TATE HOMOLOGY BEYOND GORENSTEIN RINGS

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Abstract. Tate homology—originally defined for modules over group algebras—has a straightforward generalization to Iwanaga–Gorenstein rings, and a far-reaching generalization to associative rings. We report on progress in understanding the latter.

Introduction

The theories of Tate homology and Tate cohomology go back to the early 1950s, and they were originally introduced as (co)homology theories for (modules over) group algebras. The underlying construction has since evolved through a series of generalizations to yield a theory for Iwanaga–Gorenstein rings; that is, noetherian rings with finite self-injective dimension on either side. This process was started by T. Nakayama [13] already in 1957, and the most recent developments—due to Avramov and Martsinkovsky [3], Veliche [14], and Iacob [12]—date from the 2000s. Here is the central piece of technology:

Definition 1. A complex T of projective right R-modules is called *totally acyclic* if one has $\mathrm{H}(T) = 0 = \mathrm{H}(\mathrm{Hom}_R(T, P))$ for every projective right R-module P. A *complete resolution* of a right R-module M is a diagram $T \xrightarrow{\varpi} P \xrightarrow{\pi} M$, where T is a totally acyclic complex of projective right R-modules, π is a projective resolution, and ϖ_i is an isomorphism for $i \gg 0$.

It is not evident, but every module over an Iwanaga–Gorenstein ring has a complete resolution [9, 14] (and that characterizes these rings). The next definition thus defines Tate (co)homology for every pair of modules over an Iwanaga–Gorenstein ring.

Definition 2. Let M be a right R-module with a complete resolution $T \to P \to M$. For a left R-module N, *Tate homology* $\widehat{\mathrm{Tor}}{}^R_*(M, N)$ is the homology of the complex $T \otimes_R N$, and for a right R-module N, *Tate cohomology* $\widehat{\mathrm{Ext}}{}^*_R(M, N)$ is the cohomology of the complex $\mathrm{Hom}_R(T, N)$.

It is not only possible to take Tate homology beyond group algebras, it is also useful. Here is an example due to Christensen and Jorgensen [7]. (A commutative local ring is Iwanaga–Gorenstein if and only if it is Gorenstein in the commutative algebra sense.)

Theorem 3. *Let R be a commutative Gorenstein local ring. For R-modules M and N with $\widehat{\mathrm{Tor}}{}^R_*(M, N) = 0$ one has*

$$\mathrm{depth}_R(M \otimes^{\mathbf{L}}_R N) = \mathrm{depth}_R M + \mathrm{depth}_R N - \mathrm{depth}\, R\,.$$

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The symbol $M \otimes_R^{\mathbf{L}} N$ denotes the *derived tensor product* of M and N. It can be computed as $P \otimes_R N$, where $P \to M$ is a projective resolution. The homology of $M \otimes_R^{\mathbf{L}} N$ is $\operatorname{Tor}_*^R(M,N)$, and the theorem thus generalizes Auslander's [1] *depth formula* $\operatorname{depth}_R(M \otimes_R N) = \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R$ for modules with $\operatorname{Tor}_{>0}^R(M,N) = 0$.

Stabilization of (co)homology

A view of Tate (co)homology as a *stabilization* of ordinary (co)homology emerged in the 1980s and early 1990s in work of P. Vogel, published by Goichot [10] and in work of Benson and Carlson [5]. The stable cohomology $\widetilde{\operatorname{Ext}}_R^*(M,N)$ of a pair of right R-modules is computed as follows: Let $P \to M$ and $Q \to N$ be projective resolutions. The total Hom-complex $\operatorname{Hom}_R(P,Q)$ is the product totalization of the double complex $(\operatorname{Hom}_R(P_i,Q_j))_{i,j\geq 0}$. The direct sum totalization of the same double complex yields a subcomplex of $\operatorname{Hom}_R(P,Q)$; the quotient complex is denoted $\widetilde{\operatorname{Hom}}_R(P,Q)$, and its cohomology is stable cohomology $\widetilde{\operatorname{Ext}}_R^*(M,N)$.

It was proved by Cornick and Kropholler [8] that there is an isomorphism

$$\widetilde{\operatorname{Ext}}_R^*(M,-) \cong \widehat{\operatorname{Ext}}_R^*(M,-)$$

whenever Tate cohomology $\widehat{\operatorname{Ext}}_R^*(M,-)$ is defined, i.e. M has a complete resolution. This establishes stable cohomology as a wide-ranging generalization of Tate cohomology, available over every associative ring. Moreover, a detailed study by Avramov and Veliche [4] of stable cohomology over commutative local rings has shown that the theory carries useful information beyond the setting of Gorenstein rings.

Now, what about the homological side? A theory of stable homology, also due to P. Vogel, is included in [10]. Here is the definition.

Definition 4. Let M be a right R-module and N be a left R-module. Let $P \to M$ be a projective resolution and let $N \to I$ be an injective resolution. The tensor product $P \otimes_R I$ is the direct sum totalization of the double complex $(P_i \otimes_R I^j)_{i,j\geq 0}$. It is a subcomplex of the product totalization $P \mathbin{\overline{\otimes}}_R I$ of the same double complex, and the homology of the quotient complex is *stable homology* $\widetilde{\operatorname{Tor}}_*^R(M,N)$.

While it is proved in [10] that stable homology, indeed, coincides with Tate homology over group algebras, little has been known about the general stable homology theory. The purpose of the talk is to report on progress in this direction that has been achieved in recent work of Celikbas, Christensen, Liang, and Piepmeyer [6].

Stable homology

To simplify the statements, R is now assumed to be noetherian (on either side) and all modules are tacitly assumed to be finitely generated. First of all, stable homology agrees with Tate homology whenever the latter is defined.

Theorem 5. *Let M be a right R-module with a complete resolution. For every $i \in \mathbb{Z}$ the stable homology $\widetilde{\operatorname{Tor}}_i^R(M,-)$ is naturally isomorphic to Tate homology $\widehat{\operatorname{Tor}}_i^R(M,-)$.*

One expects a homology theory to detect finiteness of homological dimensions. The next two results reflect the asymmetry in the definition of stable homology.

Proposition 6. *Let R be an Artin algebra or commutative and local. For a right R-module M, the following conditions are equivalent.*

(i) *M has finite projective dimension.*

(ii) $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$ *for all* $i \in \mathbb{Z}$.

(iii) *There is an* $i \geq 0$ *with* $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$.

Proposition 7. *Let R be an Artin algebra or commutative and local. For a left R-module N, the following conditions are equivalent.*

(i) *N has finite injective dimension.*

(ii) $\widetilde{\operatorname{Tor}}_i^R(-,N) = 0$ *for all* $i \in \mathbb{Z}$.

(iii) *There is an* $i \leq 0$ *with* $\widetilde{\operatorname{Tor}}_i^R(-,N) = 0$.

It is evident from these two vanishing results that stable homology cannot be balanced in the way Tor is balanced.

Theorem 8. *Let R be an Artin algebra or commutative. The following conditions on R are equivalent.*

(i) *R is Iwanaga–Gorenstein.*

(ii) *For all right R-modules M, all left R-modules N, and all $i \in \mathbb{Z}$ there are isomorphisms* $\widetilde{\operatorname{Tor}}_i^R(M,N) \cong \widetilde{\operatorname{Tor}}_i^{R^\circ}(N,M)$.[1]

Stable homology also detects commutative Gorenstein rings in a different, and perhaps surprising, way.

Theorem 9. *Let R be commutative. The following conditions are equivalent.*

(i) *The local ring $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p}$ in R.*

(ii) *For every R-module M one has* $\widetilde{\operatorname{Tor}}_*^R(M,R) = 0$.

The proof showcases one way in which to extract information from stable homology, or rather, vanishing of stable homology.

Proof. By a theorem of S. Goto [11], the ring R is Gorenstein at every prime ideal if and only if every R-module M has a complete resolution $T \to P \to M$. A result of Foxby, published in [2], implies that M has a complete resolution if (and only if) there is an isomorphism $M \cong \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,R),R)$ in the derived category over R.

(i) $\Longrightarrow$ (ii): Let M be an R-module, by Goto's theorem it has a complete resolution $T \to P \to M$, so by Theorem 5 one has

$$\widetilde{\operatorname{Tor}}_*^R(M,R) \cong \widehat{\operatorname{Tor}}_*^R(M,R) = \operatorname{H}(T \otimes_R R) \cong \operatorname{H}(T) = 0\,.$$

(ii) $\Longrightarrow$ (i): Let M be an R-module; by assumption one has $\widetilde{\operatorname{Tor}}_*^R(M,R) = 0$. Let $P \to M$ be a degree-wise finitely generated projective resolution and let $R \to I$ be an injective resolution. In the derived category there are isomorphisms

$$M \cong P \cong P \otimes_R R \cong P \otimes_R I \cong P \otimes_R \operatorname{Hom}_R(R,I)\,.$$

There is a natural morphism of complexes

$$P \otimes_R \operatorname{Hom}_R(R,I) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(P,R),I)\,,$$

[1]Here R° denotes the opposite ring of R.

and the right-hand complex is isomorphic to $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,R),R)$ in the derived category. However, the natural morphism is not invertible; that comes down to the left-hand complex being a direct sum totalization as opposed to the right-hand complex which is a product totalization. Now, the assumption $\widetilde{\mathrm{Tor}}{}^R_*(M,R) = 0$ yields $P \otimes_R I \cong P \mathbin{\overline{\otimes}}_R I$ in the derived category, and one has

$$P \mathbin{\overline{\otimes}}_R I \cong P \mathbin{\overline{\otimes}}_R \mathrm{Hom}_R(R,I) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(P,R),I)$$

where the last isomorphism holds as both complexes are now product totalizations. Thus, for every R-module M one has $M \cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,R),R)$ in the derived category, and it follows from the works of Foxby and Goto that R is Gorenstein at every prime. $\square$

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