ACYCLIC COMPLEXES AND REGULAR RINGS

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Abstract. A 2009 paper by Iacob and Iyengar characterizes noetherian regular rings in terms of properties of complexes of projective modules, flat modules, and injective modules. We show that the relevant properties of such complexes are equivalent without reference to regularity of the ring and that they characterize coherent regular rings and von Neumann regular rings.

Introduction

In this short paper R denotes an associative unital ring. An R-module is a left R-module, and right R-modules are considered modules over the opposite ring R°.

Following Bertin [2] and Glaz [9] we say that R is left/right *regular* if every finitely generated left/right ideal in R has finite projective dimension. We note that this definition is broader than the one used by Iacob and Iyengar [10], which implicitly includes the assumption that R is left/right noetherian.

As a corollary to our first main result, Theorem 2.1, we obtain the following:

Theorem 0. *Let R be right coherent. The following conditions are equivalent.*

(i) *R is right regular.*

(ii) *Every complex of finitely generated free R-modules is semi-projective.*

(iii) *Every complex of projective R-modules is semi-projective.*

(iv) *Every complex of injective R°-modules is semi-injective.*

(v) *Every complex of flat R-modules is semi-flat.*

(vi) *Every acyclic complex of projective R-modules is contractible.*

(vii) *Every acyclic complex of injective R°-modules is contractible.*

($viii$) *Every acyclic complex of flat R-modules is pure acyclic.*

In fact, Theorem 2.1 shows that any right regular ring satisfies the conditions above.

The equivalence of the conditions in Theorem 0 was proved for commutative noetherian rings by Christensen, Foxby, and Holm [5]. Beyond that realm it applies, for example, to von Neumann regular rings; see Example 2.5.

The equivalence of some of the conditions in Theorem 0 was proved already in [10], see Remark 1.7. Recently the equivalence of (i) and (iii)–($viii$) was proved by Gillespie and Iacob [8]; see Remark 4.4. Our proofs do not rely on these works.

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In Sections 3 and 4 we characterize right regularity of a right coherent ring R in terms of properties of complexes of flat-cotorsion R-modules and complexes of fp-injective R°-modules. One upshot is that the equivalence of conditions (*ii*)–(*viii*) in Theorem 0 can be established without invoking (*i*), the right regularity of the ring; this is part of our second main result, Theorem 4.2.

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1. Complexes of projective, injective, and flat modules

We start by recalling some facts about complexes of projective and flat modules. For convenience we give references to [5], which the reader may also consult for any unexplained notation or terminology.

1.1. Recall from [5, Prop. 5.2.10] that a complex P of projective R-modules is semi-projective if and only if $\operatorname{Hom}_R(P, A)$ is acyclic for every acyclic R-complex A.

For every R-complex X there is an exact sequence of R-complexes

$$0 \longrightarrow A \longrightarrow P \longrightarrow X \longrightarrow 0 \tag{1.1.1}$$

where P is semi-projective and A is acyclic.

(a) For an acyclic complex P of projective R-modules the next conditions are equivalent. (See [5, Prop. 4.3.29 and Cor. 5.5.26].)
   - (*i*) P is semi-projective.
   - (*ii*) P is contractible.
   - (*iii*) P is pure acyclic.

(b) If in an exact sequence $0 \to P' \to P \to P'' \to 0$ of complexes of projective R-modules two of the complexes are semi-projective, then so is the third.

1.2. Recall from [5, Prop. 5.4.9] that a complex F of flat R-modules is semi-flat if and only if $A \otimes_R F$ is acyclic for every acyclic R°-complex A.

(a) An acyclic complex of flat R-modules is pure acyclic if and only if it is semi-flat. (See [5, Thm. 5.5.22].)

(b) If in an exact sequence $0 \to F' \to F \to F'' \to 0$ of complexes of flat R-modules two of the complexes are semi-flat, then so is the third.

(c) A complex of projective R-modules is semi-flat if and only if it is semi-projective. (See [5, Cor. 5.4.10 and Thm. 5.5.27].)

1.3 Proposition. *The following conditions are equivalent.*

(P0) *Every complex of finitely generated free R-modules is semi-projective.*

(P1) *Every complex of projective R-modules is semi-projective.*

(P2) *Every acyclic complex of projective R-modules is contractible.*

(P3) *Every acyclic complex of projective R-modules is semi-projective.*

(F1) *Every complex of flat R-modules is semi-flat.*

(F2) *Every acyclic complex of flat R-modules is pure acyclic.*

(F3) *Every acyclic complex of flat R-modules is semi-flat.*

Proof. First we argue that conditions (P0)–(P3) are equivalent. In view of 1.1(a) and the fact that free modules are projective, the following implications are clear:

$$\text{(P0)} \impliedby \text{(P1)} \implies \text{(P2)} \implies \text{(P3)}\,,$$

which leaves two implications to prove.

(P3) $\implies$ (P1): Let X be a complex of projective R-modules and consider the exact sequence $0 \to A \to P \to X \to 0$ from (1.1.1). It follows that A is a complex of projective R-modules and, thus, semi-projective, so X is semi-projective by 1.1(b).

(P0) $\implies$ (P1): Let $\mathcal{S}$ be a set of representatives for the isomorphism classes of bounded above complexes of finitely generated free R-modules; it generates a cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), S^{\perp})$ in the category of R-complexes. By work of Bravo, Gillespie, and Hovey [3, Thm. A.3] the left-hand class ${}^{\perp}(\mathcal{S}^{\perp})$ consists of all complexes of projective R-modules. Thus, if every complex in $\mathcal{S}$ is semi-projective, then every acyclic R-complex belongs to $\mathcal{S}^{\perp}$, whence every complex of projective R-modules is semi-projective.

Condition (F1) evidently implies (F3). By 1.2(a) conditions (F2) and (F3) are equivalent, and by 1.1(a) they imply (P2). This leaves one implication to prove.

(P1) $\implies$ (F1): Let F be a complex of flat R-modules; by work of Neeman [11, Thm. 8.6] there is an exact sequence

$$0 \longrightarrow A \longrightarrow P \longrightarrow F \longrightarrow 0$$

where P is a complex of projective R-modules and A is a pure acyclic complex of flat R-modules. By assumption P is semi-projective and hence semi-flat, see 1.2(c). The complex A is semi-flat by 1.2(a), so F is semi-flat by 1.2(b). □

1.4. Recall from [5, Prop. 5.3.16] that a complex I of injective R°-modules is semi-injective if and only if $\operatorname{Hom}_{R^{\circ}}(A, I)$ is acyclic for every acyclic R°-complex A.

For every R°-complex X there is an exact sequence of R°-complexes

$$0 \longrightarrow X \longrightarrow I \longrightarrow A \longrightarrow 0 \tag{1.4.1}$$

where I is semi-injective and A is acyclic.

(a) For an acyclic complex I of injective R°-modules the following conditions are equivalent. (See [5, Prop. 4.3.29] and Bazzoni, Cortés-Izurdiaga, and Estrada [1, Props. 2.4(1) and 4.8(1)].)
   - (*i*) I is semi-injective.
   - (*ii*) I is contractible.
   - (*iii*) I is pure acyclic.

(b) If in an exact sequence $0 \to I' \to I \to I'' \to 0$ of complexes of injective R°-modules, two of the complexes are semi-injective, then so is the third.

1.5 Proposition. *The following conditions are equivalent.*

- (I1) *Every complex of injective R°-modules is semi-injective.*
- (I2) *Every acyclic complex of injective R°-modules is contractible.*
- (I3) *Every acyclic complex of injective R°-modules is semi-injective.*

Proof. By 1.4(a) conditions (I2) and (I3) are equivalent, and (I1) evidently implies (I3). For the converse let X be a complex of injective R°-modules and consider the exact sequence $0 \to X \to I \to A \to 0$ from (1.4.1). It follows that A is a complex of injective R°-modules and, therefore, semi-injective, so X is semi-injective by 1.4(b). □

1.6 Remark. The conditions (I1)–(I3) imply the conditions from Proposition 1.3. To see this it suffices to verify that (I3) implies (F2): Let F be an acyclic complex of flat R-modules. The character complex $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is an acyclic complex of injective R°-modules and, therefore, contractible, whence F is pure acyclic.

1.7 Remark. The equivalence of some of the conditions above are known from the literature. Conditions (P1), (P2), (F1), and (F2) are equivalent by [10, Props. 3.1, 3.3, and 3.4]. Conditions (I1) and (I2) are equivalent by [10, Prop. 2.1].

Under the assumption that R is commutative and noetherian, the equivalence of (P0) and (P1) was proved in [5, Thm. 20.2.12]. Tereshkin [13] has informed us that he knows of the equivalence for any ring, as proved in Proposition 1.3, from private communication with Positselski. The argument he implied is different from the one we give here.

2. Regular rings

2.1 Theorem. *Consider the following conditions on R.*

- (i) *R is right regular.*
- (ii) *Every finitely generated right ideal in R has finite flat dimension.*
- (iii) *One/all of conditions* (I1)–(I3) *from* 1.5 *hold.*
- (iv) *One/all of conditions* (P0)–(P3) *and* (F1)–(F3) *from* 1.3 *hold.*
- (v) *Every R°-module with a degreewise finitely generated projective resolution has finite projective dimension.*

The following implications hold,

$$(i) \implies (ii) \implies (iii) \implies (iv) \implies (v)\,,$$

and if R is right coherent then (v) implies (i), i.e. the conditions are equivalent.

Proof. Evidently (i) implies (ii), and (iii) implies (iv) by Remark 1.6 .

$(ii) \Longrightarrow (iii)$: By Proposition 1.5 it suffices to show that R satisfies (I2). Let I be an acyclic complex of injective R°-modules and set $J = \prod_{n\in\mathbb{Z}} \Sigma^n I$. It is also an acyclic complex of injective R°-modules, and one has $\mathrm{Z}_n(J) \cong \mathrm{Z}_{n-1}(J)$ for every $n \in \mathbb{Z}$, so it follows from [1, Prop. 4.8(3)] that the cycle modules $\mathrm{Z}_n(J)$ are injective. Each cycle module $\mathrm{Z}_n(I)$ is a direct summand of $\mathrm{Z}_n(J)$ and hence injective, so I is contractible.

$(iv) \Longrightarrow (v)$: By Proposition 1.3 it suffices to show that (P0)–(P3) imply that every R°-module with a degreewise finitely generated projective resolution has finite projective dimension. Let $L \xrightarrow{\simeq} M$ be such a module and resolution. Further, let E be a faithfully injective R-module and $P \xrightarrow{\simeq} E$ a projective resolution. The complex $L^* = \mathrm{Hom}_{R^\circ}(L, R)$ of finitely generated free R-modules is by (P0) semi-projective; this explains the first and third isomorphisms in the computation below. Further, L^* is by [11, Prop. 7.12] a compact object in the homotopy category of projective R-modules; this explains the second isomorphism.

$$\begin{aligned}
\mathrm{H}_0(\mathrm{Hom}_R(L^*, \textstyle\coprod_{n\in\mathbb{Z}} \Sigma^n E)) &\cong \mathrm{H}_0(\mathrm{Hom}_R(L^*, \textstyle\coprod_{n\in\mathbb{Z}} \Sigma^n P)) \\
&\cong \textstyle\coprod_{n\in\mathbb{Z}} \mathrm{H}_0(\mathrm{Hom}_R(L^*, \Sigma^n P)) \\
&\cong \textstyle\coprod_{n\in\mathbb{Z}} \mathrm{H}_0(\mathrm{Hom}_R(L^*, \Sigma^n E)) \\
&\cong \textstyle\coprod_{n\in\mathbb{Z}} \mathrm{H}_n(\mathrm{Hom}_R(L^*, E)) \\
&\cong \textstyle\coprod_{n\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{H}_n(L^*), E)\,.
\end{aligned}$$

At the same time, the canonical embedding $\coprod_{n\in\mathbb{Z}} \Sigma^n E \to \prod_{n\in\mathbb{Z}} \Sigma^n E$ is an isomorphism, so one has

$$\begin{aligned} \mathrm{H}_0(\mathrm{Hom}_R(L^*, \textstyle\coprod_{n\in\mathbb{Z}} \Sigma^n E)) &\cong \mathrm{H}_0(\mathrm{Hom}_R(L^*, \textstyle\prod_{n\in\mathbb{Z}} \Sigma^n E)) \\ &\cong \textstyle\prod_{n\in\mathbb{Z}} \mathrm{H}_0(\mathrm{Hom}_R(L^*, \Sigma^n E)) \\ &\cong \textstyle\prod_{n\in\mathbb{Z}} \mathrm{H}_n(\mathrm{Hom}_R(L^*, E)) \\ &\cong \textstyle\prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{H}_n(L^*), E)\,. \end{aligned}$$

As the relevant Hom, homology, and shift functors preserve (co)products in the sense of [5, 3.1.8 and 3.1.20], it follows that the canonical embedding

$$\textstyle\coprod_{n\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{H}_n(L^*), E) \longrightarrow \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{H}_n(L^*), E)$$

is an isomorphism. Thus, $\mathrm{Hom}_R(\mathrm{H}_n(L^*), E)$ and, therefore, $\mathrm{H}_n(L^*)$ is non-zero for only finitely many $n \in \mathbb{Z}$. Thus, for $n \ll 0$ the complex $0 \to \mathrm{Z}_n(L^*) \to (L^*)_n \to \cdots$ is acyclic, so splicing it with a projective resolution of $\mathrm{Z}_n(L^*)$ yields an acyclic complex of projective R-modules. By (P3) it is contractible, so for $n \ll 0$ the R-module $\mathrm{Z}_n(L^*)$ is projective. As one has $L \cong \mathrm{Hom}_R(L^*, R)$ the R°-module $\mathrm{Z}_n(L)$ is projective for $n \gg 0$, so M has finite projective dimension.

$(v) \Longrightarrow (i)$: Assume that R is right coherent and let $\mathfrak{a} \subset R$ be a finitely generated right ideal. The quotient $R/\mathfrak{a}$ is a finitely presented R°-module, and every such module has a degreewise finitely generated projective resolution. It follows that $R/\mathfrak{a}$ and, therefore, $\mathfrak{a}$ has finite projective dimension. □

Without the coherence assumption the last implication proved above may fail.

2.2 Example. Let k be a field. The local ring $R = k[x_1, x_2, \ldots]/(x_1, x_2, \ldots)^2$ with maximal ideal $\mathfrak{m} = (x_1, x_2, \ldots)$ is not coherent; indeed the kernel $\mathfrak{m}$ of the canonical map $R \twoheadrightarrow (x_1)$ is not finitely generated. By [3, Prop. 2.5] the only R-modules that admit a degreewise finitely generated projective resolution are the finitely generated free R-modules. However, R is not a regular ring: The proof of [3, Prop. 2.5] shows that every non-free finitely presented R-module has projective dimension at least 2. Yet, the existence of any R-module of finite projective dimension at least 1 implies the existence of an injective homomorphism $\partial\colon P \hookrightarrow Q$ of free R-modules. As R is perfect, one can assume that the image of ∂ is contained in $\mathfrak{m}Q$, see [5, Thms. B.55 and B.60], which forces the existence of a non-free finitely presented R-module of projective dimension 1; a contradiction. Thus, every non-projective R-module has infinite projective dimension.

The example above suggests that coherence is the "minimal" condition on R that makes all five conditions in Theorem 2.1 equivalent. In the proof of Theorem 2.1, the argument for the implication $(iv) \Longrightarrow (v)$ relies on Neeman's [11, Prop. 7.12], which is also used in [10, 3.5]. Let us, therefore, record that this implication, under the coherence assumption, has a more elementary proof.

2.3 Remark. Let R be right coherent and assume that it satisfies Theorem 2.1(iv). Let M be a finitely presented R°-module, and N an R-module with a semi-flat resolution $F \xrightarrow{\simeq} N$. The canonical embedding $\Phi\colon \coprod_{n\in\mathbb{Z}} \Sigma^n F \to \prod_{n\in\mathbb{Z}} \Sigma^n F$ is a quasi-isomorphism since the homology of either complex equals N in each degree. The mapping cone of Φ is by (F2) pure acyclic, so $M \otimes_R \Phi$ is a quasi-isomorphism as well; this explains the third isomorphism below.

$$\begin{aligned}\textstyle\coprod_{n\in\mathbb{Z}} \operatorname{Tor}_n^R(M,N) &\cong \textstyle\coprod_{n\in\mathbb{Z}} \mathrm{H}_0(M\otimes_R \Sigma^n F)\\ &\cong \mathrm{H}_0(M\otimes_R \textstyle\coprod_{n\in\mathbb{Z}} \Sigma^n F)\\ &\cong \mathrm{H}_0(M\otimes_R \textstyle\prod_{n\in\mathbb{Z}} \Sigma^n F)\\ &\cong \textstyle\prod_{n\in\mathbb{Z}} \mathrm{H}_0(M\otimes_R \Sigma^n F) \cong \textstyle\prod_{n\in\mathbb{Z}} \operatorname{Tor}_n^R(M,N)\end{aligned}$$

As in the proof of Theorem 2.1 it follows that $\operatorname{Tor}_n^R(M,N)$ is nonzero for only finitely many n whence M, being finitely presented, has finite projective dimension. Thus R satisfies Theorem 2.1(v).

The proof of Theorem 2.1 suggests that the equivalence, for a right coherent ring, of all ten conditions from Propositions 1.3 and 1.5 "factors through" the right regularity property of the ring, but their equivalence can, in fact, be established without reference to this property; see Remark 3.3 and Theorem 4.2.

As an application of Theorem 2.1 we offer a short proof of a result already proved by Glaz [9, Thm. 6.2.5] in the commutative case.

2.4 Proposition. *Let $R \subseteq S$ be right coherent rings such that S is faithfully flat as an R°-module. If S is right regular, then R is right regular.*

Proof. Let F be an acyclic complex of flat R-modules. As S is right regular, the acyclic complex $S\otimes_R F$ of flat S-modules is pure acyclic by Theorem 2.1; this is condition (F2). The character complex

$$\operatorname{Hom}_{\mathbb{Z}}(S\otimes_R F, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_R(F, \operatorname{Hom}_{\mathbb{Z}}(S,\mathbb{Q}/\mathbb{Z}))$$

is contractible. By [5, Prop. 1.3.49] the R-module $\operatorname{Hom}_{\mathbb{Z}}(S,\mathbb{Q}/\mathbb{Z})$ is faithfully injective, so F is pure acyclic, whence R is right regular by Theorem 2.1. □

2.5 Example. Let R be von Neumann regular, that is, every R-module is flat. It follows that a product of flat R-modules is flat, whence R is right coherent. Flatness of every R-module also means that R satisfies condition (F2), whence R is right regular by Theorem 2.1. As von Neumann regularity is a left–right symmetric property, R is also left coherent and left regular.

A von Neumann regular ring is a special case of a right coherent ring of finite weak global dimension; any such ring evidently satisfies condition (F2) and is thus right regular. Finkel Jones and Teply [6] give examples of such rings. Glaz [9, Sect. 6.2] shows that the polynomial algebra in countably many variables over a field is a coherent regular ring of infinite weak global dimension.

2.6 Corollary. *The following conditions are equivalent.*

(i) *R is von Neumann regular.*
(ii) *Every complex of finitely presented R-modules is semi-projective.*
(iii) *Every R-complex is semi-flat.*
(iv) *Every acyclic R-complex is pure acyclic.*

Proof. Condition (iii) implies (iv) by 1.2(a).

(i)$\Longrightarrow$(ii): Every R-module is flat, so every finitely presented R-module is projective. Further, R is per Example 2.5 right regular, so by Theorem 2.1 every complex of projective R-modules is semi-projective.

$(ii) \Longrightarrow (iii)$: An R-complex is a filtered colimit of complexes of finitely presented R-modules, see [5, Prop. 3.3.21], i.e. a filtered colimit of semi-flat R-complexes, see 1.2(c), and hence semi-flat by [5, Prop. 5.4.13].

$(iv) \Longrightarrow (i)$: Let M be an R-module. There is a projective R-module P and an exact sequence $0 \to K \to P \to M \to 0$; as it is pure, M is flat. □

3. Complexes of flat-cotorsion modules

A cotorsion pair (X, Y) in the category $\mathsf{Mod}(R)$ of R-modules induces two cotorsion pairs $(\mathsf{X}\text{-}\mathsf{ac}, \mathsf{semi}\text{-}\mathsf{Y})$ and $(\mathsf{semi}\text{-}\mathsf{X}, \mathsf{Y}\text{-}\mathsf{ac})$ in the category of R-complexes; this was proved by Gillespie [7, Prop. 3.6]. Here the class $\mathsf{X}\text{-}\mathsf{ac}$ consists of acyclic complexes X with cycle modules $\mathrm{Z}_n(X)$ from X, while $\mathsf{semi}\text{-}\mathsf{Y}$ consists of complexes Y of modules from Y with the property that $\mathrm{Hom}_R(X, Y)$ is acyclic for every complex X in $\mathsf{X}\text{-}\mathsf{ac}$. Similarly $\mathsf{Y}\text{-}\mathsf{ac}$ consists of acyclic complexes Y with cycle modules $\mathrm{Z}_n(Y)$ from Y, and $\mathsf{semi}\text{-}\mathsf{X}$ consists of complexes X of modules from X with the property that $\mathrm{Hom}_R(X, Y)$ is acyclic for every complex Y from $\mathsf{Y}\text{-}\mathsf{ac}$. If the cotorsion pair (X, Y) is complete and hereditary, then the induced cotorsion pairs are both complete; see Yang and Liu [14, Thm. 3.5].

Let $\mathsf{Prj}(R)$, $\mathsf{Inj}(R)$, $\mathsf{Flat}(R)$, and $\mathsf{Cot}(R)$ be the classes of projective, injective, flat, and cotorsion R-modules. For each of the cotorsion pairs $(\mathsf{Prj}(R), \mathsf{Mod}(R))$ and $(\mathsf{Mod}(R), \mathsf{Inj}(R))$ only one of the induced cotorsion pairs in the category of R-complexes is of interest, and they yield the notions of semi-projective and semi-injective complexes. We proceed to recall the key properties of the cotorsion pairs induced by the complete hereditary cotorsion pair $(\mathsf{Flat}(R), \mathsf{Cot}(R))$ in $\mathsf{Mod}(R)$. A module in $\mathsf{Flat}(R) \cap \mathsf{Cot}(R)$ is called *flat-cotorsion.*

3.1 Lemma. *For every R-complex X there are exact sequences of R-complexes*

$$(3.1.1) \qquad 0 \longrightarrow C' \longrightarrow F \longrightarrow X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X \longrightarrow C \longrightarrow F' \longrightarrow 0\,.$$

Here F is semi-flat and C' is an acyclic complex of cotorsion modules, while C is a complex of cotorsion modules, and F' is acyclic and semi-flat.

(a) *An R-complex is semi-flat if and only if it belongs to* $\mathsf{semi}\text{-}\mathsf{Flat}(\mathsf{R})$, *and a complex in* $\mathsf{Flat}(\mathsf{R})\text{-}\mathsf{ac}$ *is a pure acyclic complex of flat R-modules.*

(b) *Every complex of cotorsion R-modules belongs to* $\mathsf{semi}\text{-}\mathsf{Cot}(\mathsf{R})$ *and every acyclic complex of cotorsion R-modules belongs to* $\mathsf{Cot}(\mathsf{R})\text{-}\mathsf{ac}$.

(c) *For an acyclic complex F of flat-cotorsion R-modules the next conditions are equivalent.*

  (i) *F is semi-flat.*

  (ii) *F is contractible.*

  (iii) *F is pure acyclic.*

Proof. An acyclic complex with flat cycle modules is a pure acyclic complex of flat R-modules, and such a complex is semi-flat by 1.2(a). The remaining assertions in parts (a) and (b) hold by [1, Thm. 1.3] and [4, Prop. 1.6]. It follows that the exact sequences in (3.1.1) are standard approximation sequences associated to the induced cotorsion pairs. In part (c) conditions (i) and (iii) are equivalent by 1.2(a). Evidently, (ii) implies (iii), and the converse follows from part (b). Indeed, in a pure acyclic complex of flat modules the cycle modules are flat and by (b) they are cotorsion as well. □

3.2 Proposition. *The following conditions are equivalent and equivalent to conditions* (P0)–(P3) *and* (F1)–(F3) *from* 1.3.

(FC1) *Every complex of flat-cotorsion R-modules is semi-flat.*

(FC2) *Every acyclic complex of flat-cotorsion R-modules is contractible.*

(FC3) *Every acyclic complex of flat-cotorsion R-modules is semi-flat.*

Proof. By Lemma 3.1(c) conditions (FC2) and (FC3) are equivalent; (FC1) clearly implies (FC3). For the converse let X be a complex of flat-cotorsion R-modules and consider the exact sequence $0 \to C' \to F \to X \to 0$ from (3.1.1). It follows that C' is a complex of flat-cotorsion modules and, hence, semi-flat, so X is semi-flat by 1.2(b). Thus, conditions (FC1)–(FC3) are equivalent. Evidently (F1) implies (FC1). For the converse let X be a complex of flat R-modules and consider the exact sequence $0 \to X \to C \to F' \to 0$ from (3.1.1). It follows that C is a complex of flat-cotorsion modules and, hence, semi-flat. Now X is semi-flat by 1.2(b). □

3.3 Remark. Assume that R is right coherent. For an acyclic complex I of injective R°-modules, the character complex $\operatorname{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is an acyclic complex of flat-cotorsion R-modules. If it is contractible, then I is pure acyclic and, therefore, contractible by 1.4(a). Thus (FC2) implies (I2), i.e. conditions (P0)–(P3), (F1)–(F3), (I1)–(I3), and (FC1)–(FC3) are per Remark 1.6 and Proposition 3.2 equivalent.

We can now add a condition to the characterization of von Neumann regular rings. Over such a ring every cotorsion module is injective, so (v) below can be seen as the counterpart to (I2) in the characterization of regular rings.

3.4 Corollary. *The next condition is equivalent to conditions* (i)–(iv) *from* 2.6.

(v) *Every acyclic complex of cotorsion R-modules is contractible.*

Proof. Recall the equivalent conditions (i)–(iv) from Corollary 2.6.

$(v) \Longrightarrow (iii)$: Let M be an R-complex. For every acyclic complex C of cotorsion R-modules the complex $\operatorname{Hom}_R(M, C)$ is acyclic, so M is semi-flat by Lemma 3.1(a).

$(i) \Longrightarrow (v)$: Every R-module is flat, and R is right coherent and right regular by Example 2.5, so every acyclic complex of cotorsion R-modules is a complex of flat-cotorsion modules and hence contractible by Theorem 2.1 and Proposition 3.2. □

4. Complexes of fp-injective modules

Recall that an R°-module E is fp-injective if $\operatorname{Ext}^1_{R^\circ}(F, E) = 0$ holds for every finitely presented R°-module F. The fp-injective R°-modules constitute the right-hand class of a cotorsion pair; the modules in the left-hand class are known as fp-projective. If R is right coherent, then this cotorsion pair, $(\mathsf{FpPrj}(R^\circ), \mathsf{FpInj}(R^\circ))$, is complete and hereditary and in many ways dual to $(\mathsf{Flat}(R), \mathsf{Cot}(R))$. In the next lemma we collect the key properties of the induced cotorsion pairs in the category of R-complexes. We refer to complexes in $\mathsf{semi\text{-}FpInj}(\mathsf{R}^\circ)$ as *semi-fp-injective* and to modules in $\mathsf{FpPrj}(R^\circ) \cap \mathsf{FpInj}(R^\circ)$ as *fp-pro-injective.*

4.1 Lemma. *Let R be right coherent. For every R°-complex X there are exact sequences of R°-complexes*

$$(4.1.1) \qquad 0 \longrightarrow X \longrightarrow E \longrightarrow P' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow E' \longrightarrow P \longrightarrow X \longrightarrow 0\,.$$

Here E is semi-fp-injective and P' is an acyclic complex of fp-projective modules, while P is a complex of fp-projective modules and E' is acyclic and semi-fp-injective.

(a) *Every complex of fp-projective R°-modules belongs to* $\mathsf{semi\text{-}FpPrj}(\mathsf{R}^\circ)$ *and every acyclic complex of fp-projective R°-modules is in* $\mathsf{FpPrj}(\mathsf{R}^\circ)\text{-}\mathsf{ac}$.

(b) *An acyclic complex of fp-injective R°-modules is pure acyclic if and only if it is semi-fp-injective.*

(c) *If in an exact sequence $0 \to E' \to E \to E'' \to 0$ of complexes of fp-injective R°-modules two of the complexes are semi-fp-injective, then so is the third.*

(d) *For an acyclic complex E of fp-pro-injective R°-modules the next conditions are equivalent.*

  (i) *E is semi-fp-injective.*

  (ii) *E is contractible.*

  (iii) *E is pure acyclic.*

Proof. It follows from parts (a) and (b) that the exact sequences in (4.1.1) are standard approximation sequences associated to the induced cotorsion pairs.

The assertions in part (a) were proved by Šaroch and Šťovíček [12, Example 4.3].

(b): Let E be an acyclic complex of fp-injective R°-modules. If E is semi-fp-injective, then each cycle module $\mathrm{Z}_n(E)$ is fp-injective, see Gillespie [7, Cor. 3.13(5)] and, therefore, E is pure acyclic. Conversely, if E is pure acyclic, then the cycle modules $\mathrm{Z}_n(E)$ are fp-injective and E is semi-fp-injective by [7, Lem. 3.10].

(c): Let P be an acyclic complex of fp-projective R°-modules. There is an induced exact sequence

$$0 \longrightarrow \mathrm{Hom}_{R^\circ}(P, E') \longrightarrow \mathrm{Hom}_{R^\circ}(P, E) \longrightarrow \mathrm{Hom}_{R^\circ}(P, E'') \longrightarrow 0$$

and if two of these complexes are acyclic, then so is the third.

(d): Let E be an acyclic complex of fp-pro-injective R°-modules. Conditions (i) and (iii) are equivalent by (b), and every contractible complex is pure acyclic. If E is pure acyclic, then the cycle modules $\mathrm{Z}_n(E)$ are fp-injective, and by part (a) they are fp-projective as well, so E is contractible. □

4.2 Theorem. *Let R be right coherent. The following conditions are equivalent, and equivalent to conditions* (P0)–(P3)*,* (F1)–(F3)*,* (I1)–(I3)*, and* (FC1)–(FC3) *from* 1.3*,* 1.5*, and* 3.2*.*

(fpI1) *Every complex of fp-injective R°-modules is semi-fp-injective.*

(fpI2) *Every acyclic complex of fp-injective R°-modules is pure acyclic.*

(fpI3) *Every acyclic complex of fp-injective R°-modules is semi-fp-injective.*

(fpPI1) *Every complex of fp-pro-injective R°-modules is semi-fp-injective.*

(fpPI2) *Every acyclic complex of fp-pro-injective R°-modules is contractible.*

(fpPI3) *Every acyclic complex of fp-pro-injective R°-modules is semi-fp-injective.*

Proof. First we argue that the six conditions (fpI1)–(fpI3) and (fpPI1)–(fpPI3) are equivalent. By Lemma 4.1(b,d) one has

$$\text{(fpI1)} \implies \text{(fpI2)} \implies \text{(fpI3)} \qquad \text{and} \qquad \text{(fpPI1)} \implies \text{(fpPI2)} \implies \text{(fpPI3)}\,,$$

which leaves two implications to prove.

(fpI3) $\Longrightarrow$ (fpPI1): Let X be a complex of fp-pro-injective R°-modules and consider the exact sequence $0 \to X \to E \to P' \to 0$ from (4.1.1). It follows that P' is a complex of fp-injective R°-modules and, therefore, semi-fp-injective, so X is semi-fp-injective by Lemma 4.1(c).

(fpPI3) $\Longrightarrow$ (fpI1): Let X be a complex of fp-injective R°-modules and consider the exact sequence $0 \to X \to E \to P' \to 0$ from (4.1.1). It follows that P' is an acyclic complex of fp-pro-injective R°-modules and, therefore, semi-fp-injective, so X is semi-fp-injective by Lemma 4.1(c).

To finish, it suffices by Remark 1.6 and Proposition 3.2 to prove two implications.

(fpI2) $\Longrightarrow$ (I2): Let I be an acyclic complex of injective R°-modules. It is pure acyclic by (fpI2) and hence contractible by 1.4(a).

(FC2) $\Longrightarrow$ (fpI2): Let E be an acyclic complex of fp-injective R°-modules. The character complex $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ is an acyclic complex of flat-cotorsion R-modules and hence contractible, so E is pure acyclic. ☐

4.3 Corollary. *Let R be right coherent; it is right regular if and only if one/all of the nineteen conditions* (P0)–(P3), (F1)–(F3), (I1)–(I3), (FC1)–(FC3), (fpI1)–(fpI3), *and* (fpPI1)–(fpPI3) *from* 1.3, 1.5, 3.2, *and* 4.2 *hold.*

Proof. Combine Theorems 2.1 and 4.2. ☐

4.4 Remark. Gillespie and Iacob [8, Thms. 4.3–4.6] characterize right coherent right regular rings in terms of the equivalence of conditions (P1), (P2), (F1), (F2), (I1), (I2), (fpI1), and (fpI2) from 1.3, 1.5, and 4.2, among other conditions not considered here.

Finally we can add conditions to the characterization of von Neumann regular rings. Over such a ring every fp-projective module is projective, so (*vii*) below can be seen as the counterpart to (P2) in the characterization of regular rings.

4.5 Corollary. *The following conditions are equivalent and equivalent to conditions* (*i*)–(*v*) *from* 2.6 *and* 3.4.

(*vi*) *Every R-complex is semi-fp-injective.*

(*vii*) *Every acyclic complex of fp-projective R-modules is contractible.*

Proof. Recall the equivalent conditions (*i*)–(*v*) from Corollaries 2.6 and 3.4.

(*i*) $\Longrightarrow$ (*vii*): A finitely presented R-module is by Corollary 2.6 projective, so every R-module is fp-injective. Thus, an fp-projective module is an fp-pro-injective R-module. Further, R is by Example 2.5 left coherent and left regular, so every acyclic complex of fp-pro-injective R-modules is contractible by Corollary 4.3.

(*vii*) $\Longrightarrow$ (*vi*): Let M be an R-complex. A complex P in $\mathsf{FpPrj(R)\text{-}ac}$ is, in particular, an acyclic complex of fp-projective R-modules and, therefore, contractible. Thus $\mathrm{Hom}_R(P, M)$ is acyclic, whence M is semi-fp-injective.

(*vi*) $\Longrightarrow$ (*i*): As every R-module is fp-injective, every finitely presented R-module is projective. Since a module is a filtered colimit of its finitely presented submodules, it follows that every R-module is flat, so R is von Neumann regular. ☐

References

[1] Silvana Bazzoni, Manuel Cortés-Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes*, Algebr. Represent. Theory **23** (2020), no. 5, 1861–1883. MR4140057

[2] José Bertin, *Anneaux cohérents réguliers*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A590–A591. MR0288116

[3] Daniel Bravo, Jim Gillespie, and Mark Hovey, *The stable module category of a general ring*, 2014, preprint `arXiv.org/abs/1405.5768`.

[4] Lars Winther Christensen, Sergio Estrada, Li Liang, Peder Thompson, Dejun Wu, and Gang Yang, *A refinement of Gorenstein flat dimension via the flat–cotorsion theory*, J. Algebra **567** (2021), 346–370. MR4159258

[5] Lars Winther Christensen, Hans-Bjørn Foxby, and Henrik Holm, *Derived Category Methods in Commutative Algebra*, Springer Monographs in Mathematics, Springer, Cham, 2024. MR4890472

[6] Marsha Finkel Jones and Mark L. Teply, *Coherent rings of finite weak global dimension*, Comm. Algebra **10** (1982), no. 5, 493–503. MR0647834

[7] James Gillespie, *The flat model structure on* $\mathrm{Ch}(R)$, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390. MR2052954

[8] Jim Gillespie and Alina Iacob, *Homological dimensions of complexes over coherent regular rings*, 2024, preprint `arXiv.org/abs/2409.08393`.

[9] Sarah Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, vol. 1371, Springer-Verlag, Berlin, 1989. MR0999133

[10] Alina Iacob and Srikanth B. Iyengar, *Homological dimensions and regular rings*, J. Algebra **322** (2009), no. 10, 3451–3458. MR2568347

[11] Amnon Neeman, *The homotopy category of flat modules, and Grothendieck duality*, Invent. Math. **174** (2008), no. 2, 255–308. MR2439608

[12] Jan Šaroch and Jan Šťovíček, *Singular compactness and definability for* Σ*-cotorsion and Gorenstein modules*, Selecta Math. (N.S.) **26** (2020), no. 2, paper no. 23, 40 pp. MR4076700

[13] David Tereshkin, private communication.

[14] Gang Yang and Zhongkui Liu, *Cotorsion pairs and model structures on* $\mathrm{Ch}(R)$, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 3, 783–797. MR2837480

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