-modules*

$$0 \longrightarrow \mathrm{H}(\Omega_R^n(M))^{\oplus} \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow \mathrm{H}(M)^{\oplus} \to 0.$$

where each L_i is a free R-module.

Proof. The claim follows since for every $i \ge 0$ there is a short exact sequence

$$0 \longrightarrow \mathrm{H}(\Omega_R^{i+1}(M))^{\oplus} \longrightarrow (F^i)^{\oplus} \longrightarrow \mathrm{H}(\Omega_R^i(M))^{\oplus} \longrightarrow 0$$

from Construction 3.1. ☐

3.5. In the next several results, we make use of the following exact sequences from [5, Proposition 2.2.12]:

(1) $$0 \longrightarrow \mathrm{H}_i(M) \longrightarrow \mathrm{C}_i(M) \longrightarrow \mathrm{B}_{i-1}(M) \longrightarrow 0$$

(2) $$0 \longrightarrow \mathrm{Z}_i(M) \longrightarrow M_i \longrightarrow \mathrm{B}_{i-1}(M) \longrightarrow 0$$

(3) $$0 \longrightarrow \mathrm{B}_i(M) \longrightarrow M_i \longrightarrow \mathrm{C}_i(M) \longrightarrow 0\,.$$

Lemma 3.6. *Suppose M is a right bounded R-complex such that M_i and $\mathrm{H}_i(M)$ have finite Gorenstein dimension for all i. Then $\mathrm{B}_i(M)$, $\mathrm{Z}_i(M)$, and $\mathrm{C}_i(M)$ have finite Gorenstein dimension for all i as well.*

Proof. Shifting for convenience, we may suppose $\inf M = 0$. We proceed by induction on i to show that $\mathrm{B}_{i-1}(M)$, $\mathrm{Z}_i(M)$, and $\mathrm{C}_i(M)$ have finite Gorenstein dimension for all i. If $i = 0$, then as $\inf M = 0$, one has $\mathrm{B}_{-1}(M) = 0$ and $\mathrm{Z}_0(M) = M_0$, so these modules have finite Gorenstein dimension; moreover $\mathrm{H}_0(M) = \mathrm{C}_0(M)$ holds, so $\mathrm{C}_0(M)$ has finite Gorenstein dimension by assumption. Thus, the base case is established.

Now suppose we know $\mathrm{B}_{i-1}(M)$, $\mathrm{Z}_i(M)$, and $\mathrm{C}_i(M)$ have finite Gorenstein dimension for some $i \ge 0$. From 3.5(3), we see that $\mathrm{B}_i(M)$ has finite Gorenstein dimension, from 3.5(2) we see that $\mathrm{Z}_{i+1}(M)$ has finite Gorenstein dimension, and from 3.5(1) we see that $\mathrm{C}_{i+1}(M)$ has finite Gorenstein dimension. ☐

Proposition 3.7. *Suppose $M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ and suppose $\mathrm{H}(M)$ has finite Gorenstein dimension. Then M has finite Gorenstein dimension.*

Proof. Let F be a free resolution of M. Then since every term of F has finite Gorenstein dimension, it follows from Lemma 3.6 that $\mathrm{Z}_i(F)$, $\mathrm{B}_i(F)$, and $\mathrm{C}_i(F)$ have finite Gorenstein dimension for all i. But, $\mathrm{H}_i(F) = 0$ for $i \gg 0$, and so $\mathrm{C}_i(F)$ is Gorenstein projective for $i \gg 0$, and thus M has finite Gorenstein dimension. □

For the next result, we recall from [5] that the depth of the zero module is $+\infty$.

Lemma 3.8. *Let R be local and M a right bounded R-complex with $\operatorname{depth}_R M_i \geq \operatorname{depth} R$ for all i. If $\operatorname{depth}_R \mathrm{H}_i(M) \geq \operatorname{depth} R - 1$ holds for all i, then $\operatorname{depth}_R \mathrm{Z}_i(M) \geq \operatorname{depth} R$ and $\operatorname{depth}_R \mathrm{B}_i(M) \geq \operatorname{depth} R$ hold for all i.*

Proof. Shifting M for convenience, we may suppose $\inf M = 0$. There is nothing to show if $\operatorname{depth} R = 0$, so we may assume $\operatorname{depth} R$ is positive. We proceed by induction on i to show that $\operatorname{depth}_R \mathrm{Z}_i(M) \geq \operatorname{depth} R$ and $\operatorname{depth}_R \mathrm{B}_{i-1}(M) \geq \operatorname{depth} R$ hold for all i. When $i = 0$, we have $\mathrm{B}_{-1}(M) = 0$ and $\mathrm{Z}_0(M) = M_0$, so the claim holds when $i = 0$.

Now, suppose we have the claim for some $i \geq 0$. An application of the depth lemma to 3.5(1) yields $\operatorname{depth}_R \mathrm{C}_i(M) \geq \operatorname{depth} R - 1$, while application of the depth lemma to 3.5(3) forces $\operatorname{depth}_R \mathrm{B}_i(M) \geq \operatorname{depth} R$. Finally, applying it to 3.5(2) shows that $\operatorname{depth}_R \mathrm{Z}_{i+1}(M) \geq \operatorname{depth} R$ holds. □

Proposition 3.9. *For every complex $M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ one has*

$$\operatorname{level}_R^{\mathsf{G}(R)} M \leq \max\{2, \operatorname{Gdim}_R \mathrm{H}(M)^{\oplus} + 1\} = \max\{2, \operatorname{level}_R^{\mathsf{G}(R)} \mathrm{H}(M)^{\oplus}\}\,.$$

Proof. The equality follows from [3, Corollary 3.4], so it remains to show the inequality. There is nothing to prove if $\operatorname{Gdim}_R \mathrm{H}(M)^{\oplus}$ is infinite, so we may suppose that $\operatorname{Gdim}_R \mathrm{H}(M)^{\oplus}$ is finite. Then, it follows from Proposition 3.7 that M has finite Gorenstein dimension, so we may replace M by a G-resolution to suppose M is a bounded complex of modules from $\mathsf{G}(R)$. First, notice that if $\mathrm{Z}(M)^{\oplus}$ and $\mathrm{B}(M)^{\oplus}$ belong to $\mathsf{G}(R)$, then [2, Lemma 2.4(2)] applied to the exact sequence

$$0 \longrightarrow \mathrm{Z}(M) \longrightarrow M \longrightarrow \Sigma\,\mathrm{B}(M) \longrightarrow 0$$

yields

$$\operatorname{level}_R^{\mathsf{G}(R)} M \leq \operatorname{level}_R^{\mathsf{G}(R)} \mathrm{Z}(M) + \operatorname{level}_R^{\mathsf{G}(R)} \mathrm{B}(M) = \operatorname{level}_R^{\mathsf{G}(R)} \mathrm{Z}(M)^{\oplus} + \operatorname{level}_R^{\mathsf{G}(R)} \mathrm{B}(M)^{\oplus} \leq 2\,,$$

where the equality holds as the subcomplexes $\mathrm{Z}(M)$ and $\mathrm{B}(M)$ have zero differentials. In particular, the asserted inequality holds in case $\operatorname{Gdim}_R \mathrm{H}(M)^{\oplus} = 0$, by Lemmas 3.6 and 3.8, so we suppose $n \coloneqq \operatorname{Gdim}_R \mathrm{H}(M)^{\oplus}$ is positive.

From Lemma 3.4, we see that $\operatorname{Gdim}_R \mathrm{H}(\Omega_R^{n-1}(M))^{\oplus} = 1$. Now, let $\mathfrak{p}$ be a prime ideal. Then, $\operatorname{Gdim}_{R_\mathfrak{p}} \mathrm{H}(\Omega_R^{n-1}(M_\mathfrak{p}))^{\oplus} \leq 1$, which implies $\operatorname{depth}_{R_\mathfrak{p}} \mathrm{H}(\Omega_R^{n-1}(M_\mathfrak{p}))^{\oplus} \geq \operatorname{depth} R_\mathfrak{p} - 1$ by 2.2. Combining Lemma 3.8 with another application of 2.2, we see that $\mathrm{Z}(\Omega_R^{n-1}(M_\mathfrak{p}))^{\oplus}$ and $\mathrm{B}(\Omega_R^{n-1}(M_\mathfrak{p}))^{\oplus}$ are Gorenstein projective modules over $R_\mathfrak{p}$. Thus, $\mathrm{Z}(\Omega_R^{n-1}(M))^{\oplus}$ and $\mathrm{B}(\Omega_R^{n-1}(M))^{\oplus}$ are Gorenstein projective modules over R by [5, Proposition 19.4.12]. By the argument in the first paragraph, one now has $\operatorname{level}_R^{\mathsf{G}(R)} \Omega_R^{n-1}(M) \leq 2$, so Lemma 3.3 yields $\operatorname{level}_R^{\mathsf{G}(R)} M \leq n+1$, and the claim follows. □

The following is an immediate consequence of Proposition 3.9:

Theorem 3.10. *Let R be Gorenstein and M a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. One has*

$$\operatorname{level}_R^{\mathsf{G}(R)} M \leq \max\{2, \dim R + 1\}\,.$$

Since R is regular of finite Krull dimension if and only if there exists a uniform upper bound on the R-level of all perfect R-complexes, Theorem 3.10 tells us that over a nonregular Gorenstein ring there must exist a perfect complex P such that the inequality, $\operatorname{level}_R^{\mathsf{G}(R)} P \leq \operatorname{level}_R^R P$, which holds as R belongs to $\mathsf{G}(R)$, is strict.

Proposition 3.11. *Let R be Gorenstein and not regular and $n \geq 2$ an integer. There exists a perfect R-complex P with $\operatorname{level}_R^R P = n+1$ and $\operatorname{level}_R^{\mathsf{G}(R)} P \leq 2$.*

Proof. We build off an argument by Altmann, Grifo, Montaño, Sanders, and Vu [1, Corollary 2.3]. Let G be a module in $\mathsf{G}(R)$ and not projective, i.e. of infinite projective dimension. If $F \to G$ is a free resolution, then the truncated complex

$$P = 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

has R-level $n+1$ by the proof of [1, Corollary 2.3]. Further, there is a triangle

$$\Sigma^n \operatorname{H}_n(P) \longrightarrow P \longrightarrow \operatorname{H}_0(P) \longrightarrow$$

in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$, so $\operatorname{level}_R^{\mathsf{G}(R)} P \leq 2$ holds by [2, Lemma 2.4(2)] as both $\operatorname{H}_n(P)$ and $\operatorname{H}_0(P) \cong G$ are Gorenstein projective modules. □

The inequality in Theorem 3.9 begs the question: What happens if the $\mathsf{G}(R)$-levels of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ are at most 1?

Proposition 3.12. *If $\operatorname{level}_R^{\mathsf{G}(R)} P \leq 1$ holds for every perfect R-complex P, then R is regular of Krull dimension 0.*

Proof. It follows from Theorem 2.10 that R has Krull dimension 0. Let $\mathfrak{m}$ be a maximal ideal in R; it suffices to show that $R_\mathfrak{m}$ is regular. Let P be a perfect complex over $R_\mathfrak{m}$; per 2.8 there exists a perfect R-complex F with $F_\mathfrak{m} \simeq P$. Now, [2, Lemma 2.4(6)] gives the first inequality below

$$\operatorname{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})} P = \operatorname{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})} F_\mathfrak{m} \leq \operatorname{level}_R^{\mathsf{G}(R)} F \leq 1.$$

One can now assume that R is local. Let K be the Koszul complex on a minimal generating set for the maximal ideal of R and let k be the residue field of R. Since $\operatorname{level}_R^{\mathsf{G}(R)} K = 1$ holds, K is isomorphic in the derived category to a complex M that is a direct sum of shifts of modules from $\mathsf{G}(R)$, and so we have

$$K \simeq M \simeq \operatorname{H}(M) \simeq \operatorname{H}(K) = \bigoplus \Sigma^i \operatorname{H}_i(K)$$

Since $\operatorname{H}_0(K) = k \neq 0$, we have $\operatorname{H}_i(K) = 0$ for all $i \neq 0$ by [1, Proposition 4.7], and so $\operatorname{pd}_R k < \infty$. This implies R is regular. □

Theorem 3.13. *The following assertions hold.*

(a) *The next conditions are equivalent.*
  (*i*) $\sup\{\operatorname{level}_R^{\mathsf{G}(R)} M \mid M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)\} = 2$ *holds.*
  (*ii*) $\sup\{\operatorname{level}_R^{\mathsf{G}(R)} P \mid P$ is a perfect R-complex$\} = 2$ *holds.*
  (*iii*) *R is Gorenstein with $\dim R = 1$ or Gorenstein with $\dim R = 0$ and not regular.*

(b) *Let $d \geq 2$ be an integer. The following are equivalent.*
  (*i*) $\sup\{\operatorname{level}_R^{\mathsf{G}(R)} M \mid M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)\} = d+1$ *holds.*

(*ii*) $\sup\{\operatorname{level}_R^{\mathsf{G}(R)} P \mid P \text{ is a perfect } R\text{-complex}\} = d+1$ *holds.*

(*iii*) *R is Gorenstein with* $\dim R = d$.

Proof. (a): Assume that the equality in (*i*) holds. It follows from Theorem 2.10 that R is Gorenstein of Krull dimension at most 1. If $\dim R$ equals 1, then Lemma 2.9 implies the existence of a perfect R-complex of $\mathsf{G}(R)$-level 2. If $\dim R = 0$ holds, then R is per [2, Theorem 5.5] not regular, since there exists a complex in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$ of $\mathsf{G}(R)$-level 2 and hence R-level at least 2. Now, Proposition 3.12 implies the existence of a perfect R-complex of $\mathsf{G}(R)$-level 2. Thus, (*i*) implies both (*ii*) and (*iii*) and, more importantly, the arguments above apply verbatim to show that (*ii*) implies (*iii*). It remains to show that (*iii*) implies (*i*):

If R is Gorenstein with $\dim R \leq 1$, then 2 is by Theorem 3.10 an upper bound on the $\mathsf{G}(R)$-level of complexes in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$. If $\dim R = 1$ holds, then Lemma 2.9 implies the existence of a complex in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$ of $\mathsf{G}(R)$-level 2. If $\dim R = 0$ holds and R is not regular, then Proposition 3.12 implies the existence of a complex in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$ of $\mathsf{G}(R)$-level 2.

(b): We proceed by induction on d. Let $d = 2$. If the equality in (*i*) or (*ii*) holds, then it follows from Theorem 2.10 and part (a) that R is Gorenstein of Krull dimension 2. On the other hand, if R is such a ring, then there exists by Lemma 2.9 a perfect R-complex of $\mathsf{G}(R)$-level 3, and 3 is by Theorem 3.10 an upper bound on the $\mathsf{G}(R)$-level of complexes in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$.

Now, let $d > 2$. If the equality in (*i*) or (*ii*) holds, then it follows from Theorem 2.10 that R is Gorenstein of Krull dimension at most d, and by the induction hypothesis the Krull dimension is at least d. On the other hand, if R is such a ring, then there exists by Lemma 2.9 a perfect R-complex of $\mathsf{G}(R)$-level $d+1$, and $d+1$ is by Theorem 3.10 an upper bound on the $\mathsf{G}(R)$-level of complexes in $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R)$. □

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References

[1] Hannah Altmann, Eloísa Grifo, Jonathan Montaño, William T. Sanders, and Thanh Vu, *Lower bounds on projective levels of complexes*, J. Algebra **491** (2017), 343–356. MR 3699100

[2] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, Srikanth B. Iyengar, and Claudia Miller, *Homology of perfect complexes*, Adv. Math. **223** (2010), no. 5, 1731–1781. MR 2592508

[3] Laila Awadalla and Thomas Marley, *Level and Gorenstein projective dimension*, J. Algebra **609** (2022), 606–618. MR 4460332

[4] J. Daniel Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta*, Adv. Math. **136** (1998), no. 2, 284–339. MR 1626856

[5] Lars Winther Christensen, Hans-Bjørn Foxby, and Henrik Holm, *Derived category methods in commutative algebra*, Springer Monographs in Mathematics, Springer Cham, 2024.

[6] Lars Winther Christensen, Anders Frankild, and Henrik Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra **302** (2006), no. 1, 231–279. MR 2236602

[7] G. M. Kelly, *Chain maps inducing zero homology maps*, Proc. Cambridge Philos. Soc. **61** (1965), 847–854. MR 188273

[8] Henning Krause, *The finitistic dimension of a triangulated category*, Proc. Amer. Math. Soc. Ser. B **11** (2024), no. 49, 570–578.

[9] Janina C. Letz, *Local to global principles for generation time over commutative noetherian rings*, Homology Homotopy Appl. **23** (2021), no. 2, 165–182. MR 4259573

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