

A TEST COMPLEX FOR GORENSTEINNESS

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ABSTRACT. Let R be a commutative noetherian ring with a dualizing complex. By recent work of Iyengar and Krause [9], the difference between the category of acyclic complexes and its subcategory of totally acyclic complexes measures how far R is from being Gorenstein. In particular, R is Gorenstein if and only if every acyclic complex is totally acyclic.

In this note we exhibit a specific acyclic complex with the property that it is totally acyclic if and only if R is Gorenstein.

INTRODUCTION

Let R be a commutative noetherian ring. A complex X of R -modules is said to be *acyclic* if it has zero homology, i.e. $H(X) = 0$. An acyclic complex of projective modules is called *totally acyclic* if the acyclicity is preserved by $\mathrm{Hom}_R(-, P)$ for every projective module P . Dually, an acyclic complex of injective modules is totally acyclic if the acyclicity is preserved by $\mathrm{Hom}_R(I, -)$ for every injective module I .

Over a Gorenstein ring, every acyclic complex of projective or of injective modules is totally acyclic. Iyengar and Krause have recently proved a converse; indeed, by [9, cor. 5.5] the following are equivalent when R has a dualizing complex:

- (i) The ring R is Gorenstein.
- (ii) Every acyclic complex of projective R -modules is totally acyclic.
- (iii) Every acyclic complex of injective R -modules is totally acyclic.

Moreover, for a local ring (R, \mathfrak{m}) that is not Gorenstein and has $\mathfrak{m}^2 = 0$ there is a natural example, provided by [9, prop. 6.1(3)], of an acyclic, but not totally acyclic, complex of projective R -modules.

The purpose of this note is to prove that for every ring R with a dualizing complex D , a specific acyclic complex K , defined in 2.1, serves as a test complex for Gorensteinness in the following sense: The ring R is Gorenstein if and only if $K \otimes_R D$ is acyclic. This is achieved by Theorem 2.2. In general, K is an acyclic complex of flat R -modules. Corollary 2.6 shows that if R is an artinian local ring, then K is a complex of projective modules, and (i)–(iii) above are equivalent with

- (iv) The complex K is totally acyclic.

Test complexes of injective modules can be obtained directly from K (Corollary 2.5) or through a potentially different construction explored in Section 3. The authors of [9] have pointed out that the latter is of particular interest, as it yields a generator for $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj} R) / \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj} R)$, the Verdier quotient of acyclic complexes modulo totally acyclic complexes in the homotopy category of injective R -modules. This is proved in Theorem 3.5.

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1. BACKGROUND

Throughout this paper R is a commutative noetherian ring. The notation (R, \mathfrak{m}, k) means R is local with maximal ideal \mathfrak{m} and residue field k .

Complexes of R -modules (R -complexes for short) are graded homologically,

$$X = \cdots \rightarrow X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \rightarrow \cdots.$$

The *suspension* of X is denoted ΣX ; it is the complex with $(\Sigma X)_i = X_{i-1}$ and differential $\partial^{\Sigma X} = -\partial^X$. A complex X is said to be *bounded* if $X_i = 0$ for $|i| \gg 0$.

An isomorphism between R -complexes is denoted by a ' \cong '; we write $X \cong Y$ if there exists an isomorphism $X \xrightarrow{\cong} Y$.

A morphism between R -complexes is called a *quasi-isomorphism*, and denoted $X \xrightarrow{\cong} Y$ if the induced map in homology, $H(X) \rightarrow H(Y)$, is an isomorphism. Following [1, sec. 1] we write $X \simeq Y$, if X and Y can be linked by a sequence of quasi-isomorphisms with arrows in alternating directions. Recall that a morphism $X \rightarrow Y$ is a quasi-isomorphism if and only if its mapping cone, written $\text{Cone}(X \rightarrow Y)$, is acyclic.

1.1. Resolutions. The following facts are established in [1, sec. 1]¹ and [2].

Every R -complex X has a semi-projective resolution. That is, there is a quasi-isomorphism $P \xrightarrow{\cong} X$, where P is a complex of projective R -modules such that $\text{Hom}_R(P, -)$ preserves quasi-isomorphisms. For such a complex, also the functor $- \otimes_R P$ preserves quasi-isomorphisms. In particular, for any R -complexes $Y \simeq Z$ we have $\text{Hom}_R(P, Y) \simeq \text{Hom}_R(P, Z)$ and $Y \otimes_R P \simeq Z \otimes_R P$.

If there is an l such that $H_i(X) = 0$ for $i < l$, then X has a semi-projective resolution P with $P_i = 0$ for $i < l$. If, in addition, $H_i(X)$ is finitely generated for all i , then P can be chosen with all modules P_i finitely generated.

Every R -complex X has a semi-injective resolution. That is, there is a quasi-isomorphism $X \xrightarrow{\cong} J$, where J is a complex of injective R -modules such that $\text{Hom}_R(-, J)$ preserves quasi-isomorphisms. In particular, for such a complex J and any R -complexes $Y \simeq Z$ we have $\text{Hom}_R(Y, J) \simeq \text{Hom}_R(Z, J)$.

1.2. Lemma. *Let X and Y be R -complexes such that either $X_i = 0$ for all $i \ll 0$ or $Y_i = 0$ for all $i \gg 0$. If $H(X_i \otimes_R Y) = 0$ for all $i \in \mathbb{Z}$, then $H(X \otimes_R Y) = 0$.*

Proof. Let E be a faithfully injective R -module. The complex $X \otimes_R Y$ is acyclic if and only if $\text{Hom}_R(X \otimes_R Y, E) \cong \text{Hom}_R(X, \text{Hom}_R(Y, E))$ is so. The claim is now immediate from [5, lem. (2.4)]. \square

1.3. Dualizing complexes. Following [8, V.§2], a *dualizing complex* for R is a bounded complex D of injective R -modules such that $H_i(D)$ is finitely generated for all $i \in \mathbb{Z}$, and the homothety morphism

$$\chi^D: R \longrightarrow \text{Hom}_R(D, D)$$

is a quasi-isomorphism.

Let (R, \mathfrak{m}, k) be a local ring with a dualizing complex D . After suspensions we can assume D is *normalized*, cf. [8, V.§5], in which case [8, prop. V.3.4] yields

$$(1.3.1) \quad H(\text{Hom}_R(k, D)) \cong k.$$

¹ Where semi-projective/injective resolutions are called DG-projective/injective.

If R is artinian, then $E_R(k)$, the injective hull of the residue field, is a normalized dualizing complex for R .

2. A TEST COMPLEX OF FLAT MODULES

2.1. A distinguished complex of flat modules. Assume that R has a dualizing complex D , and let $\pi: P \xrightarrow{\simeq} D$ be a semi-projective resolution. By 1.1 we can assume that P consists of finitely generated modules with $P_i = 0$ for all $i \ll 0$. The functors $\text{Hom}_R(P, -)$ and $\text{Hom}_R(-, D)$ preserve quasi-isomorphisms, so the commutative diagram

$$\begin{array}{ccc} \text{Hom}_R(P, P) & \xrightarrow[\simeq]{\text{Hom}_R(P, \pi)} & \text{Hom}_R(P, D) \\ \chi^P \uparrow & & \simeq \uparrow \text{Hom}_R(\pi, D) \\ R & \xrightarrow[\simeq]{\chi^D} & \text{Hom}_R(D, D) \end{array}$$

shows that also the homothety map χ^P is a quasi-isomorphism. In particular,

$$K = \text{Cone}(R \xrightarrow{\chi^P} \text{Hom}_R(P, P))$$

is acyclic. The modules in $\text{Hom}_R(P, P)$ are direct products of modules of the form $\text{Hom}_R(P_i, P_{i+n})$, and each such module is flat. Thus, χ^P is a quasi-isomorphism between complexes of flat R -modules, and the mapping cone K is, therefore, an acyclic complex of flat R -modules.

We can now state the main result; the proof is given at the end of the section.

2.2. Theorem. *Let R be a commutative noetherian ring with a dualizing complex D , and let K be the acyclic complex of flat modules defined in 2.1. The ring R is Gorenstein if and only if the complex $K \otimes_R D$ is acyclic.*

2.3. Remark. While also $C = \text{Cone } \chi^D$ is an acyclic complex of flat R -modules, it cannot detect Gorensteinness. Indeed, C is bounded, so $C \otimes_R X$ is acyclic for every R -complex X by Lemma 1.2. If R is artinian, then C is even split exact.

2.4. Remark. In the theory of Gorenstein dimensions, there is a notion of a *complete flat resolution*—due to Enochs, Jenda, and Torrecillas [6]—namely an acyclic complex F of flat modules such that $F \otimes_R I$ is acyclic for every injective module I .

If R is Gorenstein, then every acyclic complex of flat R -modules is a complete flat resolution. Indeed, every injective R -module I has finite flat dimension, and then it is straightforward to verify that the functor $- \otimes_R I$ preserves acyclicity of complexes of flat modules. On the other hand, let K and D be as in Theorem 2.2. If K is a complete flat resolution, then $K \otimes_R D$ is acyclic by Lemma 1.2.

Thus, the following assertions are equivalent:

- (i) The ring R is Gorenstein.
- (ii) The complex K is a complete flat resolution.
- (iii) Every acyclic complex of flat modules is a complete flat resolution.

The complex K defined in 2.1 appears to be a natural test object for Gorensteinness. However, it might in the context of [9] be of interest to exhibit a test complex of injective or of projective modules.

To this end, we first note that the next corollary to Theorem 2.2 is immediate in view of Remark 2.4 and [3, prop. (6.4.1)]. See Section 3 for a further discussion of test complexes of injective modules.

2.5. Corollary. *Let R be a commutative noetherian ring with a dualizing complex. Let K be the acyclic complex of flat modules defined in 2.1, and let E be a faithfully injective R -module. The complex $\mathrm{Hom}_R(K, E)$ is an acyclic complex of injective modules, and R is Gorenstein if and only if $\mathrm{Hom}_R(K, E)$ is totally acyclic. \square*

For artinian local rings (R, \mathfrak{m}) , Theorem 2.2 provides a test complex of projective modules. In particular, for R with $\mathfrak{m}^2 = 0$ the following recovers [9, prop. 6.1(3)].

2.6. Corollary. *Let R be an artinian local ring. The complex K defined in 2.1 is an acyclic complex of projective R -modules, and R is Gorenstein if and only if K is totally acyclic.*

Proof. When R is artinian and local, every flat R -module is projective. Thus, K is an acyclic complex of projective modules.

The “only if” part is well-known. To prove “if”, assume K is totally acyclic and recall from 1.3 that the module $E = \mathbb{E}_R(k)$ is dualizing for R . The first of the following isomorphisms is induced by χ^E , and the second is Hom-tensor adjointness

$$\mathrm{Hom}_R(K, R) \cong \mathrm{Hom}_R(K, \mathrm{Hom}_R(E, E)) \cong \mathrm{Hom}_R(K \otimes_R E, E).$$

The complex $\mathrm{Hom}_R(K, R)$ is acyclic and E is faithfully injective, so $K \otimes_R E$ is acyclic and, therefore, R is Gorenstein by Theorem 2.2. \square

For the proof of Theorem 2.2 we need a technical lemma.

2.7. Lemma. *Let P be an R -complex of finitely generated projective modules, X be any R -complex, and B be a bounded R -complex of finitely generated modules. There is an isomorphism of R -complexes*

$$\omega: \mathrm{Hom}_R(P, X) \otimes_R B \xrightarrow{\cong} \mathrm{Hom}_R(P, X \otimes_R B).$$

Proof. It is straightforward to check that the assignment

$$\omega(\phi \otimes b)(p) = (-1)^{|p||b|} \phi(p) \otimes b,$$

where $|\cdot|$ denotes the degree of an element, defines a morphism between the relevant complexes. By assumption, there exist integers $l \leq u$ such that $B_h = 0$ when $h < l$ or $h > u$. For every $n \in \mathbb{Z}$ we have

$$\begin{aligned} (\mathrm{Hom}_R(P, X) \otimes_R B)_n &= \bigoplus_{i=n-u}^{n-l} \mathrm{Hom}_R(P, X)_i \otimes_R B_{n-i} \\ &= \bigoplus_{i=n-u}^{n-l} \left(\prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(P_j, X_{j+i}) \right) \otimes_R B_{n-i} \\ &\cong \bigoplus_{i=n-u}^{n-l} \prod_{j \in \mathbb{Z}} (\mathrm{Hom}_R(P_j, X_{j+i}) \otimes_R B_{n-i}) \\ &\cong \bigoplus_{i=n-u}^{n-l} \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(P_j, X_{j+i} \otimes_R B_{n-i}) \end{aligned}$$

$$\begin{aligned}
&\cong \prod_{j \in \mathbb{Z}} \text{Hom}_R(P_j, \bigoplus_{i=n-u}^{n-l} X_{j+i} \otimes_R B_{n-i}) \\
&= \prod_{j \in \mathbb{Z}} \text{Hom}_R(P_j, (X \otimes_R B)_{j+n}) \\
&= \text{Hom}_R(P, X \otimes_R B)_n.
\end{aligned}$$

Since the modules B_{n-i} are finitely generated, the functors $- \otimes_R B_{n-i}$ commute with arbitrary products for every i ; this explains the first isomorphism. The modules P_j are finitely generated and projective, so for all i, j , and n the homomorphism of modules

$$\text{Hom}_R(P_j, X_{j+i}) \otimes_R B_{n-i} \xrightarrow{\omega_{ij}^n} \text{Hom}_R(P_j, X_{j+i} \otimes_R B_{n-i})$$

is invertible, and this accounts for the second isomorphism. Thus, ω is an isomorphism of graded modules, and the sign in the definition of ω ensures that it commutes with the differentials. \square

Proof of Theorem 2.2. The “only if” part was settled in Remark 2.4.

For the “if” part, assume that the complex $K \otimes_R D$ is acyclic; the isomorphism $\text{Cone}(\chi^P \otimes_R D) \cong K \otimes_R D$ implies that

$$(1) \quad \chi^P \otimes_R D: D \longrightarrow \text{Hom}_R(P, P) \otimes_R D$$

is a quasi-isomorphism.

Choose an n such that $H_i(D) = 0$ for all $i > n$, and let B be the soft truncation of P on the left at n :

$$B = 0 \longrightarrow \text{Coker } \partial_{n+1}^P \xrightarrow{\bar{\partial}_n^P} P_{n-1} \xrightarrow{\partial_{n-1}^P} P_{n-2} \longrightarrow \cdots$$

There are quasi-isomorphisms $B \xleftarrow{\simeq} P \xrightarrow{\simeq} D$ and, hence, a quasi-isomorphism $\beta: B \xrightarrow{\simeq} D$; see [1, 1.1.I.(1) and 1.4.I]. Since the mapping cone of β is a bounded acyclic complex, and $\text{Hom}_R(P, P)$ is a complex of flat modules, Lemma 1.2 applies to show that also $\text{Hom}_R(P, P) \otimes_R \text{Cone}(\beta)$ is acyclic. Thus, the isomorphism $\text{Cone}(\text{Hom}_R(P, P) \otimes_R \beta) \cong \text{Hom}_R(P, P) \otimes_R \text{Cone}(\beta)$ implies that also

$$(2) \quad \text{Hom}_R(P, P) \otimes_R \beta: \text{Hom}_R(P, P) \otimes_R B \longrightarrow \text{Hom}_R(P, P) \otimes_R D$$

is a quasi-isomorphism.

By the choice of P , cf. 2.1, the bounded complex B consists of finitely generated modules, and Lemma 2.7 yields an isomorphism

$$(3) \quad \omega: \text{Hom}_R(P, P) \otimes_R B \xrightarrow{\cong} \text{Hom}_R(P, P \otimes_R B).$$

Finally, let $\iota: P \otimes_R B \xrightarrow{\simeq} J$ be a semi-injective resolution; the quasi-isomorphism ι is preserved by $\text{Hom}_R(P, -)$, and the resulting quasi-isomorphism combined with (1), (2), and (3) yields

$$(4) \quad D \simeq \text{Hom}_R(P, J).$$

It suffices to prove that $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal \mathfrak{m} of R . Let \mathfrak{m} be a maximal ideal; the complex $D_{\mathfrak{m}}$ is dualizing for $R_{\mathfrak{m}}$, see [8, cor. V.2.3]. Set $k = R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \cong R/\mathfrak{m}$. We may, after suspensions, assume $D_{\mathfrak{m}}$ is normalized, so $H(\text{Hom}_{R_{\mathfrak{m}}}(k, D_{\mathfrak{m}})) \cong k$; see (1.3.1). Moreover, there are isomorphisms $\text{Hom}_R(k, D) \cong \text{Hom}_R(k, D) \otimes_R R_{\mathfrak{m}} \cong \text{Hom}_{R_{\mathfrak{m}}}(k, D_{\mathfrak{m}})$, so we have

$$(5) \quad k \cong H(\text{Hom}_R(k, D)).$$

Let $v: Q \xrightarrow{\simeq} k$ be a semi-projective resolution of k over R . As $\mathrm{Hom}_R(-, D)$ preserves quasi-isomorphisms, we have

$$(6) \quad \mathrm{Hom}_R(k, D) \xrightarrow{\simeq} \mathrm{Hom}_R(Q, D).$$

Also $\mathrm{Hom}_R(Q, -)$ preserves quasi-isomorphisms, and from (4) we get

$$(7) \quad \mathrm{Hom}_R(Q, D) \simeq \mathrm{Hom}_R(Q, \mathrm{Hom}_R(P, J)) \cong \mathrm{Hom}_R(Q \otimes_R P, J),$$

where the isomorphism is Hom-tensor adjointness. Finally, $v \otimes_R P$ is a quasi-isomorphism, and hence so is

$$(8) \quad \mathrm{Hom}_R(v \otimes_R P, J): \mathrm{Hom}_R(k \otimes_R P, J) \xrightarrow{\simeq} \mathrm{Hom}_R(Q \otimes_R P, J).$$

Combining (5)–(8) and again using Hom-tensor adjointness, we obtain

$$\begin{aligned} k &\cong \mathrm{H}(\mathrm{Hom}_R(k \otimes_R P, J)) \\ &\cong \mathrm{H}(\mathrm{Hom}_R((k \otimes_R P) \otimes_k k, J)) \\ &\cong \mathrm{H}(\mathrm{Hom}_k(k \otimes_R P, \mathrm{Hom}_R(k, J))) \\ &\cong \mathrm{Hom}_k(\mathrm{H}(k \otimes_R P), \mathrm{H}(\mathrm{Hom}_R(k, J))). \end{aligned}$$

Thus, $\mathrm{Hom}_k(\mathrm{H}(k \otimes_R P), \mathrm{H}(\mathrm{Hom}_R(k, J)))$ is a finitely generated k -vector space; in particular, $\mathrm{H}(k \otimes_R P)$ must be finitely generated. Note that $\mathrm{H}_i(k \otimes_R P) \cong \mathrm{H}_i(k \otimes_{R_m} P_m)$ for all $i \in \mathbb{Z}$; it follows that $\mathrm{H}_i(k \otimes_{R_m} P_m) = 0$ for all $i \gg 0$. By [1, prop. 5.5] the dualizing R_m -complex D_m then has finite flat dimension, and hence R_m is Gorenstein; see [7, thm. (17.23)] or [4, thm. (8.1)]. \square

3. A TEST COMPLEX OF INJECTIVE MODULES

The next construction is another source for test complexes.

3.1. A distinguished complex of injective modules. Assume R has a dualizing complex D . As in 2.1, let $\pi: P \xrightarrow{\simeq} D$ be a semi-projective resolution of D consisting of finitely generated modules with $P_i = 0$ for all $i \ll 0$. The assignment $\varphi \otimes p \mapsto \varphi(p)$ defines a morphism of complexes, ε , such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_R(P, D) \otimes_R P & \xrightarrow{\varepsilon} & D \\ \mathrm{Hom}_R(P, \pi) \otimes_R P \uparrow \simeq & & \simeq \uparrow \pi \\ \mathrm{Hom}_R(P, P) \otimes_R P & \xleftarrow[\chi^P \otimes_R P]{\simeq} & R \otimes_R P. \end{array}$$

Thus, ε is a quasi-isomorphism between complexes of injective R -modules, and the mapping cone

$$M = \mathrm{Cone}(\mathrm{Hom}_R(P, D) \otimes_R P \xrightarrow{\varepsilon} D)$$

an acyclic complex of injective R -modules.

An argument similar to the proof of Theorem 2.2 yields the next result, which is also a corollary of Theorem 3.5.

3.2. Theorem. *Let R be a commutative noetherian ring with a dualizing complex, and let M be the acyclic complex of injective modules defined in 3.1. The ring R is Gorenstein if and only if M is totally acyclic.* \square

3.3. Remark. If (R, \mathfrak{m}, k) is an artinian local ring, then there is an isomorphism

$$K \cong \Sigma \mathrm{Hom}_R(M, \mathbf{E}_R(k))$$

where K and M are the complexes from 2.1 and 3.1, and $\mathbf{E}_R(k)$ is the injective hull of k . Indeed, with $E = \mathbf{E}_R(k)$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(E, E) & \xrightarrow[\cong]{\mathrm{Hom}_R(\varepsilon, E)} & \mathrm{Hom}_R(\mathrm{Hom}_R(P, E) \otimes_R P, E) \\ \cong \uparrow \chi^E & & \cong \uparrow \\ & & \mathrm{Hom}_R(P, \mathrm{Hom}_R(\mathrm{Hom}_R(P, E), E)) \\ R & \xrightarrow[\cong]{\chi^P} & \mathrm{Hom}_R(P, P). \end{array}$$

The vertical maps on the right are the natural isomorphisms, and because E is a module, also the homothety map χ^E is a genuine isomorphism. The diagram induces the desired isomorphism between the complexes $K = \mathrm{Cone}(\chi^P)$ and $\mathrm{Cone}(\mathrm{Hom}_R(\varepsilon, E)) \cong \Sigma \mathrm{Hom}_R(\mathrm{Cone}(\varepsilon), E) = \Sigma \mathrm{Hom}_R(M, E)$.

When R is not artinian, we do not know if the complexes K and M are related.

Using [5, prop. (5.1)] it is not hard to prove the next parallel to Corollary 2.5.

3.4. Corollary. *Let R be a commutative noetherian ring with a dualizing complex. Let M be the acyclic complex of injective modules defined in 3.1, and let E be a faithfully injective R -module. The complex $\mathrm{Hom}_R(M, E)$ is an acyclic complex of flat modules, and R is Gorenstein if and only if $\mathrm{Hom}_R(M, E) \otimes_R I$ is acyclic for every injective module I . \square*

In conversations, the authors of [9] have informed us of Theorem 3.5 below; note that it contains Theorem 3.2. For notation and terminology we refer to [9].

3.5. Theorem. *Let R be a commutative noetherian ring with a dualizing complex. The acyclic complex M of injective modules defined in 3.1 generates the quotient category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj} R) / \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj} R)$.*

Proof. By [9, 1.7, 5.4, and 5.9(3)] the quotient category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj} R) / \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj} R)$ is generated by the image of the dualizing complex D under the equivalence $\mathbf{D}^f(R) \xrightarrow{\sim} \mathbf{K}^c(\mathrm{Inj} R)$, cf. [9, 2.3(2)].

Let $P \xrightarrow{\sim} D$ be a semi-projective resolution. The functor $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms, so the composite

$$R \xrightarrow{\sim} \mathrm{Hom}_R(P, P) \xrightarrow{\sim} \mathrm{Hom}_R(P, D)$$

provides an injective resolution $R \xrightarrow{\sim} iR = \mathrm{Hom}_R(P, D)$. Since iR is a compact object in $\mathbf{K}(\mathrm{Inj} R)$, the inclusion of the localizing subcategory $\mathrm{Loc}(iR) \subseteq \mathbf{K}(\mathrm{Inj} R)$ admits a right adjoint $\rho: \mathbf{K}(\mathrm{Inj} R) \rightarrow \mathrm{Loc}(iR)$; see [9, 1.5.1]. By [9, 2.3(2)] the image of D in $\mathbf{K}(\mathrm{Inj} R)$ is

$$\mathrm{Cone}(\rho(D) \xrightarrow{\xi} D),$$

where ξ is the natural map.

It remains to show that $M \cong \text{Cone}(\rho(D) \xrightarrow{\xi} D)$. It suffices to establish a commutative diagram,

$$\begin{array}{ccc} \rho(D) & \xrightarrow{\xi} & D \\ \downarrow \cong & \searrow \varepsilon & \\ \text{Hom}_R(P, D) \otimes_R P & & \end{array}$$

The complex $\text{Hom}_R(P, D) \otimes_R P = iR \otimes_R P$ is in $\text{Loc}(iR)$, and since ε is a quasi-isomorphism, $\text{Hom}_{\mathbf{K}(\text{Inj } R)}(iR, \varepsilon)$ is an isomorphism, cf. [9, 2.2]. The existence of the desired isomorphism $\rho(D) \cong \text{Hom}_R(P, D) \otimes_R P$ now follows from [9, 1.4]. \square

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