BUILDING MODULES FROM THE SINGULAR LOCUS

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Abstract. A finitely generated module over a commutative noetherian ring of finite Krull dimension can be built from the prime ideals in the singular locus by iteration of three procedures: taking extensions, direct summands, and cosyzygies. In 2003 Schoutens gave a bound on the number of iterations required to build any module, and in this note we determine the exact number. This building process yields a stratification of the module category, which we study in detail for local rings that have an isolated singularity.

Introduction

Let R be a commutative noetherian ring of finite Krull dimension. In [3] Schoutens shows that starting from the set of singular primes in R, one can build the entire category of finitely generated R-modules by way of extensions, direct summands, and cosyzygies. Schoutens's result gives a bound, in terms of the Krull dimension of R, on the number of times these procedures must be repeated to complete the building process. In this paper we give an improved bound on this number and show that it is sharp. In the process we give a condensed proof of the original result.

From the building process one gets a stratification of the module category into full subcategories that we call "tiers". Over a regular ring the tiers simply sort the modules by projective dimension, but over singular rings the picture remains opaque. We describe the tiers explicitly for a local ring with an isolated singularity.

1. Tiers of modules

In this paper R is a commutative noetherian ring, and $\mathsf{mod}\,R$ denotes the category of finitely generated R-modules. By a subcategory of $\mathsf{mod}\,R$ we always mean a full subcategory closed under isomorphisms. By $\operatorname{Reg} R$ we denote the *regular locus* of R; that is, the set $\operatorname{Reg} R = \{\mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p} \text{ is regular}\}$. The *singular locus* of R is the complementary set $\operatorname{Sing} R = \operatorname{Spec} R \setminus \operatorname{Reg} R$.

Definition 1. Let S be a subcategory of $\mathsf{mod}\,R$.

- Denote by $\langle \mathsf{S} \rangle$ the smallest subcategory of $\mathsf{mod}\,R$ that contains $\mathsf{S} \cup \{0\}$ and is closed under extensions and direct summands.
- Denote by $\mathsf{cosyz}\,\mathsf{S}$ the subcategory whose objects are modules X such that there exists an exact sequence $0 \to S \to P \to X \to 0$ where S is in S and P is finitely generated and projective.
- Set $\mathsf{tier}_{-1}\mathsf{S} = \langle \mathsf{S} \rangle$, $\mathsf{tier}_0\mathsf{S} = \langle \mathsf{S} \cup \mathsf{cosyz}\,\langle \mathsf{S} \rangle \rangle$ and for $n \in \mathbb{N}$ set

$$\mathsf{tier}_n\mathsf{S} = \langle \mathsf{tier}_{n-1}\mathsf{S} \cup \mathsf{cosyz}\,(\mathsf{tier}_{n-1}\mathsf{S}) \rangle.$$

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Let $\mathsf{S}(R)$ be the subcategory of $\mathsf{mod}\,R$ with skeleton $\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R\}$; we consider the question of which, if any, of the subcategories in the chain

$$\langle \mathsf{S}(R) \rangle \;=\; \mathsf{tier}_{-1}\mathsf{S}(R) \;\subseteq \cdots \subseteq\; \mathsf{tier}_n\mathsf{S}(R) \;\subseteq\; \mathsf{tier}_{n+1}\mathsf{S}(R) \;\subseteq \cdots$$

is the entire module category $\mathsf{mod}\,R$.

In terms of of tiers, Schoutens's result [3, Theorem VI.8] can be stated as follows. If R has finite Krull dimension d, then one has $\mathsf{tier}_d\mathsf{S}(R) = \mathsf{mod}\,R$, and if R is local and singular, then one has $\mathsf{tier}_{d-1}\mathsf{S}(R) = \mathsf{mod}\,R$. For regular rings, Schoutens's bound is the best possible. Our theorem below sharpens the bound for singular rings: We replace d (in the local case $d-1$) by $c = \operatorname{codim}(\operatorname{Sing} R)$, the *codimension* of the singular locus, which is -1 if $\operatorname{Reg} R$ is empty and otherwise given by

$$c = \sup\{\operatorname{ht}_R \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Reg} R\}.$$

Theorem 2. *Let R be a commutative noetherian ring and set*

$$\mathsf{S}(R) = \{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R\}.$$

If $c = \operatorname{codim}(\operatorname{Sing} R)$ is finite, then there is an equality $\mathsf{tier}_c\mathsf{S}(R) = \mathsf{mod}\,R$.

Proof. As every R-module has a prime filtration and tiers are closed under extensions, it is sufficient to prove that every cyclic module $R/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in R, is in $\mathsf{tier}_c\mathsf{S}(R)$. For a prime ideal $\mathfrak{p} \in \operatorname{Reg} R$, set

$$n(\mathfrak{p}) = \max\{\dim(\mathfrak{q}/\mathfrak{p}) \mid \mathfrak{p} \subseteq \mathfrak{q} \text{ and } \mathfrak{q} \text{ is minimal in } \operatorname{Sing} R\}.$$

For $\mathfrak{p} \in \operatorname{Sing} R$, set $n(\mathfrak{p}) = 0$; we proceed by induction on $n(\mathfrak{p})$. By definition, $R/\mathfrak{p}$ is in $\mathsf{S}(R)$ and, therefore, in $\mathsf{tier}_c\mathsf{S}(R)$ if $n(\mathfrak{p})$ is 0. Let $n \geq 1$ and assume that $R/\mathfrak{p}$ is in $\mathsf{tier}_c\mathsf{S}(R)$ for all $\mathfrak{p}$ with $n(\mathfrak{p}) < n$. Fix a prime ideal $\mathfrak{p}$ with $n(\mathfrak{p}) = n$ and set $h = \operatorname{ht}_R \mathfrak{p}$. Since $R_\mathfrak{p}$ is regular, one can choose elements $x_1, \dots, x_h$ in $\mathfrak{p}$ such that the ideal $I = (x_1, \dots, x_h)$ has height h and the equality

$$IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p} \tag{1}$$

holds. As $\mathfrak{p}/I$ is a minimal prime ideal in R/I, there exists an element $a \in R$ with $\mathfrak{p} = (I \colon a)$, and it follows from (1) that a is not in $\mathfrak{p}$. It is now elementary to verify the equality $I = (I + (a)) \cap \mathfrak{p}$, which yields a Mayer–Vietoris exact sequence

$$0 \longrightarrow R/I \longrightarrow R/\mathfrak{p} \oplus R/(I + (a)) \longrightarrow R/(\mathfrak{p} + (a)) \longrightarrow 0. \tag{2}$$

The support of the module $R/(\mathfrak{p} + (a))$ consists of prime ideals that strictly contain $\mathfrak{p}$. Thus, $R/(\mathfrak{p} + (a))$ has a prime filtration with subquotients of the form $R/\mathfrak{q}$, where each $\mathfrak{q}$ satisfies the inequality $n(\mathfrak{q}) < n(\mathfrak{p})$. By the induction hypothesis, these subquotients $R/\mathfrak{q}$ are in $\mathsf{tier}_c\mathsf{S}(R)$ and hence so is $R/(\mathfrak{p} + (a))$.

By (2) it now suffices to show that R/I is in $\mathsf{tier}_c\mathsf{S}(R)$. To this end, consider the Koszul complex $K = K(x_1, \dots, x_h)$ on the generators of I. For $\mathfrak{q} \in \operatorname{Reg} R$, the non-units among the elements $x_1/1, \dots, x_h/1$ in $R_\mathfrak{q}$ form a regular sequence. It follows that the homology modules $\mathrm{H}_i(K)$ for $i > 0$ have support in $\operatorname{Sing} R$, see [2, Theorem 16.5], and therefore that they are in $\mathsf{tier}_{-1}\mathsf{S}(R)$. Let $d_1, \dots, d_h$ denote the differential maps on K. The modules K_i in the Koszul complex are free, and the module $\operatorname{Ker} d_h = \mathrm{H}_h(K)$ is in $\mathsf{tier}_{-1}\mathsf{S}(R)$. It now follows from the exact sequences

$$0 \longrightarrow \operatorname{Im} d_{i+1} \longrightarrow \operatorname{Ker} d_i \longrightarrow \mathrm{H}_i(K) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ker} d_i \longrightarrow K_i \longrightarrow \operatorname{Im} d_i \longrightarrow 0$$

that $\operatorname{Im} d_i$ is in $\mathsf{tier}_{h-i}\mathsf{S}(R)$ for $h \geq i \geq 1$. In particular, the ideal $I = \operatorname{Im} d_1$ is in $\mathsf{tier}_{h-1}\mathsf{S}(R)$. Thus the cosyzygy R/I is in $\mathsf{tier}_h\mathsf{S}(R)$ and clearly one has $h \leq c$. $\square$

The proof above is quite close to Schoutens's original argument.

Remark 3. One cannot leave out of any of the three procedures—adding cosyzygies, closing up under extensions, or closing up under summands—from the definition of tiers and still generate the entire module category. For the sake of the argument, let R be an isolated curve singularity, i.e. a one-dimensional Cohen–Macaulay local ring R with $\mathsf{S}(R) = \{k\}$, where k is the residue field of R.

- Without adding cosyzygies, one does not move beyond the category $\langle \mathsf{S}(R) \rangle$, which contains only the R-modules of finite length and hence not R.
- The R-module k is simple and cannot be embedded in a free R-module. Furthermore, R is indecomposable as an R-module. It follows that by adding cosyzygies and closing up under summands one only gets k and modules of projective dimension at most 1. Thus, extensions are needed.
- Summands cannot be dispensed with either. The closure E of $\mathsf{S}(R)$ under extensions is the subcategory of modules of finite length. Since no such module can be embedded in a free R-module, $\mathsf{cosyz}\,\mathsf{E}$ contains exactly the finitely generated free modules. If the closure under extensions of $\mathsf{E} \cup \mathsf{cosyz}\,\mathsf{E}$ is the entire module category $\mathsf{mod}\,R$—or if $\mathsf{mod}\,R$ can be attained by alternately closing up under extensions and taking syzygies a finite number of times—then the Grothendieck group of R is generated by k and R. However for any even integer $n \geq 4$, the Grothendieck group of the D_n singularity, $\mathbb{C}[\![x,y]\!]/(x^2y + y^{n-1})$, requires three generators; see [4, Lemma (13.2) and Proposition (13.10)].

2. The codimension of $\operatorname{Sing} R$ is the best possible bound

We now show that the bound provided by Theorem 2 is optimal; that is, $\mathsf{tier}_n\mathsf{S}(R)$ for $n < c$ is a proper subcategory of $\mathsf{mod}\,R$. First note that if R is regular, then $\operatorname{Sing} R$ and hence $\mathsf{S}(R)$ is empty. Thus $\mathsf{tier}_{-1}\mathsf{S}(R)$ contains only the zero module, and it follows from the definition that $\mathsf{tier}_n\mathsf{S}(R)$ for $n \geq 0$ contains precisely the modules of projective dimension at most n. The next lemma shows that, to some extent, this simple observation carries over to general rings.

Lemma 4. *For a finitely generated R-module M the following assertions hold.*

(a) *M is in $\mathsf{tier}_{-1}\mathsf{S}(R)$ if and only if one has $M_\mathfrak{p} = 0$ for every $\mathfrak{p} \in \operatorname{Reg} R$.*

(b) *If M is in $\mathsf{tier}_n\mathsf{S}(R)$, then $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq n$ holds for every $\mathfrak{p} \in \operatorname{Reg} R$.*

Proof. As $\operatorname{Sing} R$ is a specialization closed subset of $\operatorname{Spec} R$, one has $(R/\mathfrak{q})_\mathfrak{p} = 0$ for every $\mathfrak{q} \in \operatorname{Sing} R$ and every $\mathfrak{p} \in \operatorname{Reg} R$. It follows that $M_\mathfrak{p}$ is 0 for every $M \in \mathsf{tier}_{-1}\mathsf{S}(R)$ and every $\mathfrak{p} \in \operatorname{Reg} R$. Conversely, if one has $M_\mathfrak{p} = 0$ for every $\mathfrak{p} \in \operatorname{Reg} R$, then M has a prime filtration with subquotients $R/\mathfrak{q}$ in $\mathsf{S}(R)$, so M is in $\mathsf{tier}_{-1}\mathsf{S}(R)$. This proves part (a).

(b): Assume that X is in $\mathsf{cosyz}\,(\mathsf{tier}_{-1}\mathsf{S}(R))$, then there is exact sequence

$$0 \longrightarrow S \longrightarrow P \longrightarrow X \longrightarrow 0,$$

where P is a finitely generated projective module and S is in $\mathsf{tier}_{-1}\mathsf{S}(R)$. It follows that X is free at every $\mathfrak{p} \in \operatorname{Reg} R$, and hence so are all modules in $\mathsf{tier}_0\mathsf{S}(R)$.

Let $n \geq 1$ and assume that the inequality $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} \leq n-1$ holds for all modules X in $\mathsf{tier}_{n-1}\mathsf{S}(R)$ and for every $\mathfrak{p} \in \operatorname{Reg} R$. It follows that every module in $\mathsf{cosyz}\,(\mathsf{tier}_{n-1}\mathsf{S}(R))$ has projective dimension at most n at every $\mathfrak{p} \in \operatorname{Reg} R$, and hence the desired inequality holds for all modules in $\mathsf{tier}_n\mathsf{S}(R)$. $\square$

Up to $\mathsf{tier}_c\mathsf{S}(R)$ each tier strictly contains the previous one.

Proposition 5. *If $c = \operatorname{codim}(\operatorname{Sing} R)$ is finite, then there are strict inclusions*

$$\mathsf{tier}_{-1}\mathsf{S}(R) \subset \mathsf{tier}_0\mathsf{S}(R) \subset \cdots \subset \mathsf{tier}_{c-1}\mathsf{S}(R) \subset \mathsf{tier}_c\mathsf{S}(R)$$

of subcategories of $\mathsf{mod}\, R$.

Proof. Let S be any subcategory of $\mathsf{mod}\, R$; if one has $\mathsf{tier}_n\mathsf{S} = \mathsf{tier}_{n+1}\mathsf{S}$ for some $n \geq -1$, then it follows from the definition that $\mathsf{tier}_n\mathsf{S}$ equals $\mathsf{tier}_m\mathsf{S}$ for all $m \geq n$.

Thus, it is sufficient to show that $\mathsf{tier}_{c-1}\mathsf{S}(R)$ is not the entire category $\mathsf{mod}\, R$. To this end choose a prime ideal $\mathfrak{p}$ in $\operatorname{Reg} R$ of height c. By the Auslander–Buchsbaum Equality one has $\operatorname{pd}_{R_\mathfrak{p}}(R/\mathfrak{p})_\mathfrak{p} = c$, so it follows from Lemma 4 that $R/\mathfrak{p}$ does not belong to $\mathsf{tier}_{c-1}\mathsf{S}(R)$. □

Our proof of Theorem 2 only shows that every finitely generated R-module is in $\mathsf{tier}_c\mathsf{S}(R)$; it gives no information on the least tier to which a given module M belongs, but Lemma 4 provides a lower bound, namely $\sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Reg} R\}$.

Recall that a module $M \in \mathsf{mod}\, R$ is called is *maximal Cohen–Macaulay* if the equality $\operatorname{depth}_R M = \dim R$ holds. Such a module M is free on the regular locus; indeed, the Auslander–Buchsbaum Equality yields $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq 0$ for all $\mathfrak{p}$ in $\operatorname{Reg} R$. We show in the next section that over certain Cohen–Macaualay local rings R there are maximal Cohen–Macaulay modules which are not in $\mathsf{tier}_0\mathsf{S}(R)$. Thus, the lower bound provided by Lemma 4 is not sharp, and we ask the question:

Question 6. Let R be a Cohen–Macaulay local ring and denote by $\mathsf{CM}(R)$ the subcategory of $\mathsf{mod}\, R$ consisting of all maximal Cohen–Macaulay modules. What is the following number?

$$\varepsilon(R) = \min\{n \geq -1 \mid \mathsf{CM}(R) \subseteq \mathsf{tier}_n\mathsf{S}(R)\}.$$

If R is a regular local ring, then $\varepsilon(R)$ is 0 and we show in the next section that it may be as big as $c = \operatorname{codim}(\operatorname{Sing} R)$ for a singular ring. A broader question is, of course, given a module, how can one determine the least tier it belongs to?

3. Isolated singularities

A local ring R is Cohen–Macaulay if R is a maximal Cohen–Macaulay R-module, and R is said to have an *isolated singularity* if R is singular but $R_\mathfrak{p}$ is regular for every non-maximal prime ideal in R. In this section we give a description of the subcategories $\mathsf{tier}_n\mathsf{S}(R)$ for a local ring R with an isolated singularity; one that is explicit enough to answer Question 6 for a Cohen–Macaualy local ring with an isolated singularity.

For a subcategory S of $\mathsf{mod}\, R$, every module in $\langle \mathsf{S} \rangle$ can be reached by alternately taking summands and extensions; to discuss this we recall some notation from [1].

Definition 7. Let S and T be subcategories of $\mathsf{mod}\, R$.

(1) Denote by $\mathsf{add}\,\mathsf{S}$ the additive closure of S, that is, the smallest subcategory of $\mathsf{mod}\, R$ containing S and closed under finite direct sums and direct summands.
(2) Denote by $\mathsf{S} \circ \mathsf{T}$ the subcategory of $\mathsf{mod}\, R$ consisting of the R-modules M that fit into an exact sequence $0 \to S \to M \to T \to 0$ with $S \in \mathsf{S}$ and $T \in \mathsf{T}$.

(3) Set $\mathsf{S} \bullet \mathsf{T} = \mathsf{add}\,(\mathsf{add}\,\mathsf{S} \circ \mathsf{add}\,\mathsf{T})$, and for integers $m \geq 1$, set

$$|\mathsf{S}|_m = \begin{cases} \mathsf{add}\,\mathsf{S} & \text{for } m = 1, \\ |\mathsf{S}|_{m-1} \bullet \mathsf{S} & \text{for } m \geq 2. \end{cases}$$

Remark 8. Let S and T be subcategories of $\mathsf{mod}\,R$. A module M in $\mathsf{mod}\,R$ belongs to $\mathsf{S} \bullet \mathsf{T}$ if and only if there is an exact sequence $0 \to S \to E \to T \to 0$ with $S \in \mathsf{add}\,\mathsf{S}$ and $T \in \mathsf{add}\,\mathsf{T}$ such that M is a direct summand of E. Moreover, one has $|\mathsf{S}|_m \bullet |\mathsf{S}|_{m'} = |\mathsf{S}|_{m+m'}$ for all $m, m' \geq 1$; see [1].

Lemma 9. *For every subcategory* S *of* $\mathsf{mod}\,R$ *one has* $\langle \mathsf{S} \rangle = \bigcup_{m \geq 1} |\mathsf{S}|_m$.

Proof. Set $\mathsf{T} = \bigcup_{m \geq 1} |\mathsf{S}|_m$. Evidently one has $\mathsf{S} \subseteq \mathsf{T} \subseteq \langle \mathsf{S} \rangle$, and T is by construction closed under direct summands. Let

$$0 \longrightarrow T \longrightarrow E \longrightarrow T' \longrightarrow 0$$

be an exact sequence in $\mathsf{mod}\,R$ with T and T' in T. There are integers $m, m' \geq 1$ with $T \in |\mathsf{S}|_m$ and $T' \in |\mathsf{S}|_{m'}$, and hence E is in $|\mathsf{S}|_{m+m'}$. Thus, T is also closed under extensions, and by the definition of $\langle S \rangle$ it follows that one has $T = \langle S \rangle$. $\square$

Let R be a local ring with residue field k. Denote by $\mathsf{fln}(R)$ the subcategory of $\mathsf{mod}\,R$ whose objects are all modules of finite length. For $n \geq -1$ denote by $\mathsf{fpd}_n(R)$ the subcategory of $\mathsf{mod}\,R$ whose objects are all modules of projective dimension at most n. Note that one has $\mathsf{fln}(R) = \langle \{k\} \rangle$ and $\mathsf{fpd}_{-1}(R) = \{0\}$.

Theorem 10. *Let R be a local ring with residue field k. For $-1 \leq n \leq \operatorname{depth} R - 1$ there are equalities of subcategories of $\mathsf{mod}\,R$,*

$$\mathsf{tier}_n\{k\} = \langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R) \rangle = \langle \{k\} \cup \mathsf{fpd}_n(R) \rangle,$$

and for $-1 \leq n \leq \operatorname{depth} R - 2$ the category $\mathsf{tier}_n\{k\}$ contains precisely the modules M such that there is an exact sequence

$$0 \longrightarrow L \longrightarrow M \oplus M' \longrightarrow P \longrightarrow 0$$

in $\mathsf{mod}\,R$ with $L \in \mathsf{fln}(R)$ and $P \in \mathsf{fpd}_n(R)$.

Proof. First we show that every module in $\langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R) \rangle$ for $-1 \leq n \leq \operatorname{depth} R - 2$ fits in an exact sequence $0 \to L \to M \oplus M' \to P \to 0$ with $L \in \mathsf{fln}(R)$ and $P \in \mathsf{fpd}_n(R)$. The assertion is trivial for $n = -1$, so let $0 \leq n \leq \operatorname{depth} R - 2$. Fix a module M in $\langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R) \rangle$; by Lemma 9 it belongs to $|\,\mathsf{fln}(R) \cup \mathsf{fpd}_n(R)|_m$ for some $m \geq 1$. We now argue by induction on m that M fits in an exact sequence of the prescribed form.

For $m = 1$ one has $M \in \mathsf{add}\,(\mathsf{fln}(R) \cup \mathsf{fpd}_n(R))$, whence there is an isomorphism $M \oplus M' \cong L \oplus P$ for modules $M' \in \mathsf{mod}\,R$, $L \in \mathsf{fln}(R)$, and $P \in \mathsf{fpd}_n(R)$.

For $m \geq 2$ there is an exact sequence

$$0 \longrightarrow X \longrightarrow M \oplus M' \longrightarrow Y \longrightarrow 0 \tag{1}$$

in $\mathsf{mod}\,R$ with $X \in |\,\mathsf{fln}(R) \cup \mathsf{fpd}_n(R)|_{m-1}$ and $Y \in \mathsf{add}\,(\mathsf{fln}(R) \cup \mathsf{fpd}_n(R))$. The base and hypothesis of induction yield an isomorphism $Y \oplus Y' \cong L \oplus P$ and an exact sequence $0 \to L' \to X \oplus X' \to P' \to 0$, with L and L' in $\mathsf{fln}(R)$ and with P and P' in $\mathsf{fpd}_n(R)$. Combined with (1) they yield an exact sequence

$$0 \longrightarrow X \oplus X' \longrightarrow X' \oplus M \oplus M' \oplus Y' \longrightarrow L \oplus P \longrightarrow 0.$$

Set $V = X' \oplus M \oplus M' \oplus Y'$. Consider the pushout diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L' \arrow[r, equal] \arrow[d] & L' \arrow[d] & & \\
0 \arrow[r] & X \oplus X' \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & L \oplus P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & L \oplus P \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
\tag{2}
$$

and the pullback diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P' \arrow[r] \arrow[d, equal] & P'' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] & W \arrow[r] \arrow[d] & L \oplus P \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{3}
$$

Note from the top row in (3) that the module P'' is in $\mathsf{fpd}_n(R)$. From the inequality $n \leq \operatorname{depth} R - 2$ and the Auslander–Buchsbaum Equality one gets $\operatorname{depth}_R P'' \geq 2$. By the cohomological characterization of depth [2, Theorem 16.6] this implies $\operatorname{Ext}^1_R(k, P'') = 0$ and, therefore, $\operatorname{Ext}^1_R(L, P'') = 0$. Thus, the middle column in (3) is split exact, and the middle column in (2) becomes $0 \to L' \to V \to L \oplus P'' \to 0$. Consider the pullback diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L' \arrow[r] \arrow[d, equal] & L'' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L' \arrow[r] & V \arrow[r] \arrow[d] & L \oplus P'' \arrow[r] \arrow[d] & 0 \\
 & & P'' \arrow[r, equal] \arrow[d] & P'' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{4}
$$

Note from the top row that L'' is in $\mathsf{fln}(R)$. As M is a direct summand of V, the middle column is a desired exact sequence.

Clearly, one has $\langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R)\rangle = \langle \{k\} \cup \mathsf{fpd}_n(R)\rangle$; to finish the proof we show by induction that $\mathsf{tier}_n\{k\} = \langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R)\rangle$ holds for $-1 \le n \le \operatorname{depth} R - 1$. For $n = -1$, one has $\mathsf{tier}_n\{k\} = \mathsf{fln}(R) = \langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R)\rangle$. Let $n \ge 0$ and assume that $\mathsf{tier}_{n-1}\{k\} = \langle \mathsf{fln}(R) \cup \mathsf{fpd}_{n-1}(R)\rangle$ holds. By definition one then has

$$\mathsf{tier}_n\{k\} = \langle \langle \mathsf{fln}(R) \cup \mathsf{fpd}_{n-1}(R)\rangle \cup \mathsf{cosyz}\langle \mathsf{fln}(R) \cup \mathsf{fpd}_{n-1}(R)\rangle\rangle,$$

whence it suffices to establish the equality

$$\mathsf{cosyz}\langle \mathsf{fln}(R) \cup \mathsf{fpd}_{n-1}(R)\rangle = \mathsf{fpd}_n(R).$$

The inclusion "$\supseteq$" is clear because a module in $\mathsf{fpd}_n(R)$ is a cosyzygy of its first syzygy, which is in $\mathsf{fpd}_{n-1}(R)$. For the opposite inclusion, let M be a module in $\mathsf{cosyz}\langle \mathsf{fln}(R) \cup \mathsf{fpd}_{n-1}(R)\rangle$. There is an exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is free and N is in $\langle \mathsf{fln}(R) \cup \mathsf{fpd}_{n-1}(R)\rangle$. From the inequalities $-1 < n \le \operatorname{depth} R - 1$ follows that R and hence F has positive depth, whence also N has positive depth. Moreover, one has $-1 \le n-1 \le \operatorname{depth} R - 2$, so it follows from the first part of the proof that there is an exact sequence in $\mathsf{mod}\, R$,

$$0 \longrightarrow L \xrightarrow{\binom{\alpha}{\beta}} N \oplus N' \longrightarrow P \longrightarrow 0,$$

with $L \in \mathsf{fln}(R)$ and $P \in \mathsf{fpd}_{n-1}(R)$. Since L has finite length and N has positive depth, the map α is zero. Thus, there is an isomorphism $P \cong N \oplus C$, where C is the cokernel of β. Thus N belongs to $\mathsf{fpd}_{n-1}(R)$, and hence M is in $\mathsf{fpd}_n(R)$. □

For a local ring R with an isolated singularity one has $\mathsf{S}(R) = \{k\}$, so Theorems 2 and 10 combine to yield:

Corollary 11. *Let R be a d-dimensional local ring with an isolated singularity. For every $n \ge -1$ one has*

$$\mathsf{tier}_n\mathsf{S}(R) = \langle \{k\} \cup \mathsf{fpd}_n(R)\rangle;$$

in particular, one has

$$\mathsf{mod}\, R = \langle \{k\} \cup \mathsf{fpd}_{d-1}(R)\rangle. \qquad \square$$

To answer Question 6 for a Cohen–Macaualy local ring with an isolated singularity, we record another consequence of Theorem 10. Denote by $\mathsf{dep}(R)$ the subcategory of $\mathsf{mod}\, R$ whose objects are all modules of positive depth; it includes the zero module as it has infinite depth by convention.

Proposition 12. *Let R be a local ring. For $-1 \le n \le \operatorname{depth} R - 2$ one has*

$$\mathsf{tier}_n\{k\} \cap \mathsf{dep}(R) = \mathsf{fpd}_n(R).$$

Proof. By Theorem 10 one has $\mathsf{tier}_n\{k\} = \langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R)\rangle$ and it follows from the inequality $n \le \operatorname{depth} R - 2$ and the Auslander–Buchsbaum Equality that every module in $\mathsf{fpd}_n(R)$ has positive depth. This proves the inclusion "$\supseteq$". To show the opposite inclusion, fix a module M in $\langle \mathsf{fln}(R) \cup \mathsf{fpd}_n(R)\rangle \cap \mathsf{dep}(R)$. It follows from Theorem 10 that there is an exact sequence in $\mathsf{mod}\, R$

$$0 \longrightarrow L \xrightarrow{\binom{\alpha}{\beta}} M \oplus M' \longrightarrow P \longrightarrow 0$$

with $L \in \mathsf{fln}(R)$ and $P \in \mathsf{fpd}_n(R)$. Since M has positive depth, the map α is zero and it follows that M is a direct summand of P, whence M is in $\mathsf{fpd}_n(R)$. $\square$

Corollary 13. *Let R be a d-dimensional Cohen–Macaulay local ring.*
(a) *If $d \geq 1$, then every maximal Cohen–Macaulay module in $\mathsf{tier}_{d-2}\{k\}$ is free.*
(b) *If R has an isolated singularity, then one has*

$$\varepsilon(R) = d - 1.$$

Proof. (a) By Proposition 12 one has

$$\mathsf{CM}(R) \cap \mathsf{tier}_{d-2}\{k\} = \mathsf{CM}(R) \cap (\mathsf{tier}_{d-2}\{k\} \cap \mathsf{dep}(R)) = \mathsf{CM}(R) \cap \mathsf{fpd}_{d-2}(R).$$

In $\mathsf{CM}(R) \cap \mathsf{fpd}_{d-2}(R)$ is only 0 if $d = 1$ and precisely the free R-modules if $d \geq 2$.

(b) The equality is trivial for $d = 0$ and it follows from (a) for $d \geq 1$. $\square$

The corollary shows that the lower bound that Lemma 4 gives for which tier a module M can belong to, $\sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Reg} R\}$, is far from being sharp.

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