

THREE TAKES ON ALMOST COMPLETE INTERSECTION IDEALS OF GRADE 3

LARS WINTHER CHRISTENSEN, OANA VELICHE, AND JERZY WEYMAN

ABSTRACT. We are interested in the structure of almost complete intersection ideals of grade 3. We give three constructions of these ideals and their free resolutions: one from the commutative algebra point of view, an equivariant construction giving a nice canonical form, and finally an interpretation in terms of open sets in certain Schubert varieties.

CONTENTS

1. Almost complete intersections following Avramov and Brown	2
2. Generic almost complete intersections	4
3. The equivariant form of the format $(1, 4, n, n - 3)$	13
4. Schubert varieties in orthogonal Grassmannians	17
Appendix A. Pfaffian identities following Knuth	27
Appendix B. Minors via Pfaffians following Brill	32
Appendix C. Generic almost complete intersections: the proofs	35
Acknowledgments	46
References	46

INTRODUCTION

Let R be a commutative noetherian local ring. A celebrated result of Buchsbaum and Eisenbud [10] states that every Gorenstein ideal in R of grade 3 is generated by the $2m \times 2m$ Pfaffians of a $(2m + 1) \times (2m + 1)$ skew symmetric matrix. Later, Avramov [2] and Brown [6] independently proved a similar result for almost complete intersections. Their proofs are based on the fact that an almost complete intersection ideal is linked to a Gorenstein ideal; this means that their descriptions of the resolutions depend on certain choices, so they are not coordinate free.

In this paper we take three approaches to almost complete intersection ideals of grade 3. They involve different languages, so they can be appreciated by different audiences. However we show how these three approaches intertwine and influence each other.

The first approach uses only commutative and linear algebra. The main theorems about grade 3 almost complete intersection ideals in the local ring R are stated in Section 1. They are proved in Section 2 and Appendices A–C by specialization from the generic case. In the generic case we use the Buchsbaum–Eisenbud Acyclicity

Date: 10 December 2021.

2020 Mathematics Subject Classification. 13C05; 13H10.

Key words and phrases. Almost complete intersection ideal, minimal free resolution.

L.W.C. was partly supported by Simons Foundation collaboration grant 428308 and J.W. was partly supported by NSF DMS grant 1802067.

Criterion and a computation with Pfaffians inspired by the Buchsbaum–Eisenbud Structure Theorems, see Remark 2.4, to construct the minimal free resolutions. We emphasize that our description of the resolutions in the generic case does not depend on linkage; this avoids an implicit change of basis present in [6], see Remark 2.8. Under this first and purely algebraic approach all statements are given full proofs; the next two approaches offer interpretations of the same statements.

The second approach, taken in Section 3, is to provide canonical equivariant forms of almost complete intersections. The ideals one obtains depend on a skew symmetric matrix and three vectors. This view of almost complete intersections was reached by analyzing the generic ring \hat{R}_{gen} constructed by Weyman [27]. The idea was to look for an open set in \hat{R}_{gen} of points where the corresponding resolution is a resolution of a perfect ideal. This set can be explicitly described as the points where localization of certain complex over \hat{R}_{gen} is split exact. Calculating this “splitting form” of an ideal of grade 3 with four generators led to our form of almost complete intersection. One could use the geometric technique of calculating syzygies to prove the acyclicity of these complexes but they are identical to those from the commutative algebra approach so we do not follow through on that. The advantage of this method is that one can give a geometric interpretation of the zero set of almost complete intersection ideals. Moreover the fact, first noticed in [2], that the skew symmetric matrix associated to an almost complete intersection ideal can be chosen with a 3×3 block of zeros on the diagonal is particularly natural under this approach.

Finally, in Section 4, we give a geometric interpretation of both Gorenstein ideals and almost complete intersections of grade 3. It turns out that they are intersections of the so-called big open cell with two Schubert varieties of codimension 3 in the connected component of the orthogonal Grassmannian $OGr(n, 2n)$ of isotropic subspaces of dimension n in a $2n$ -dimensional orthogonal space. It is interesting that in this construction the two Schubert varieties appear together with a regular sequence by which they are linked. This pattern generalizes from the D_n root system to E_6 , E_7 and E_8 ; see Sam and Weyman [24]. We show that the defining ideals are exactly the same as in commutative algebra approach, but we indicate how one could see the graded format of the finite free resolutions just from representation theory viewpoint. Also, the fact about three submaximal Pfaffians forming a regular sequence get a clear geometric interpretation, as one can see geometrically that their zero set has codimension 3.

1. ALMOST COMPLETE INTERSECTIONS FOLLOWING AVRAMOV AND BROWN

For a grade 3 perfect ideal \mathfrak{a} in a commutative noetherian local ring (R, \mathfrak{m}, k) , the minimal free resolution of the quotient ring R/\mathfrak{a} has the form

$$F = 0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0,$$

and we refer to the rank of F_3 as the *type* of R/\mathfrak{a} ; if R is Cohen–Macaulay, then this is indeed the Cohen–Macaulay type. Throughout the paper we treat quotients of odd and even type separately.

By a result of Buchsbaum and Eisenbud [10] the minimal free resolution F has a structure of a skew commutative differential graded algebra. This structure is not unique, but the induced skew commutative algebra structure on $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$ is

unique. It provides for a classification of quotients R/\mathfrak{a} as worked out Weyman [26] and by Avramov, Kustin, and Miller [3].

To state the main theorems about grade 3 almost complete intersection ideals in local rings we introduce some matrix-related notation.

1.1 Notation. Let M be an $m \times n$ matrix with entries in a commutative ring. For subsets

$$I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, m\} \quad \text{and} \quad J = \{j_1, \dots, j_l\} \subseteq \{1, \dots, n\}$$

with $i_1 < \dots < i_k$ and $j_1 < \dots < j_l$ we write $M[i_1 \dots i_k; j_1 \dots j_l]$ for the submatrix of M obtained by taking the rows indexed by I and the columns indexed by J . At times, it is more convenient to specify a submatrix in terms of removal of rows and columns: The symbol $M[\overline{i_1 \dots i_k}; \overline{j_1 \dots j_l}]$ specifies the submatrix of M obtained by removing the rows indexed by I and the columns indexed by J . These notations can also be combined: For example, $M[\overline{i_1 \dots i_k}; j_1 \dots j_l]$ is the submatrix obtained by taking the rows indexed by the complement of I and the columns indexed by J .

For an $n \times n$ skew symmetric matrix T , the Pfaffian of T is written $\text{Pf}(T)$. For a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ the Pfaffian of the submatrix $T[i_1 \dots i_k; i_1 \dots i_k]$ is written $\text{Pf}_{i_1 \dots i_k}(T)$ while the Pfaffian of $T[\overline{i_1 \dots i_k}; \overline{i_1 \dots i_k}]$ is written $\text{Pf}_{\overline{i_1 \dots i_k}}(T)$.

1.2 Theorem. *Let $n \geq 5$ be an odd number. Let (R, \mathfrak{m}, k) be a local ring and $\mathfrak{a} \subset R$ a grade 3 almost complete intersection ideal such that R/\mathfrak{a} is of type $n - 3$. There exists an $n \times n$ skew symmetric block matrix*

$$U = \left(\begin{array}{c|ccc} O & & & \\ \hline -B^T & A & & \end{array} \right) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & t_{14} & t_{15} & \dots \\ 0 & 0 & 0 & t_{24} & t_{25} & \dots \\ 0 & 0 & 0 & t_{34} & t_{35} & \dots \\ \hline -t_{14} & -t_{24} & -t_{34} & 0 & t_{45} & \dots \\ -t_{15} & -t_{25} & -t_{35} & -t_{45} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right)$$

with entries in \mathfrak{m} such that the minimal free resolution of R/\mathfrak{a} ,

$$F = 0 \longrightarrow R^{n-3} \xrightarrow{\partial_3} R^n \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R,$$

has differentials

$$\partial_3 = \left(\frac{B}{A} \right),$$

$$\partial_2 = \begin{pmatrix} \text{Pf}(A) & 0 & 0 & -\text{Pf}_{\overline{234}}(U) & \text{Pf}_{\overline{235}}(U) & \dots & \text{Pf}_{\overline{23n}}(U) \\ 0 & \text{Pf}(A) & 0 & -\text{Pf}_{\overline{134}}(U) & \text{Pf}_{\overline{135}}(U) & \dots & \text{Pf}_{\overline{13n}}(U) \\ 0 & 0 & \text{Pf}(A) & -\text{Pf}_{\overline{124}}(U) & \text{Pf}_{\overline{125}}(U) & \dots & \text{Pf}_{\overline{12n}}(U) \\ \text{Pf}_{\overline{1}}(U) & -\text{Pf}_{\overline{2}}(U) & \text{Pf}_{\overline{3}}(U) & -\text{Pf}_{\overline{4}}(U) & \text{Pf}_{\overline{5}}(U) & \dots & \text{Pf}_{\overline{n}}(U) \end{pmatrix},$$

and

$$\partial_1 = (-\text{Pf}_{\overline{1}}(U) \quad \text{Pf}_{\overline{2}}(U) \quad -\text{Pf}_{\overline{3}}(U) \quad \text{Pf}(A)).$$

In particular, \mathfrak{a} is generated by $\text{Pf}_{\overline{1}}(U)$, $\text{Pf}_{\overline{2}}(U)$, $\text{Pf}_{\overline{3}}(U)$, and $\text{Pf}(A)$. Moreover, the multiplicative structure on $\text{Tor}_*^R(R/\mathfrak{a}, k)$ is of class $\mathbf{H}(3, 2)$ if R/\mathfrak{a} is of type 2 and otherwise of class $\mathbf{H}(3, 0)$.

The proof of this theorem is given in 2.7 and the next theorem is proved in 2.12.

1.3 Theorem. *Let $n \geq 6$ be an even number. Let (R, \mathfrak{m}, k) be a local ring and $\mathfrak{a} \subset R$ a grade 3 almost complete intersection ideal such that R/\mathfrak{a} is of type $n - 3$. There exists an $n \times n$ skew symmetric block matrix U as in Theorem 1.2 such that the minimal free resolution of R/\mathfrak{a} ,*

$$F = 0 \longrightarrow R^{n-3} \xrightarrow{\partial_3} R^n \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R,$$

has differentials

$$\partial_3 = \begin{pmatrix} B \\ A \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & -\text{Pf}_{1234}(U) & \text{Pf}_{1235}(U) & \cdots & -\text{Pf}_{123n}(U) \\ \text{Pf}_{13}(U) & -\text{Pf}_{23}(U) & 0 & \text{Pf}_{34}(U) & -\text{Pf}_{35}(U) & \cdots & \text{Pf}_{3n}(U) \\ -\text{Pf}_{12}(U) & 0 & \text{Pf}_{23}(U) & -\text{Pf}_{24}(U) & \text{Pf}_{25}(U) & \cdots & -\text{Pf}_{2n}(U) \\ 0 & \text{Pf}_{12}(U) & -\text{Pf}_{13}(U) & \text{Pf}_{14}(U) & -\text{Pf}_{15}(U) & \cdots & \text{Pf}_{1n}(U) \end{pmatrix},$$

and

$$\partial_1 = (\text{Pf}(U) \quad \text{Pf}_{12}(U) \quad \text{Pf}_{13}(U) \quad \text{Pf}_{23}(U)).$$

In particular, \mathfrak{a} is generated by $\text{Pf}(U)$, $\text{Pf}_{12}(U)$, $\text{Pf}_{13}(U)$, and $\text{Pf}_{23}(U)$. Moreover, the multiplicative structure on $\text{Tor}_*^R(R/\mathfrak{a}, k)$ is of class \mathbf{T} .

The proofs Theorems 1.2 and 1.3 have been deferred to the next section because we obtain them by specialization of statements about generic almost complete intersections.

2. GENERIC ALMOST COMPLETE INTERSECTIONS

In this section and the appendices we deal extensively with relations between Pfaffians of submatrices $T[i_1 \dots i_k; i_1 \dots i_k]$ of a fixed skew symmetric matrix T . It is, therefore, convenient to have the following variation on the notation from 1.1:

$$(2.0.1) \quad \text{pf}_T(i_1 \dots i_k) = \text{Pf}_{i_1 \dots i_k}(T) \quad \text{and} \quad \text{pf}_T(\overline{i_1 \dots i_k}) = \text{Pf}_{\overline{i_1 \dots i_k}}(T).$$

It emphasizes the subset, which changes, over the matrix, which is fixed; for homogeneity we set $\text{pf}_T = \text{Pf}(T)$.

2.1 Setup. Let n be a natural number and $\mathcal{R} = \mathbb{Z}[\tau_{ij} \mid 1 \leq i < j \leq n]$ the polynomial algebra in indeterminates τ_{ij} over \mathbb{Z} . Let \mathcal{T} be the $n \times n$ skew symmetric matrix with entries $\mathcal{T}[i; j] = \tau_{ij} = -\mathcal{T}[j; i]$ for $1 \leq i < j \leq n$ and zeros on the diagonal. It looks like this:

$$\mathcal{T} = \begin{pmatrix} 0 & \tau_{12} & \tau_{13} & \tau_{14} & \cdots \\ -\tau_{12} & 0 & \tau_{23} & \tau_{24} & \cdots \\ -\tau_{13} & -\tau_{23} & 0 & \tau_{34} & \cdots \\ -\tau_{14} & -\tau_{24} & -\tau_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

2.2 Lemma. *Adopt the setup from 2.1 and denote by \mathfrak{M} the ideal generated by the indeterminates τ_{ij} . Let $\partial: \mathcal{R}^{n-3} \longrightarrow \mathcal{R}^n$ be the linear map given by the matrix $\mathcal{T}[1 \dots n; 4 \dots n]$. One has $\partial(\mathcal{R}^{n-3}) \cap \mathfrak{M}^2 \mathcal{R}^n = \partial(\mathfrak{M} \mathcal{R}^{n-3})$.*

Proof. Let g_4, \dots, g_n and f_1, \dots, f_n be the standard bases for the free modules \mathcal{R}^{n-3} and \mathcal{R}^n . Let $x = \sum_{i=4}^n a_i g_i$ be an element of \mathcal{R}^{n-3} ; one has

$$\partial(x) = \sum_{i=4}^n a_i \left(\sum_{j=1}^{i-1} \tau_{ji} f_j - \sum_{j=i+1}^n \tau_{ij} f_j \right) = \sum_{j=1}^n \left(\sum_{i=4}^{j-1} -a_i \tau_{ij} + \sum_{i=j+1}^n a_i \tau_{ji} \right) f_j.$$

Thus, if $\partial(x)$ is contained in $\mathfrak{M}^2 \mathcal{R}^n$, then all the elements $a_i \tau_{ij}$ and $a_i \tau_{ji}$ belong to \mathfrak{M}^2 , which implies that the every a_i is in \mathfrak{M} . Thus, $\partial(\mathcal{R}^{n-3}) \cap \mathfrak{M}^2 \mathcal{R}^n$ is contained in $\partial(\mathfrak{M} \mathcal{R}^{n-3})$, and the opposite containment is trivial. \square

Quotients of even type.

2.3 Theorem. *Let $n \geq 5$ be an odd number; consider the ring \mathcal{R} and the $n \times n$ matrix \mathcal{T} from 2.1. The homomorphisms given by the matrices*

$$\partial_3 = \mathcal{T}[1 \dots n; 4 \dots n],$$

$$\partial_2 = \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{123}) & 0 & 0 & -\text{pf}_{\mathcal{T}}(\overline{234}) & \text{pf}_{\mathcal{T}}(\overline{235}) & \cdots & \text{pf}_{\mathcal{T}}(\overline{23n}) \\ 0 & \text{pf}_{\mathcal{T}}(\overline{123}) & 0 & -\text{pf}_{\mathcal{T}}(\overline{134}) & \text{pf}_{\mathcal{T}}(\overline{135}) & \cdots & \text{pf}_{\mathcal{T}}(\overline{13n}) \\ 0 & 0 & \text{pf}_{\mathcal{T}}(\overline{123}) & -\text{pf}_{\mathcal{T}}(\overline{124}) & \text{pf}_{\mathcal{T}}(\overline{125}) & \cdots & \text{pf}_{\mathcal{T}}(\overline{12n}) \\ \text{pf}_{\mathcal{T}}(\overline{1}) & -\text{pf}_{\mathcal{T}}(\overline{2}) & \text{pf}_{\mathcal{T}}(\overline{3}) & -\text{pf}_{\mathcal{T}}(\overline{4}) & \text{pf}_{\mathcal{T}}(\overline{5}) & \cdots & \text{pf}_{\mathcal{T}}(\overline{n}) \end{pmatrix},$$

and

$$\partial_1 = (-\text{pf}_{\mathcal{T}}(\overline{1}) \quad \text{pf}_{\mathcal{T}}(\overline{2}) \quad -\text{pf}_{\mathcal{T}}(\overline{3}) \quad \text{pf}_{\mathcal{T}}(\overline{123})),$$

define an exact sequence

$$\mathcal{F} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}.$$

That is, denoting by \mathfrak{A}_n the ideal generated by the entries of ∂_1 , the complex \mathcal{F} is a free resolution of $\mathcal{R}/\mathfrak{A}_n$. Moreover, the ideal \mathfrak{A}_n is perfect of grade 3.

The proof of this theorem relies on a series of technical results that we defer to Appendix C. The proof shows how they come together.

Proof. It follows from Lemma C.1 that \mathcal{F} is a complex. The expected ranks of the homomorphisms ∂_3 , ∂_2 , and ∂_1 are $n-3$, 3, and 1. To show that the complex is exact at \mathcal{R}^{n-3} , \mathcal{R}^n , and \mathcal{R}^4 it suffices by the Buchsbaum–Eisenbud Acyclicity Criterion [8] to verify the inequalities

$$\text{grade}_{\mathcal{R}}(I_{n-3}(\partial_3)) \geq 3, \quad \text{grade}_{\mathcal{R}}(I_3(\partial_2)) \geq 2, \quad \text{and} \quad \text{grade}_{\mathcal{R}}(I_1(\partial_1)) \geq 1,$$

where as usual $I_r(\partial)$ denotes the ideal generated by the $r \times r$ minors of ∂ . By Lemma C.2 the ideal $I_1(\partial_1) = \mathfrak{A}_n$ has grade at least 3. By Lemma C.3 the radical $\sqrt{\mathfrak{A}_n}$ is contained in $\sqrt{I_{n-3}(\partial_3)}$, so $\text{grade}_{\mathcal{R}}(I_{n-3}(\partial_3)) \geq \text{grade}_{\mathcal{R}}(\mathfrak{A}_n) \geq 3$ holds. By Proposition C.4, the generators of $I_3(\partial_2)$ are products of generators of the ideals $I_{n-3}(\partial_3)$ and $I_1(\partial_1) = \mathfrak{A}_n$. It follows that the radical $\sqrt{I_3(\partial_2)}$ contains $\sqrt{\mathfrak{A}_n}$, so one also has $\text{grade}_{\mathcal{R}}(I_3(\partial_2)) \geq 3$. Thus, \mathcal{F} is a free resolution of $\mathcal{R}/\mathfrak{A}_n$; in particular, the projective dimension of $\mathcal{R}/\mathfrak{A}_n$ is at most 3. As the grade of \mathfrak{A}_n is at least 3, it follows that \mathfrak{A}_n is perfect of grade 3. \square

The following commentary also applies to the proof of Theorem 2.9.

2.4 Remark. The proof of Theorem 2.3 is based on establishing containments among radicals to ensure that the rank conditions in the Buchsbaum–Eisenbud

Acyclicity Criterion are met; per [9, Theorem 2.1] the conclusion that \mathcal{F} is exact implies that the radicals $\sqrt{I_{n-3}(\partial_3)}$, $\sqrt{I_3(\partial_2)}$, and $\sqrt{I_1(\partial_1)}$ agree.

The inspiration for the pivotal Proposition C.4 came from the same paper, namely from the Buchsbaum–Eisenbud Structure Theorems which, in the guise of [9, Corollary 5.1], say that for \mathcal{F} to be a resolution the equality $I_{n-3}(\partial_3)I_1(\partial_1) = I_3(\partial_2)$ must hold. The vehicle for the proof of Proposition C.4 is a relation between the sub-Pfaffians and general minors of a skew symmetric matrix; it was first discovered by Brill [4] and reproved by us in [11] using Knuth’s [17] combinatorial approach to Pfaffians in the same vein as in Appendices A–C.

As noticed in [2] one can replace the upper left 3×3 block in \mathcal{T} with a block of zeros without changing the ideal \mathfrak{A}_n .

2.5 Lemma. *Let $n \geq 5$ be an odd number and $T = (t_{ij})$ an $n \times n$ skew symmetric matrix with entries in a commutative ring R . Let U be the matrix obtained from T by replacing the upper left 3×3 block by a block of zeros; i.e.*

$$U = \begin{pmatrix} 0 & 0 & 0 & t_{14} & t_{15} & \dots \\ 0 & 0 & 0 & t_{24} & t_{25} & \dots \\ 0 & 0 & 0 & t_{34} & t_{35} & \dots \\ -t_{14} & -t_{24} & -t_{34} & 0 & t_{45} & \dots \\ -t_{15} & -t_{25} & -t_{35} & -t_{45} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

There is an equality of ideals in R ,

$$(\text{Pf}_{\overline{1}}(T), \text{Pf}_{\overline{2}}(T), \text{Pf}_{\overline{3}}(T), \text{Pf}_{\overline{123}}(T)) = (\text{Pf}_{\overline{1}}(U), \text{Pf}_{\overline{2}}(U), \text{Pf}_{\overline{3}}(U), \text{Pf}_{\overline{123}}(U)).$$

Proof. Notice that $\text{Pf}_{\overline{12i}}(T) = \text{Pf}_{\overline{12i}}(U)$ holds for $i \in \{3, \dots, n\}$. Lemma A.2 applied with $u_1 \dots u_k = 2 \dots n$ and $\ell = 1$ now yields

$$\begin{aligned} \text{Pf}_{\overline{1}}(T) &= \sum_{i=3}^n t_{2i}(-1)^{i-1} \text{Pf}_{\overline{12i}}(T) \\ &= t_{23} \text{Pf}_{\overline{123}}(U) + \sum_{i=4}^n t_{2i}(-1)^{i-1} \text{Pf}_{\overline{12i}}(U) = t_{23} \text{Pf}_{\overline{123}}(U) + \text{Pf}_{\overline{1}}(U). \end{aligned}$$

Similarly, one gets

$$\begin{aligned} \text{Pf}_{\overline{2}}(T) &= \sum_{i=3}^n t_{1i}(-1)^{i-1} \text{Pf}_{\overline{12i}}(T) \\ &= t_{13} \text{Pf}_{\overline{123}}(U) + \sum_{i=4}^n t_{1i}(-1)^{i-1} \text{Pf}_{\overline{12i}}(U) = t_{13} \text{Pf}_{\overline{123}}(U) + \text{Pf}_{\overline{2}}(U) \end{aligned}$$

and

$$\begin{aligned} \text{Pf}_{\overline{3}}(T) &= t_{12} \text{Pf}_{\overline{123}}(T) + \sum_{i=4}^n t_{1i}(-1)^{i-1} \text{Pf}_{\overline{13i}}(T) \\ &= t_{12} \text{Pf}_{\overline{123}}(U) + \sum_{i=4}^n t_{1i}(-1)^{i-1} \text{Pf}_{\overline{13i}}(U) = t_{12} \text{Pf}_{\overline{123}}(U) + \text{Pf}_{\overline{3}}(U). \end{aligned}$$

The asserted equality of ideals is immediate from these three expressions. \square

2.6 Proposition. *Let $n \geq 5$ be an odd number. Consider the ring \mathcal{R} and the $n \times n$ matrix \mathcal{T} from 2.1 as well as the ideal \mathfrak{A}_n from Theorem 2.3. Let \mathcal{U} be the matrix obtained from \mathcal{T} by replacing the upper left 3×3 block by a block of zeros; i.e.*

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 0 & \tau_{14} & \tau_{15} & \cdots \\ 0 & 0 & 0 & \tau_{24} & \tau_{25} & \cdots \\ 0 & 0 & 0 & \tau_{34} & \tau_{35} & \cdots \\ -\tau_{14} & -\tau_{24} & -\tau_{34} & 0 & \tau_{45} & \cdots \\ -\tau_{15} & -\tau_{25} & -\tau_{35} & -\tau_{45} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The homomorphisms given by the matrices

$$\partial_3 = \mathcal{U}[1 \dots n; 4 \dots n],$$

$$\partial_2 = \begin{pmatrix} \text{pf}_{\mathcal{U}}(\overline{123}) & 0 & 0 & -\text{pf}_{\mathcal{U}}(\overline{234}) & \text{pf}_{\mathcal{U}}(\overline{235}) & \cdots & \text{pf}_{\mathcal{U}}(\overline{23n}) \\ 0 & \text{pf}_{\mathcal{U}}(\overline{123}) & 0 & -\text{pf}_{\mathcal{U}}(\overline{134}) & \text{pf}_{\mathcal{U}}(\overline{135}) & \cdots & \text{pf}_{\mathcal{U}}(\overline{13n}) \\ 0 & 0 & \text{pf}_{\mathcal{U}}(\overline{123}) & -\text{pf}_{\mathcal{U}}(\overline{124}) & \text{pf}_{\mathcal{U}}(\overline{125}) & \cdots & \text{pf}_{\mathcal{U}}(\overline{12n}) \\ \text{pf}_{\mathcal{U}}(\overline{1}) & -\text{pf}_{\mathcal{U}}(\overline{2}) & \text{pf}_{\mathcal{U}}(\overline{3}) & -\text{pf}_{\mathcal{U}}(\overline{4}) & \text{pf}_{\mathcal{U}}(\overline{5}) & \cdots & \text{pf}_{\mathcal{U}}(\overline{n}) \end{pmatrix},$$

and

$$\partial_1 = (-\text{pf}_{\mathcal{U}}(\overline{1}) \quad \text{pf}_{\mathcal{U}}(\overline{2}) \quad -\text{pf}_{\mathcal{U}}(\overline{3}) \quad \text{pf}_{\mathcal{U}}(\overline{123})),$$

define a free resolution $\mathcal{L} = 0 \rightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}$ of $\mathcal{R}/\mathfrak{A}_n$. In particular, the ideal \mathfrak{A}_n is generated by $\text{pf}_{\mathcal{U}}(\overline{1})$, $\text{pf}_{\mathcal{U}}(\overline{2})$, $\text{pf}_{\mathcal{U}}(\overline{3})$, and $\text{pf}_{\mathcal{U}}(\overline{123})$.

Proof. By Lemma 2.5 one has

$$(\text{pf}_{\mathcal{U}}(\overline{1}), \text{pf}_{\mathcal{U}}(\overline{2}), \text{pf}_{\mathcal{U}}(\overline{3}), \text{pf}_{\mathcal{U}}(\overline{123})) = (\text{pf}_{\mathcal{T}}(\overline{1}), \text{pf}_{\mathcal{T}}(\overline{2}), \text{pf}_{\mathcal{T}}(\overline{3}), \text{pf}_{\mathcal{T}}(\overline{123})) = \mathfrak{A}_n.$$

Next we show how to obtain the free resolution \mathcal{L} from the resolution \mathcal{F} from Theorem 2.3. To distinguish the differentials on the resolutions we introduce superscripts \mathcal{L} and \mathcal{F} . Consider the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \tau_{23} & -\tau_{13} & \tau_{12} & 1 \end{pmatrix} \quad \text{with inverse} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\tau_{23} & \tau_{13} & -\tau_{12} & 1 \end{pmatrix}.$$

As in the proof of Lemma 2.5 one has

- (1) $\text{pf}_{\mathcal{T}}(\overline{1}) = \tau_{23} \text{pf}_{\mathcal{U}}(\overline{123}) + \text{pf}_{\mathcal{U}}(\overline{1}),$
- (2) $\text{pf}_{\mathcal{T}}(\overline{2}) = \tau_{13} \text{pf}_{\mathcal{U}}(\overline{123}) + \text{pf}_{\mathcal{U}}(\overline{2}),$
- (3) $\text{pf}_{\mathcal{T}}(\overline{3}) = \tau_{12} \text{pf}_{\mathcal{U}}(\overline{123}) + \text{pf}_{\mathcal{U}}(\overline{3}),$ and
- (4) $\text{pf}_{\mathcal{T}}(\overline{123}) = \text{pf}_{\mathcal{U}}(\overline{123}).$

These identities show that one has $\partial_1^{\mathcal{L}} = \partial_1^{\mathcal{F}} S$. Thus, the matrices

$$\partial_3^{\mathcal{F}}, \quad S^{-1} \partial_2^{\mathcal{F}}, \quad \text{and} \quad \partial_1^{\mathcal{F}} S$$

determine a free resolution of $\mathcal{R}/\mathfrak{A}_n$. As the submatrices $\partial_3^{\mathcal{L}} = \mathcal{U}[1 \dots n; 4 \dots n]$ and $\partial_3^{\mathcal{F}} = \mathcal{T}[1 \dots n; 4 \dots n]$ agree, it suffices to show that $\partial_2^{\mathcal{L}} = S^{-1} \partial_2^{\mathcal{F}}$ holds.

For indices $1 \leq i \leq n$ one has

$$\text{pf}_{\mathcal{T}}(\overline{12i}) = \text{pf}_{\mathcal{U}}(\overline{12i}), \quad \text{pf}_{\mathcal{T}}(\overline{13i}) = \text{pf}_{\mathcal{U}}(\overline{13i}), \quad \text{and} \quad \text{pf}_{\mathcal{T}}(\overline{23i}) = \text{pf}_{\mathcal{U}}(\overline{23i}).$$

It follows that the first three rows in the matrices $\partial_2^{\mathcal{L}}$, $\partial_2^{\mathcal{F}}$, and $S^{-1}\partial_2^{\mathcal{F}}$ agree. We now focus of the fourth rows of $\partial_2^{\mathcal{L}}$ and $S^{-1}\partial_2^{\mathcal{F}}$. The first three entries in the fourth rows agree by the identities (1), (2), and (3). Now fix $j \in \{4, \dots, n\}$. Another application of Lemma A.2 yields

$$\begin{aligned} \text{pf}_{\mathcal{T}}(\bar{j}) &= \tau_{12} \text{pf}_{\mathcal{T}}(\overline{12j}) - \tau_{13} \text{pf}_{\mathcal{T}}(\overline{13j}) \\ &\quad + \sum_{i=4}^{j-1} (-1)^i \tau_{1i} \text{pf}_{\mathcal{T}}(\overline{1ij}) + \sum_{i=j+1}^n (-1)^{i-1} \tau_{1i} \text{pf}_{\mathcal{T}}(\overline{1ij}). \end{aligned}$$

One has $\text{pf}_{\mathcal{T}}(\overline{1ij}) = \tau_{23} \text{pf}_{\mathcal{T}}(\overline{123ij}) + \text{pf}_{\mathcal{U}}(\overline{1ij})$, again by Lemma A.2, and therefore

$$\begin{aligned} &\text{pf}_{\mathcal{T}}(\bar{j}) - \tau_{12} \text{pf}_{\mathcal{T}}(\overline{12j}) + \tau_{13} \text{pf}_{\mathcal{T}}(\overline{13j}) \\ &= \tau_{23} \left(\sum_{i=4}^{j-1} (-1)^i \tau_{1i} \text{pf}_{\mathcal{T}}(\overline{123ij}) + \sum_{i=j+1}^n (-1)^{i-1} \tau_{1i} \text{pf}_{\mathcal{T}}(\overline{123ij}) \right) \\ &\quad + \sum_{i=1}^{j-1} (-1)^i \tau_{1i} \text{pf}_{\mathcal{U}}(\overline{1ij}) + \sum_{i=j+1}^n (-1)^{i-1} \tau_{1i} \text{pf}_{\mathcal{U}}(\overline{1ij}) \\ &= \tau_{23} \text{pf}_{\mathcal{T}}(\overline{23j}) + \text{pf}_{\mathcal{U}}(\bar{j}), \end{aligned}$$

where the last equality follows from two applications of Lemma A.2. This identity shows that the fourth row entries of $\partial_2^{\mathcal{L}}$ and $S^{-1}\partial_2^{\mathcal{F}}$ agree in column j . \square

One could also establish Proposition 2.6 as follows: After invoking Lemma 2.5, repeat the proof of Theorem 2.3 noticing at every step that the conclusions remain valid after evaluation at $\tau_{12} = \tau_{13} = \tau_{23} = 0$.

2.7 Proof of Theorem 1.2. An almost complete intersection ideal of grade 3 is by [10, Proposition 5.2] linked to a Gorenstein ideal of grade 3, and Brown [6, Proposition 4.3] uses this to show that there exists a skew symmetric matrix T with entries in \mathfrak{m} such that \mathfrak{a} is generated by the Pfaffians $\text{Pf}_{\overline{1}}(T)$, $\text{Pf}_{\overline{2}}(T)$, $\text{Pf}_{\overline{3}}(T)$, and $\text{Pf}_{\overline{123}}(T)$. Lemma 2.5 shows that one can replace the upper left 3×3 block in T with zeroes and arrive at the asserted block matrix U .

Adopt Setup 2.1. Let $\mathcal{R} \rightarrow R$ be given by $\tau_{ij} \mapsto t_{ij}$; it makes R an \mathcal{R} -algebra and maps Pfaffians of submatrices of \mathcal{T} to the corresponding Pfaffians of submatrices of U , i.e. $\text{pf}_{\mathcal{T}}(\overline{123})$ maps to $\text{Pf}_{\overline{123}}(U)$ etc. Let \mathcal{F} be the free resolution of $\mathcal{R}/\mathfrak{a}_n$ from Theorem 2.3. As one has $R/\mathfrak{a} = \mathcal{R}/\mathfrak{a}_n \otimes_{\mathcal{R}} R$ and \mathfrak{a} has grade 3 it follows from Bruns and Vetter [7, Theorem 3.5] that

$$(0) \quad F = \mathcal{F} \otimes_{\mathcal{R}} R$$

is a free resolution of R/\mathfrak{a} over R , and it is minimal as the differentials are given by matrices with entries in \mathfrak{m} .

We now establish parts of a multiplicative structure on F : just enough to determine the multiplicative structure on the k -algebra $\text{Tor}_*^R(R/\mathfrak{a}, k)$. Let e_1, \dots, e_4 , f_1, \dots, f_n , and g_1, \dots, g_{n-3} be the standard bases for the free modules F_1 , F_2 , and F_3 . From the three obvious Koszul relations one gets

$$\begin{aligned} \partial_2(e_4 e_1) &= \text{Pf}_{\overline{123}}(U) e_1 + \text{Pf}_{\overline{1}}(U) e_4 = \partial_2(f_1), \\ \partial_2(e_4 e_2) &= \text{Pf}_{\overline{123}}(U) e_2 - \text{Pf}_{\overline{2}}(U) e_4 = \partial_2(f_2), \text{ and} \\ \partial_2(e_4 e_3) &= \text{Pf}_{\overline{123}}(U) e_3 + \text{Pf}_{\overline{3}}(U) e_4 = \partial_2(f_3). \end{aligned}$$

Thus one can set

$$(1) \quad e_4 e_1 = f_1, \quad e_4 e_2 = f_2, \quad \text{and} \quad e_4 e_3 = f_3.$$

These three products in F induce non-trivial products in $\text{Tor}_*^R(R/\mathfrak{a}, k)$. Applying Lemma A.2 the same way as in the proof of Lemma 2.5 one gets:

$$\begin{aligned} \partial_2(e_1 e_2) &= -\text{Pf}_{\overline{1}}(U)e_2 - \text{Pf}_{\overline{2}}(U)e_1 = \partial_2\left(\sum_{i=4}^n t_{3i} f_i\right), \\ \partial_2(e_2 e_3) &= \text{Pf}_{\overline{2}}(U)e_3 + \text{Pf}_{\overline{3}}(U)e_2 = \partial_2\left(\sum_{i=4}^n t_{1i} f_i\right), \text{ and} \\ \partial_2(e_3 e_1) &= -\text{Pf}_{\overline{3}}(U)e_1 + \text{Pf}_{\overline{1}}(U)e_3 = \partial_2\left(\sum_{i=4}^n t_{2i} f_i\right). \end{aligned}$$

Thus one can set

$$(2) \quad e_1 e_2 = \sum_{i=4}^n t_{3i} f_i, \quad e_2 e_3 = \sum_{i=4}^n t_{1i} f_i, \quad \text{and} \quad e_3 e_1 = \sum_{i=4}^n t_{2i} f_i.$$

The products (2) induce trivial products in $\text{Tor}_*^R(R/\mathfrak{a}, k)$. We have now accounted for all products of elements from F_1 , so R/\mathfrak{a} is of class $\mathbf{H}(3, q)$, where q denotes the dimension of the subspace $\text{Tor}_1^R(R/\mathfrak{a}, k) \cdot \text{Tor}_2^R(R/\mathfrak{a}, k)$ of $\text{Tor}_3^R(R/\mathfrak{a}, k)$; see [3, Theorem 2.1].

For $j \in \{1, 2, 3\}$ one has

$$\partial_3(e_4 f_j) = \text{Pf}_{\overline{123}}(U)f_j - e_4(\text{Pf}_{\overline{123}}(U)e_j \pm \text{Pf}_{\overline{j}}(U)e_4) = 0$$

by (1). Since ∂_3 is injective, one has

$$e_4 f_1 = e_4 f_2 = e_4 f_3 = 0.$$

For $j \in \{4, \dots, n\}$ one gets

$$\begin{aligned} \partial_3(e_4 f_j) &= \text{Pf}_{\overline{123}}(U)f_j \\ (3) \quad &+ (-1)^j e_4(\text{Pf}_{\overline{23j}}(U)e_1 + \text{Pf}_{\overline{13j}}(U)e_2 + \text{Pf}_{\overline{12j}}(U)e_3 \text{Pf}_{\overline{j}}(U)e_4) \\ &= (-1)^j (\text{Pf}_{\overline{23j}}(U)f_1 + \text{Pf}_{\overline{13j}}(U)f_2 + \text{Pf}_{\overline{12j}}(U)f_3) - \text{Pf}_{\overline{123}}(U)f_j, \end{aligned}$$

again by (1). Thus, for $n \geq 7$ one has $\partial_3(e_4 f_j) \in \mathfrak{m}^2 F_2$. In this case it follows from Lemma 2.2 and (0) that there is an element $x_j \in \mathfrak{m} F_3$ with $\partial_3(x_j) = \partial_3(e_4 f_j)$, so by injectivity of ∂_3 one has $e_4 f_j = x_j$; in particular this product induces a trivial product in $\text{Tor}_*^R(R/\mathfrak{a}, k)$. For $n = 5$ one has $j \in \{4, 5\}$ and (3) specializes to

$$\begin{aligned} \partial_3(e_4 f_4) &= t_{15} f_1 + t_{25} f_2 + t_{35} f_3 - t_{45} f_4 = \partial_3(g_2) \quad \text{and} \\ \partial_3(e_4 f_5) &= -(t_{14} f_1 + t_{24} f_2 + t_{34} f_3 + t_{45} f_5) = -\partial_3(g_1). \end{aligned}$$

As ∂_3 is injective, this shows that in this case one has

$$e_4 f_4 = g_2 \quad \text{and} \quad e_4 f_5 = -g_1.$$

To prove the assertion about the multiplicative structure on $\text{Tor}_*^R(R/\mathfrak{a}, k)$, it suffices to show that there are no further non-zero products in $\text{Tor}_1^R(R/\mathfrak{a}, k) \cdot \text{Tor}_2^R(R/\mathfrak{a}, k)$. To this end it suffices by Lemma 2.2 and (0) to show that $\partial_3(e_i f_j)$ belongs to $\mathfrak{m}^2 F_2$ for indices $1 \leq i \leq 3$ and $1 \leq j \leq n$. For $1 \leq i, j \leq 3$ one has

$$\partial_3(e_i f_j) = (-1)^i \text{Pf}_{\overline{i}}(U)f_j - e_i(\text{Pf}_{\overline{123}}(U)e_j + (-1)^{j-1} \text{Pf}_{\overline{j}}(U)e_4).$$

This is indeed in $\mathfrak{m}^2 F_2$ as $\text{Pf}_{\bar{i}}(U)$ and $\text{Pf}_{\bar{j}}(U)$ belong to \mathfrak{m}^2 , and $e_i e_j \in \mathfrak{m} F_2$ by (2). For indices $1 \leq i \leq 3$ and $4 \leq j \leq n$ one has

$$\begin{aligned} \partial_3(e_i f_j) &= (-1)^i \text{Pf}_{\bar{i}}(U) f_j \\ &\quad - e_i (-1)^{j-1} (\text{Pf}_{\overline{23j}}(U) e_1 + \text{Pf}_{\overline{13j}}(U) e_2 + \text{Pf}_{\overline{12j}}(U) e_3 + \text{Pf}_{\bar{j}}(U) e_4). \end{aligned}$$

As above $\text{Pf}_{\bar{i}}(U)$ and $\text{Pf}_{\bar{j}}(U)$ belong to \mathfrak{m}^2 , and the products $e_i e_1$, $e_i e_2$, and $e_i e_3$ belong to $\mathfrak{m} F_2$ by (2). \square

Quotients of odd type.

2.8 Remark. In Brown's work [6], the statements to the effect that all almost complete intersection ideals come from skew symmetric matrices—Propositions 4.2 and 4.3 in *loc. cit.* as cited in our proofs of Theorems 1.2 and 1.3—are separated from the descriptions of the free resolutions: Propositions 3.2 and 3.3 in *loc. cit.* The proofs of all four statements in [6] rely on the fact from [10] that almost complete intersection ideals are linked to Gorenstein ideals, but compare the proofs of [6, Propositions 3.2 and 4.2] for almost complete intersections of odd type: The linking sequence used in the description of the free resolution is different from the one used to associate a skew symmetric matrix; a change of basis argument is thus required to reconcile the two. Our explicit construction of the free resolution in the generic case, Theorems 2.3 and 2.9, allows us to avoid such issues in the proofs of Theorems 1.2 and 1.3.

2.9 Theorem. *Let $n \geq 6$ be an even number; consider the ring \mathcal{R} and the $n \times n$ matrix \mathcal{T} from 2.1. The homomorphisms given by the matrices*

$$\partial_3 = \mathcal{T}[1 \dots n; 4 \dots n],$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & -\text{pf}_{\mathcal{T}}(\overline{1234}) & \text{pf}_{\mathcal{T}}(\overline{1235}) & \cdots & -\text{pf}_{\mathcal{T}}(\overline{123n}) \\ \text{pf}_{\mathcal{T}}(\overline{13}) & -\text{pf}_{\mathcal{T}}(\overline{23}) & 0 & \text{pf}_{\mathcal{T}}(\overline{34}) & -\text{pf}_{\mathcal{T}}(\overline{35}) & \cdots & \text{pf}_{\mathcal{T}}(\overline{3n}) \\ -\text{pf}_{\mathcal{T}}(\overline{12}) & 0 & \text{pf}_{\mathcal{T}}(\overline{23}) & -\text{pf}_{\mathcal{T}}(\overline{24}) & \text{pf}_{\mathcal{T}}(\overline{25}) & \cdots & -\text{pf}_{\mathcal{T}}(\overline{2n}) \\ 0 & \text{pf}_{\mathcal{T}}(\overline{12}) & -\text{pf}_{\mathcal{T}}(\overline{13}) & \text{pf}_{\mathcal{T}}(\overline{14}) & -\text{pf}_{\mathcal{T}}(\overline{15}) & \cdots & \text{pf}_{\mathcal{T}}(\overline{1n}) \end{pmatrix},$$

and

$$\partial_1 = (\text{pf}_{\mathcal{T}} \quad \text{pf}_{\mathcal{T}}(\overline{12}) \quad \text{pf}_{\mathcal{T}}(\overline{13}) \quad \text{pf}_{\mathcal{T}}(\overline{23}))$$

define an exact sequence

$$\mathcal{F} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}.$$

That is, denoting by \mathfrak{A}_n the ideal generated by the entries of ∂_1 , the complex \mathcal{F} is a free resolution of $\mathcal{R}/\mathfrak{A}_n$. Moreover, the ideal \mathfrak{A}_n is perfect of grade 3.

Proof. The proof of Theorem 2.3 applies, one only needs to replace the references to C.1–C.4 with references to C.5–C.8. \square

2.10 Lemma. *Let $n \geq 6$ be an even number and $T = (t_{ij})$ an $n \times n$ skew symmetric matrix with entries in a commutative ring R . Let U be the matrix obtained from T by replacing the upper left 3×3 block by a block of zeros; see Lemma 2.5. There is an equality of ideals in R*

$$(\text{Pf}(T), \text{Pf}_{\overline{12}}(T), \text{Pf}_{\overline{13}}(T), \text{Pf}_{\overline{23}}(T)) = (\text{Pf}(U), \text{Pf}_{\overline{12}}(U), \text{Pf}_{\overline{13}}(U), \text{Pf}_{\overline{23}}(U)).$$

Proof. First notice that one has

$$(1) \quad \text{Pf}_{\overline{12}}(T) = \text{Pf}_{\overline{12}}(U), \quad \text{Pf}_{\overline{13}}(T) = \text{Pf}_{\overline{13}}(U), \quad \text{and} \quad \text{Pf}_{\overline{23}}(T) = \text{Pf}_{\overline{23}}(U).$$

Lemma A.2 applied with $u_1 \dots u_k = 1 \dots n$ and $\ell = 1$ yields

$$(2) \quad \begin{aligned} \text{Pf}(T) &= \sum_{i=2}^n t_{1i} (-1)^i \text{Pf}_{\overline{1i}}(T) \\ &= t_{12} \text{Pf}_{\overline{12}}(T) - t_{13} \text{Pf}_{\overline{13}}(T) + \sum_{i=4}^n t_{1i} (-1)^i \text{Pf}_{\overline{1i}}(T). \end{aligned}$$

For $i \geq 4$ the same lemma applied with $u_1 \dots u_k = 2 \dots n \setminus i$ and $\ell = 2$ yields

$$(3) \quad \text{Pf}_{\overline{1i}}(T) = t_{23} \text{Pf}_{\overline{123i}}(T) + \text{Pf}_{\overline{1i}}(U).$$

From (2), (3), and further applications of A.2 one now gets

$$(4) \quad \begin{aligned} \text{Pf}(T) - t_{12} \text{Pf}_{\overline{12}}(T) + t_{13} \text{Pf}_{\overline{13}}(T) \\ &= t_{23} \sum_{i=4}^n t_{1i} (-1)^i \text{Pf}_{\overline{123i}}(T) + \sum_{i=4}^n t_{1i} (-1)^i \text{Pf}_{\overline{1i}}(U) \\ &= t_{23} \text{Pf}_{\overline{23}}(T) + \text{Pf}(U). \end{aligned}$$

The asserted equality of ideals is immediate from (1) and (4). \square

2.11 Proposition. *Let $n \geq 6$ be an even number. Consider the ring \mathcal{R} and the $n \times n$ matrix \mathcal{T} from 2.1 as well as the ideal \mathfrak{A}_n from Theorem 2.9. Let \mathcal{U} be the matrix obtained from \mathcal{T} by replacing the upper left 3×3 block by a block of zeros; see Proposition 2.6. The homomorphisms given by the matrices*

$$\begin{aligned} \partial_3 &= \mathcal{U}[1, \dots, n; 4, \dots, n], \\ \partial_2 &= \begin{pmatrix} 0 & 0 & 0 & -\text{pf}_{\mathcal{U}}(\overline{1234}) & \text{pf}_{\mathcal{U}}(\overline{1235}) & \cdots & -\text{pf}_{\mathcal{U}}(\overline{123n}) \\ \text{pf}_{\mathcal{U}}(\overline{13}) & -\text{pf}_{\mathcal{U}}(\overline{23}) & 0 & \text{pf}_{\mathcal{U}}(\overline{34}) & -\text{pf}_{\mathcal{U}}(\overline{35}) & \cdots & \text{pf}_{\mathcal{U}}(\overline{3n}) \\ -\text{pf}_{\mathcal{U}}(\overline{12}) & 0 & \text{pf}_{\mathcal{U}}(\overline{23}) & -\text{pf}_{\mathcal{U}}(\overline{24}) & \text{pf}_{\mathcal{U}}(\overline{25}) & \cdots & -\text{pf}_{\mathcal{U}}(\overline{2n}) \\ 0 & \text{pf}_{\mathcal{U}}(\overline{12}) & -\text{pf}_{\mathcal{U}}(\overline{13}) & \text{pf}_{\mathcal{U}}(\overline{14}) & -\text{pf}_{\mathcal{U}}(\overline{15}) & \cdots & \text{pf}_{\mathcal{U}}(\overline{1n}) \end{pmatrix} \end{aligned}$$

and

$$\partial_1 = (\text{pf}_{\mathcal{U}} \quad \text{pf}_{\mathcal{U}}(\overline{12}) \quad \text{pf}_{\mathcal{U}}(\overline{13}) \quad \text{pf}_{\mathcal{U}}(\overline{23}))$$

define a free resolution $\mathcal{L} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}$ of $\mathcal{R}/\mathfrak{A}_n$. In particular, the ideal \mathfrak{A}_n is generated by $\text{pf}_{\mathcal{U}}$, $\text{pf}_{\mathcal{U}}(\overline{12})$, $\text{pf}_{\mathcal{U}}(\overline{13})$, and $\text{pf}_{\mathcal{U}}(\overline{23})$.

Proof. By Lemma 2.10 one has

$$(\text{pf}_{\mathcal{U}}, \text{pf}_{\mathcal{U}}(\overline{12}), \text{pf}_{\mathcal{U}}(\overline{13}), \text{pf}_{\mathcal{U}}(\overline{23})) = (\text{pf}_{\mathcal{T}}, \text{pf}_{\mathcal{T}}(\overline{12}), \text{pf}_{\mathcal{T}}(\overline{13}), \text{pf}_{\mathcal{T}}(\overline{23})) = \mathfrak{A}_n.$$

As in the proof of Proposition 2.6, we proceed to show how the free resolution \mathcal{L} is obtained from the resolution \mathcal{F} from Theorem 2.9. To distinguish the differentials on the resolutions we introduce superscripts \mathcal{L} and \mathcal{F} . Consider the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\tau_{12} & 1 & 0 & 0 \\ \tau_{13} & 0 & 1 & 0 \\ -\tau_{23} & 0 & 0 & 1 \end{pmatrix} \quad \text{with inverse} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_{12} & 1 & 0 & 0 \\ -\tau_{13} & 0 & 1 & 0 \\ \tau_{23} & 0 & 0 & 1 \end{pmatrix}.$$

Notice that one has

$$(1) \quad \text{pf}_{\mathcal{T}}(\overline{12}) = \text{pf}_{\mathcal{U}}(\overline{12}), \quad \text{pf}_{\mathcal{T}}(\overline{13}) = \text{pf}_{\mathcal{U}}(\overline{13}), \quad \text{and} \quad \text{pf}_{\mathcal{T}}(\overline{23}) = \text{pf}_{\mathcal{U}}(\overline{23}).$$

As in the proof of Lemma 2.10 one gets

$$(2) \quad \text{pf}_{\mathcal{T}} = \tau_{23} \text{pf}_{\mathcal{T}}(\overline{23}) + \text{pf}_{\mathcal{U}}.$$

These identities yield $\partial_1^{\mathcal{L}} = \partial_1^{\mathcal{F}} S$. Thus, the matrices $\partial_3^{\mathcal{F}}$, $S^{-1}\partial_2^{\mathcal{F}}$, and $\partial_1^{\mathcal{F}} S$ determine a free resolution of $\mathcal{R}/\mathfrak{A}_n$. As the matrices $\partial_3^{\mathcal{L}}$ and $\partial_3^{\mathcal{F}}$ agree, it suffices to show that $\partial_2^{\mathcal{L}} = S^{-1}\partial_2^{\mathcal{F}}$ holds.

By (1) the first three columns of the matrices $\partial_2^{\mathcal{L}}$, $\partial_2^{\mathcal{F}}$, and $S^{-1}\partial_2^{\mathcal{F}}$ agree. For indices $1 \leq i \leq n$ one has $\text{pf}_{\mathcal{T}}(\overline{123i}) = \text{pf}_{\mathcal{U}}(\overline{123i})$, so also the top rows in the matrices $\partial_2^{\mathcal{L}}$, $\partial_2^{\mathcal{F}}$, and $S^{-1}\partial_2^{\mathcal{F}}$ agree. Now fix $i \in \{4, \dots, n\}$. The $(4, i)$ entry in the matrix $S^{-1}\partial_2^{\mathcal{F}}$ is

$$\tau_{23}(-1)^{i-1} \text{pf}_{\mathcal{T}}(\overline{123i}) + (-1)^i \text{pf}_{\mathcal{T}}(\overline{1i}).$$

To see that this is indeed $(-1)^{i-1} \text{pf}_{\mathcal{U}}(\overline{123i})$, the $(4, i)$ entry in $\partial_2^{\mathcal{L}}$, apply Lemma A.2 with $u_1 \dots u_k = 2 \dots n \setminus i$ and $\ell = 2$ to get

$$\text{pf}_{\mathcal{T}}(\overline{1i}) = \tau_{23} \text{pf}_{\mathcal{T}}(\overline{123i}) + \text{pf}_{\mathcal{U}}(\overline{1i}).$$

Similar applications of Lemma A.2 yield the identities

$$\text{pf}_{\mathcal{T}}(\overline{2i}) = \tau_{13} \text{pf}_{\mathcal{T}}(\overline{123i}) + \text{pf}_{\mathcal{U}}(\overline{2i}) \quad \text{and} \quad \text{pf}_{\mathcal{T}}(\overline{3i}) = \tau_{12} \text{pf}_{\mathcal{T}}(\overline{123i}) + \text{pf}_{\mathcal{U}}(\overline{3i}),$$

which show that also the $(3, i)$ and $(2, i)$ entries in the two matrices agree. \square

2.12 Proof of Theorem 1.3. An almost complete intersection ideal of grade 3 is by [10, Proposition 5.2] linked to a Gorenstein ideal of grade 3, and Brown [6, Proposition 4.2] uses this to show that there exists a skew symmetric matrix T with entries in \mathfrak{m} such that \mathfrak{a} is generated by the Pfaffians $\text{Pf}(T)$, $\text{Pf}_{\overline{12}}(T)$, $\text{Pf}_{\overline{13}}(T)$, and $\text{Pf}_{\overline{23}}(T)$. Lemma 2.10 shows that one can replace the upper left 3×3 block in T with zeroes and arrive at the asserted block matrix U . Adopt Setup 2.1 and let \mathcal{F} be the free resolution of $\mathcal{R}/\mathfrak{A}_n$ from Theorem 2.9. As in the proof of Theorem 1.2 one sees that

$$(0) \quad F = \mathcal{F} \otimes_{\mathcal{R}} R$$

is a minimal free resolution of R/\mathfrak{a} over R .

As in the proof of Theorem 1.2 we proceed to determine enough of a multiplicative structure on F to recognize the multiplicative structure on $\text{Tor}_*^R(R/\mathfrak{a}, k)$. Let e_1, \dots, e_4 , f_1, \dots, f_n , and g_1, \dots, g_{n-3} be the standard bases for the free modules F_1 , F_2 , and F_3 . From the three obvious Koszul relations one gets

$$\begin{aligned} \partial_2(e_2e_3) &= \text{Pf}_{\overline{12}}(U)e_3 - \text{Pf}_{\overline{13}}(U)e_2 = \partial_2(-f_1), \\ \partial_2(e_3e_4) &= \text{Pf}_{\overline{13}}(U)e_4 - \text{Pf}_{\overline{23}}(U)e_3 = \partial_2(-f_3), \quad \text{and} \\ \partial_2(e_4e_2) &= \text{Pf}_{\overline{23}}(U)e_2 - \text{Pf}_{\overline{13}}(U)e_4 = \partial_2(-f_2). \end{aligned}$$

Thus one can set

$$(1) \quad e_2e_3 = -f_1, \quad e_3e_4 = -f_3, \quad \text{and} \quad e_4e_2 = -f_2.$$

These three products in F induce non-trivial products in $\text{Tor}_*^R(R/\mathfrak{a}, k)$. Repeated applications of Lemma A.2 yield:

$$\begin{aligned} \partial_2(e_1e_2) &= \text{Pf}(T)e_2 - \text{Pf}_{\overline{12}}(U)e_1 \\ &= -\left(\sum_{i=4}^n t_{3i}(-1)^i \text{Pf}_{\overline{123i}}(U)\right)e_1 + \left(\sum_{i=4}^n t_{3i}(-1)^i \text{Pf}_{\overline{3i}}(U)\right)e_2 \end{aligned}$$

$$\begin{aligned}
&= \partial_2 \left(\sum_{i=4}^n t_{3i} f_i \right), \\
\partial_2(e_1 e_3) &= \text{Pf}(T) e_3 - \text{Pf}_{\overline{13}}(U) e_3 = \partial_2 \left(\sum_{i=4}^n t_{2i} f_i \right), \text{ and} \\
\partial_2(e_1 e_4) &= \text{Pf}(T) e_4 - \text{Pf}_{\overline{23}}(U) e_1 = \partial_2 \left(\sum_{i=4}^n t_{1i} f_i \right).
\end{aligned}$$

Thus one can set

$$(2) \quad e_1 e_2 = \sum_{i=4}^n t_{3i} f_i, \quad e_1 e_3 = \sum_{i=4}^n t_{2i} f_i, \quad \text{and} \quad e_1 e_4 = \sum_{i=4}^n t_{1i} f_i.$$

The products (2) induce trivial products in $\text{Tor}_*^R(R/\mathfrak{a}, k)$. We have now accounted for all products of elements from F_1 . To prove that R/\mathfrak{a} is of class \mathbf{T} it suffices to show that all products of the form $e_i f_j$ induce the zero product in $\text{Tor}_*^R(R/\mathfrak{a}, k)$; see [3, Theorem 2.1]. To this end, it suffices by Lemma 2.2 and (0) to show that $\partial_3(e_i f_j)$ belongs to $\mathfrak{m}^2 F_2$ for all indices $1 \leq i \leq 4$ and $1 \leq j \leq n$. One has

$$\partial_3(e_i f_j) = \partial_1(e_i) f_j - \partial_2(f_j) e_i.$$

For all i one has $\partial_1(e_i) \in \mathfrak{m}^2$, and for $1 \leq j \leq 3$ also $\partial_2(f_j)$ belongs to \mathfrak{m}^2 . For $4 \leq j \leq n$ one has

$$\partial_2(f_j) e_i = (-1)^{j-1} (\text{Pf}_{\overline{123j}}(U) e_1 - \text{Pf}_{\overline{3j}}(U) e_2 + \text{Pf}_{\overline{2j}}(U) e_3 - \text{Pf}_{\overline{1j}}(U) e_4) e_i.$$

This too is in $\mathfrak{m}^2 F_2$ as the coefficients $\text{Pf}_{\overline{1j}}(U)$, $\text{Pf}_{\overline{2j}}(U)$, and $\text{Pf}_{\overline{3j}}(U)$ belong to \mathfrak{m}^2 and the product $e_1 e_i$ is in $\mathfrak{m} F_2$. \square

3. THE EQUIVARIANT FORM OF THE FORMAT $(1, 4, n, n-3)$

In this section we give an equivariant interpretation of generic four generated perfect ideals of codimension three. These ideals were already considered from a purely algebraic point of view in Section 2, and they will be treated as linear sections of Schubert varieties in Section 4.

Quotients of even type. Let $n = 2m + 3$ where m is a natural number. Consider a $2m \times 2m$ generic skew symmetric matrix $A = (c_{ij})$ and a $3 \times 2m$ generic matrix $B = (u_{ki})$. Thus we work over a ring

$$\mathcal{A} = \text{Sym}_{\mathbb{Z}}(\bigwedge^2 F \oplus F \otimes G) \cong \mathbb{Z}[c_{ij}, u_{ki}]$$

where $F = \mathbb{Z}^{2m}$ and $G = \mathbb{Z}^3$ are free \mathbb{Z} -modules. The ring \mathcal{A} has an obvious bigrading with $|c_{ij}| = (1, 0)$ and $|u_{ki}| = (0, 1)$.

3.1 Proposition. *Let $\{g_1, \dots, g_{2m}\}$ be a basis for F and set*

$$C = \sum_{1 \leq i < j \leq 2m} c_{ij} g_i \wedge g_j \quad \text{and} \quad u_k = \sum_{i=1}^{2m} u_{ki} g_i \quad \text{for } 1 \leq k \leq 3.$$

We denote by C^j the j^{th} exterior power in $\bigwedge^{2j} F$. The ideal

$$\mathcal{I}_n = (C^m, C^{m-1} \wedge u_1 \wedge u_2, C^{m-1} \wedge u_1 \wedge u_3, C^{m-1} \wedge u_2 \wedge u_3)$$

is a grade 3 almost complete intersection ideal of type $2m = n - 3$.

Proof. The exterior powers C^j have a natural description in terms of Pfaffians of the matrix A ,

$$C^j = \sum_{1 \leq i_1 < \dots < i_{2j} \leq 2m} \text{Pf}_{i_1 \dots i_{2j}}(A) \cdot g_{i_1} \wedge \dots \wedge g_{i_{2j}}.$$

Plugging these in, we see that we get the generators of the ideals described in this proposition from the matrix \mathcal{U} in Proposition 2.6 via the substitutions $c_{ij} = \tau_{i+3, j+3}$ and $u_{i,j} = \tau_{i, j+3}$. \square

Let us work out the minimal free resolution of the ideal defined above.

3.2 Proposition. *Let $n = 2m + 3$ and \mathcal{I}_n be the ideal from Proposition 3.1. The minimal graded free resolution of the cyclic \mathcal{A} -module $\mathcal{A}/\mathcal{I}_n$ is*

$$\begin{aligned} \mathcal{F}_\bullet : 0 \longrightarrow & F \otimes \left(\bigwedge^{2m} F \right)^{\otimes 2} \otimes \bigwedge^3 G \otimes \mathcal{A}(-2m+1, -3) \xrightarrow{\partial_3} \\ & \left(\bigwedge^{2m} F \right)^{\otimes 2} \otimes \bigwedge^2 G \otimes \mathcal{A}(-2m+1, -2) \oplus \bigwedge^{2m} F \otimes \bigwedge^{2m-1} F \otimes \bigwedge^3 G \otimes \mathcal{A}(-2m+2, -3) \\ & \xrightarrow{\partial_2} \bigwedge^{2m} F \otimes \mathcal{A}(-m, 0) \oplus \bigwedge^{2m} F \otimes \bigwedge^2 G \otimes \mathcal{A}(-m+1, -2) \xrightarrow{\partial_1} \mathcal{A}. \end{aligned}$$

The differentials ∂_3 , ∂_2 , ∂_1 are described in the proof below. For every field k the resolution $\mathcal{F}_\bullet \otimes_{\mathbb{Z}} k$ is minimal over

$$\mathcal{A}_k = \text{Sym}_k(\bigwedge^2 \bar{F} \oplus \bar{F} \otimes \bar{G}) \cong k[c_{ij}, u_{ki}],$$

with $\bar{F} = F \otimes_{\mathbb{Z}} k$ and $\bar{G} = G \otimes_{\mathbb{Z}} k$. The ideal $\mathcal{I}_n \otimes_{\mathbb{Z}} k$ is thus perfect of grade three.

Proof. Let us first describe the differentials in the complex \mathcal{F}_\bullet in this setting. The last differential ∂_3 is just a $(2m+3) \times 2m$ matrix with the $2m \times 2m$ block given by the matrix A and the $3 \times 2m$ block given by the matrix B . The differential ∂_2 can be expressed in block form as

$$\partial_2 = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right)$$

where A_{21} is given by multiplication by the representation $\bigwedge^{2m} F$ occurring in the degree $(m, 0)$ component of \mathcal{A} . The matrix A_{22} is given by multiplication by the representation $\bigwedge^{2m-1} F \otimes G$ occurring in the degree $(m-1, 1)$ of \mathcal{A} . The matrix A_{11} is given by multiplication by the representation $\bigwedge^{2m} F \otimes \bigwedge^2 G$ occurring in the degree $(m-1, 2)$ component of \mathcal{A} and A_{12} is given by multiplication by the representation $\bigwedge^{2m-1} F \otimes \bigwedge^3 G$ occurring in the degree $(m-2, 3)$ component of \mathcal{A} . The relations coming from the second summand are three Koszul relations between the last generator and the three others. The differential ∂_1 is given by the generators of \mathcal{I}_n .

The matrices of the differentials are:

$$\partial_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1\ 2m} \\ u_{21} & u_{22} & u_{23} & \cdots & u_{2\ 2m} \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3\ 2m} \\ 0 & c_{12} & c_{13} & \cdots & c_{1\ 2m} \\ -c_{12} & 0 & c_{23} & \cdots & c_{2\ 2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1\ 2m} & -c_{2\ 2m} & -c_{3\ 2m} & \cdots & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} -x_1 & -x_2 & -x_3 & w_1 & w_2 & \cdots & w_{2m} \\ x_4 & 0 & 0 & v_{\{2,3\}1} & v_{\{2,3\}2} & \cdots & v_{\{2,3\}2m} \\ 0 & x_4 & 0 & v_{\{1,3\}1} & -v_{\{1,3\}2} & \cdots & -v_{\{1,3\}2m} \\ 0 & 0 & x_4 & v_{\{1,2\}1} & v_{\{1,2\}2} & \cdots & v_{\{1,2\}2m} \end{pmatrix},$$

and

$$\partial_1 = (x_1 \ x_2 \ x_3 \ x_4)$$

with entries as defined below

$$x_1 = C^m, \ x_2 = C^{m-1} \wedge u_2 \wedge u_3, \ x_3 = C^{m-1} \wedge u_1 \wedge u_3, \ x_4 = C^{m-1} \wedge u_1 \wedge u_2,$$

$$v_{\{\alpha,\beta\}i} = \sum_j u_{\gamma j} \text{Pf}_{i\bar{j}}(C), \quad \text{and} \quad w_i = \sum_{j,k,l} \pm \Delta^{j,k,l} \text{Pf}_{i\bar{j}kl}(C).$$

Here $\Delta^{j,k,l}$ is a 3×3 minor of the $3 \times (2m)$ matrix B on columns j, k, l . Finally γ is the complement of $\{\alpha, \beta\}$ in the set $\{1, 2, 3\}$.

The exterior powers C^j have a natural description in terms of Pfaffians of the matrix A ,

$$C^j = \sum_{1 \leq i_1 < \cdots < i_{2j} \leq 2m} \text{Pf}_{i_1 \dots i_{2j}}(A) \cdot g_{i_1} \wedge \cdots \wedge g_{i_{2j}}.$$

Plugging these in, we see that we get the generators of the ideals described in Proposition 3.1 from the matrix \mathcal{U} from Proposition 2.6 via the substitution $c_{ij} = \tau_{i+3, j+3}$ and $u_{i,j} = \tau_{i, j+3}$. Using this substitution we see our complex is just the complex described in Proposition 2.6. \square

Notice that the representation theory dictates what the differentials should be, as each component of $\partial_3, \partial_2, \partial_1$ is determined by the equivariance property with respect to $\text{GL}(F) \times \text{GL}(G)$ up to a non-zero scalar.

Quotients of odd type. There is a nice analogue in the odd case. Let $n = 2m + 4$ where m is a natural number. Consider a $(2m+1) \times (2m+1)$ generic skew symmetric matrix $A = (c_{ij})$ and a $3 \times (2m+1)$ generic matrix $B = (u_{ki})$. Thus we work over a ring $\mathcal{A} = \text{Sym}_{\mathbb{Z}}(\bigwedge^2 F \oplus F \otimes G) \cong \mathbb{Z}[c_{ij}, u_{ki}]$ where $F = \mathbb{Z}^{2m+1}$ and $G = \mathbb{Z}^3$ are free \mathbb{Z} -modules.

3.3 Proposition. *Let $\{g_1, \dots, g_{2m+1}\}$ be a basis for F and set*

$$C = \sum_{1 \leq i < j \leq 2m+1} c_{ij} g_i \wedge g_j \quad \text{and} \quad u_k = \sum_{i=1}^{2m+1} u_{ki} g_i \quad \text{for } 1 \leq k \leq 3.$$

Again we denote by C^j the j -th exterior power of C in $\bigwedge^{2j} F$. The ideal

$$\mathcal{I}_n = (C^{m-1} \wedge u_1 \wedge u_2 \wedge u_3, C^m \wedge u_1, C^m \wedge u_2, C^m \wedge u_3)$$

is a grade 3 almost complete intersection of type $2m + 1 = n - 3$.

Proof. The exterior powers C^j have a natural description in terms of Pfaffians of the matrix A .

$$C^j = \sum_{1 \leq i_1 < \dots < i_{2j} \leq 2m} \text{Pf}_{i_1 \dots i_{2j}}(A) \cdot g_{i_1} \wedge \dots \wedge g_{i_{2j}}.$$

Plugging these in, we see that we get the generators of the ideals described in this proposition from the matrix \mathcal{U} in Proposition 2.11 via the substitution $c_{ij} = \tau_{i+3, j+3}$ and $u_{i, j} = \tau_{i, j+3}$. \square

Let us work out the minimal free resolution of the ideal defined above.

3.4 Proposition. *Let $n = 2m + 4$ and \mathcal{I}_n be the ideal from Proposition 3.3. The minimal graded free resolution of the cyclic \mathcal{A} -module $\mathcal{A}/\mathcal{I}_n$ is*

$$\begin{aligned} \mathcal{F}_\bullet : 0 \longrightarrow & F \otimes \left(\bigwedge^{2m+1} F \right)^{\otimes 2} \otimes \bigwedge^3 G \otimes \mathcal{A}(-2m, -3) \xrightarrow{\partial_3} \\ & \left(\bigwedge^{2m+1} F \right)^{\otimes 2} \otimes \bigwedge^2 G \otimes \mathcal{A}(-2m, -2) \oplus \bigwedge^{2m+1} F \otimes \bigwedge^{2m} F \otimes \bigwedge^3 G \otimes \mathcal{A}(-2m+1, -3) \\ & \xrightarrow{\partial_2} \bigwedge^{2m+1} F \otimes \bigwedge^3 G \otimes \mathcal{A}(-m+1, -3) \oplus \bigwedge^{2m+1} F \otimes G \otimes \mathcal{A}(-m, -1) \xrightarrow{\partial_1} \mathcal{A}. \end{aligned}$$

The differentials $\partial_3, \partial_2, \partial_1$ are described in the proof below. For every field k the resolution $\mathcal{F}_\bullet \otimes_{\mathbb{Z}} k$ is minimal over

$$\mathcal{A}_k = \text{Sym}_k(\bigwedge^2 \bar{F} \oplus \bar{F} \otimes \bar{G}) \cong k[c_{ij}, u_{ki}],$$

with $\bar{F} = F \otimes_{\mathbb{Z}} k$ and $\bar{G} = G \otimes_{\mathbb{Z}} k$. The ideal $\mathcal{I}_n \otimes_{\mathbb{Z}} k$ is thus perfect of grade three.

Proof. Let us describe the differentials of the resolution in this setting. The last differential ∂_3 is just a $(2m+3) \times 2m$ matrix with the $2m \times 2m$ block given by the matrix A and the $3 \times 2m$ block given by three vectors. The differential ∂_2 can be expressed in block form as

$$\partial_2 = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right)$$

where A_{21} is zero. The matrix A_{22} is given by multiplication by the representation $\bigwedge^{2m} F$ occurring in the degree $(m, 0)$ component of \mathcal{A} . The matrix A_{11} is given by multiplication by the representation $\bigwedge^{2m+1} F \otimes G$ occurring in the degree $(m, 1)$ component of \mathcal{A} and A_{12} is given by multiplication by the representation $\bigwedge^{2m} F \otimes \bigwedge^2 G$ occurring in the degree $(m-1, 2)$ component of \mathcal{A} . The differential ∂_1 is given by the generators of \mathcal{I}_n .

The matrices of differentials are:

$$\partial_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1, 2m+1} \\ u_{21} & u_{22} & u_{23} & \dots & u_{2, 2m+1} \\ u_{31} & u_{32} & u_{33} & \dots & u_{3, 2m+1} \\ 0 & c_{12} & c_{13} & \dots & c_{1, 2m+1} \\ -c_{12} & 0 & c_{23} & \dots & c_{2, 2m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1, 2m+1} & -c_{2, 2m+1} & -c_{3, 2m+1} & \dots & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & w_1 & w_2 & \cdots & w_{2m+1} \\ x_3 & x_4 & 0 & v_{11} & v_{12} & \cdots & v_{1\ 2m+1} \\ -x_2 & 0 & x_4 & v_{21} & v_{22} & \cdots & v_{2\ 2m+1} \\ 0 & -x_2 & -x_3 & v_{31} & v_{32} & \cdots & v_{3\ 2m+1} \end{pmatrix},$$

and

$$\partial_1 = (x_1 \ x_2 \ x_3 \ x_4)$$

where

$$x_1 = C^{m-1} \wedge u_1 \wedge u_2 \wedge u_3, \quad x_2 = C^m \wedge u_1, \quad x_3 = C^m \wedge u_2, \quad x_4 = C^m \wedge u_3,$$

$$w_i = \pm \text{Pf}_{\bar{i}}(C), \quad \text{and} \quad v_{\gamma i} = \sum_{j,k} \pm \Delta_{\alpha,\beta}^{j,k} \text{Pf}_{\overline{ijk}}(C).$$

Here $\Delta_{\alpha,\beta}^{j,k}$ is a 2×2 minor of the $3 \times (2m+1)$ matrix B on rows α, β and columns j, k . Finally γ is the complement of $\{\alpha, \beta\}$ in the set $\{1, 2, 3\}$.

In order to prove exactness, notice that the exterior powers C^j have a natural description in terms of Pfaffians of the matrix A .

$$C^j = \sum_{1 \leq i_1 < \cdots < i_{2j} \leq 2m} \text{Pf}_{i_1 \dots i_{2j}}(A) \cdot g_{i_1} \wedge \cdots \wedge g_{i_{2j}}.$$

Plugging these in we see that we get the generators of the ideals described in Proposition 3.3 from the matrix \mathcal{U} from Proposition 2.11, with substitutions $c_{ij} = \tau_{i+3, j+3}$ and $u_{i,j} = \tau_{i, j+3}$. Using this substitution we see our complex is just the complex described in Proposition 2.11. \square

Notice that the representation theory dictates what the differentials should be, as each component of $\partial_3, \partial_2, \partial_1$ is determined by the equivariance property with respect to $\text{GL}(F) \times \text{GL}(G)$ up to a non-zero scalar.

4. SCHUBERT VARIETIES IN ORTHOGONAL GRASSMANNIANS VS. ALMOST COMPLETE INTERSECTION AND GORENSTEIN IDEALS OF CODIMENSION 3

In this section we discuss connections between the ideals described in the previous sections and Schubert varieties in the isotropic Grassmannian of even dimensional orthogonal space. We start with a $2n$ -dimensional vector space \mathbb{W} over a field k . We denote by $Q(\cdot, \cdot)$ a non-degenerate quadratic form on \mathbb{W} that admits a hyperbolic basis $\{e_1, e_2, \dots, e_n, \bar{e}_n, \dots, \bar{e}_2, \bar{e}_1\}$. We deal with the special orthogonal group $\text{SO}(\mathbb{W})$ of isometries of \mathbb{W} of determinant 1, and its double cover $\text{Spin}(\mathbb{W})$. The maximal torus $T \cong (k^*)^n$ is contained in $\text{SO}(\mathbb{W})$ as the diagonal matrices \underline{t} acting on \mathbb{W} as follows

$$\underline{t}(e_i) = t_i e_i \quad \text{and} \quad \underline{t}(\bar{e}_i) = t_i^{-1} \bar{e}_i \quad \text{for} \quad 1 \leq i \leq n.$$

We consider the lattice of integral weights for T , which is a free \mathbb{Z} -module with coordinate basis $\{\epsilon_1, \dots, \epsilon_n\}$. We identify ϵ_i with the weight of e_i under this action; the weight of \bar{e}_i is $-\epsilon_i$.

There is an associated root system of type D_n with roots

$$\{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}.$$

Simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. If $R(D_n)$ is a \mathbb{Z} -submodule of \mathbb{Z}^n generated by roots, then the *fundamental weights* ω_i in the

dual \mathbb{Z} -module $(\mathbb{Z}^n)^*$ are generators of the dual lattice Λ , called the *weight lattice*, defined by $\omega_i(\alpha_j) = \delta_{i,j}$. We see that

$$\begin{aligned}\omega_i &= \epsilon_1 + \cdots + \epsilon_i \quad \text{for } 1 \leq i \leq n-2, \\ \omega_{n-1} &= \frac{1}{2} \sum_{i=1}^n \epsilon_i, \quad \text{and} \\ \omega_n &= \frac{1}{2} \sum_{i=1}^{n-1} \epsilon_i - \frac{1}{2} \epsilon_n.\end{aligned}$$

The action of the Weyl group. The Weyl group $W(D_n)$ acts on Λ by linear maps that permute the roots. It is a subgroup of index 2 in a hyperoctahedral group. $W(D_n)$ is generated by simple reflections s_1, s_2, \dots, s_n . For $1 \leq i \leq n-1$ the reflection s_i simply permutes ϵ_i and ϵ_{i+1} , and s_n acts as follows: $s_n(\epsilon_i) = \epsilon_i$ for $1 \leq i \leq n-2$, $s_n(\epsilon_{n-1}) = -\epsilon_n$, and $s_n(\epsilon_n) = -\epsilon_{n-1}$. It contains the permutation group $W(A_{n-1})$ on n elements generated by simple reflections s_1, \dots, s_{n-2}, s_n .

Over the field of complex numbers, one can classify representations of the group $\mathrm{SO}(\mathbb{W})$ and its double cover $\mathrm{Spin}(\mathbb{W})$. First, the category of representations of the Spin group is semi-simple, so every representation is a direct sum of irreducible ones. The irreducible representations are so-called highest weight representations $V(\lambda)$, where $\lambda = \sum_{i=1}^n \lambda_i \omega_i$ is an integral linear combination of fundamental weights with non-negative coefficients λ_i . The representations of $\mathrm{Spin}(\mathbb{W})$ are direct sums of only those irreducibles $V(\lambda)$ for which λ written in terms of ϵ_i 's belongs to Λ . Over other fields, and over \mathbb{Z} , one can define appropriate analogues of highest weight representations.

We are interested in two particular representations: the half-spinor representations $V(\omega_{n-1})$ and $V(\omega_n)$. They are closely connected, as we will show, to the space of skew symmetric matrices. To that end we recall some generalities about homogeneous spaces.

Let us work over an algebraically closed field k . Let G be a reductive algebraic group and let $P_i \subset G$ be a parabolic subgroup stabilizing a fundamental weight $\omega_i \in \Lambda$. It is well-known that there is a canonical embedding of G/P_i into $\mathbb{P}(V(\omega_i))$. To describe this embedding, consider the Weyl group W , which naturally acts on Λ , and in it the stabilizer W_{ω_i} of the i^{th} fundamental weight. For each $w \in W/W_{\omega_i}$, let $\dot{w} \in W$ be the unique minimal length representative. There is a cell decomposition

$$G/P_i = \bigsqcup_{w \in W/W_{\omega_i}} B\dot{w}P_i$$

called the Bruhat decomposition, where B is the Borel subgroup contained in P_i . The embedding $G/P_i \hookrightarrow \mathbb{P}(V(\omega_i))$ is given by $b\dot{w} \mapsto [b\dot{w}\omega_i]$. In fact, we know that $G/P = \overline{G \cdot v_{\omega_i}}$, where v_{ω_i} is the highest weight vector in $V(\omega_i)$.

The cardinality of W/W_{ω_i} is the same as the cardinality of the orbit $W \cdot \omega_i$. Now, if the fundamental weight ω_i is *minuscule*, then this number coincides with the dimension $\dim_k V(\omega_i)$ of the fundamental representation. This implies that the Bruhat graph of the Bruhat interval in the Coxeter group (W, S) corresponding to the minimal length representatives of the elements in W/W_{ω_i} coincides with the *crystal graph* associated to the representation $V(\omega_i)$.

Throughout the rest of this section we are interested in the case of a root system of type D_n and the parabolic subgroup P_{n-1} , the homogeneous space $\mathrm{Spin}(2n)/P_{n-1}$ is one of the two connected components of the isotropic Grassmannian $\mathrm{OGr}(n, 2n)$.

It is well-known, see for example Laksmibai and Raghavan [18, Section 3.3], that the homogeneous coordinate ring of the connected component of $\text{OGr}(n, 2n)$, considered as a projective subvariety of the projective space $\mathbb{P}(V(\omega_{n-1}))$, has a decomposition

$$k[\text{OGr}(n, 2n)] = \bigoplus_{d \geq 0} V(d\omega_{n-1}).$$

into irreducible representations of $\text{Spin}(\mathbb{W})$, so each graded component of this ring is irreducible. The half-spinor representation $V(\omega_{n-1})$ is a representation of dimension 2^{n-1} whose weights with respect to the Cartan subalgebra are $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with an even number of minuses. It has a twin representation $V(\omega_n)$ of dimension 2^{n-1} whose weights with respect to the Cartan subalgebra are $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with an odd number of minuses. Both half-spinor representations are constructed from the Clifford algebra of the quadratic form Q .

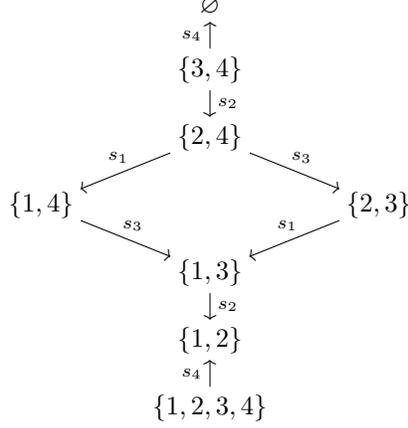
It is also known—Kostant’s Theorem, see Garfinkle’s dissertation [14]—that as a factor of $\text{Sym}_k(V(\omega_{n-1}))$ the coordinate ring $k[\text{OGr}(n, 2n)]$ is generated by quadratic equations. The generators of $k[\text{OGr}(n, 2n)]$ are the spinor coordinates; they can be indexed by the cosets $W(D_n)/W(A_{n-1})$. We denote by q_w the spinor coordinate corresponding to $w \in W(D_n)/W(A_{n-1})$; the Schubert varieties in $\text{OGr}(n, 2n)$ are also indexed by $W(D_n)/W(A_{n-1})$. There is a natural partial order on these coordinates, which in the case of Schubert varieties corresponds to the inclusion order. This partially ordered set has two combinatorial interpretations; it is a set of 2^{n-1} elements.

The first interpretation of $W(D_n)/W(A_{n-1})$ is as the set \mathcal{PE}_n of even cardinality subsets of $\{1, 2, \dots, n\}$. The Weyl group $W(D_n)$ acts on this set as follows. For $1 \leq i \leq n-1$ the simple reflection s_i acts by switching the numbers i and $i+1$. This means the subset is fixed by s_i if it contains both or none of the numbers i and $i+1$. The reflection s_n acts non-trivially only on subsets either containing or not intersecting the subset $\{n-1, n\}$. It either adds the numbers $n-1$ and n or takes them away. For a subset $I \in \mathcal{PE}_n$ let $\ell(I)$ denote the length of a minimal representative of the corresponding coset in $W(D_n)/W(A_{n-1})$. For a reflection s_i such that $s_i(I) \neq I$ one can prove that $\ell(I) = \ell(s_i(I)) \pm 1$. The partial order is generated by comparing I and $s_i(I)$ according to the length. In the case at hand, there is a concrete description: The induced partial order \mathcal{PE}_n compares subsets of a given cardinality as usual by setting

$$\{i_1, \dots, i_r\} \leq \{i_1, \dots, i_{j-1}, i_j + 1, i_{j+1}, \dots, i_r\}$$

for $1 \leq i_1 < i_2 < \dots < i_r$ and $i_j + 1 < i_{j+1}$. The partial order is generated by these inequalities together with the inequalities $\{i_1, \dots, i_r\} \leq \{i_1, \dots, i_r, n-1, n\}$ for $1 \leq i_1 < \dots < i_r < n-1$; this includes $\emptyset \leq \{n-1, n\}$.

4.1 Example. The induced partial order on \mathcal{PE}_4 is illustrated below where the arrows are directed such that $s_i(I) \leq I$ holds.



In the second interpretation one views $W(D_n)/W(A_{n-1})$ as a $W(D_n)$ -orbit of the weight—thought of as assigning an integer to each node of the Dynkin diagram D_n —under the natural action of $W(D_n)$ on these weights. The action of the simple reflection s_i on a weight

$$\begin{array}{cccccc}
 & & & & & a_{n-1} \\
 & & & & & \uparrow \\
 a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & \\
 & & & & & \downarrow \\
 & & & & & a_n
 \end{array}$$

changes a_i to $-a_i$ and adds a_i to the value at all neighboring nodes. The partial order is generated by setting $s_i(w) > w$ if and only if $s_i(w) \neq w$ and the node $w(i)$ is positive.

4.2 Example. The bijection between the set \mathcal{PE}_4 and the $W(D_4)$ -orbit of the weight

$$w = \begin{matrix} 0 \\ 0 & 0 \\ 1 \end{matrix}$$

is as follows

$$\begin{array}{ll}
 \emptyset \leftrightarrow w = \begin{matrix} 0 \\ 0 & 0 \\ 1 \end{matrix} & \{3, 4\} \leftrightarrow s_4(w) = \begin{matrix} 0 \\ 0 & 1 \\ -1 \end{matrix} \\
 \{2, 4\} \leftrightarrow s_2 s_4(w) = \begin{matrix} 1 & -1 \\ & 0 \end{matrix} & \{1, 4\} \leftrightarrow s_1 s_2 s_4(w) = \begin{matrix} -1 & 0 \\ & 1 \\ 0 \end{matrix} \\
 \{2, 3\} \leftrightarrow s_3 s_2 s_4(w) = \begin{matrix} -1 \\ 1 & 0 \\ 0 \end{matrix} & \{1, 3\} \leftrightarrow \begin{matrix} s_3 s_1 s_2 s_4(w) \\ s_1 s_3 s_2 s_4(w) \end{matrix} = \begin{matrix} -1 & -1 \\ & 1 \\ 0 \end{matrix} \\
 \{1, 2\} \leftrightarrow s_2 s_3 s_1 s_2 s_4(w) = \begin{matrix} 0 & -1 \\ & 0 \\ 1 \end{matrix} & \{1, 2, 3, 4\} \leftrightarrow s_4 s_2 s_3 s_1 s_2 s_4(w) = \begin{matrix} 0 \\ 0 & 0 \\ -1 \end{matrix}
 \end{array}$$

Notice that this bijection commutes with the Weyl group action and preserves the associated partial order.

Schubert varieties. It is known, see for example [18, Section 3.3], that the defining ideal in $k[\text{OGr}(n, 2n)]$ of every Schubert variety Ω_w is generated by spinor coordinates q_v for $v \not\leq w$ in the associated partial order. In our case this translates as follows. Consider the big open cell Y in $\text{OGr}(n, 2n)$ consisting of points with

Plücker coordinate $p_{id} \neq 0$. Recall the hyperbolic basis $\{e_1, \dots, e_n, \bar{e}_n, \dots, \bar{e}_1\}$ of \mathbb{W} . To every subspace $V \in \text{OGr}(n, 2n)$ and every basis $\{v_1, \dots, v_n\}$ of V we associate an $n \times 2n$ matrix M whose i^{th} row consists of the coordinates of the vector v_i written in the basis $\{e_1, \dots, e_n, \bar{e}_n, \dots, \bar{e}_1\}$. The big open cell Y in $\text{OGr}(n, 2n)$ discussed above consists of subspaces V such that for every basis $\{v_1, \dots, v_n\}$ of V the corresponding matrix M has a minor corresponding to columns e_1, \dots, e_n not equal to zero. The set Y is an affine space of dimension $\binom{n}{2}$ as for $V \in Y$ we can find a unique basis of V such that the corresponding matrix has a form

$$M = \left(\begin{array}{ccccc|ccccc} 0 & 0 & \cdots & 0 & 1 & 0 & x_{12} & \cdots & x_{1n-1} & x_{1n} \\ 0 & 0 & \cdots & 1 & 0 & -x_{12} & 0 & \cdots & x_{2n-1} & x_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & -x_{1n-1} & -x_{2n-1} & \cdots & 0 & x_{n-1n} \\ 1 & 0 & \cdots & 0 & 0 & -x_{1n} & -x_{2n} & \cdots & -x_{n-1n} & 0 \end{array} \right).$$

We refer to the skew symmetric $n \times n$ block as X . The restrictions to Y of the spinor coordinates correspond to sub-Pfaffians of X of all possible sizes; see for example Manivel [19]. More precisely, the weights of the half-spinor representation correspond to the subsets I of the set $\{1, \dots, n\}$ of even cardinality. For a given I , the corresponding weight is a vector $w_I = (\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with n coordinates and an even number of minuses occurring in the positions determined by I . The corresponding spinor coordinate q_I restricts to Y as the Pfaffian of the skew symmetric matrix obtained by picking from a generic $n \times n$ skew symmetric matrix the rows and columns determined by I . Thus, the quadratic equations generating the defining ideal of the homogeneous coordinate ring $k[\text{OGr}(n, 2n)]$ are just the quadratic equations in Pfaffians of all sizes of a generic skew symmetric matrix.

There are more facts that are known about Schubert varieties, the reference for this is [18, Chapter 7]. The half-spinor representation $V(\omega_{n-1})$ is an example of so-called *minuscule* representation. This means all its weight vectors are in one $W(D_n)$ -orbit. This implies that the defining ideals of Schubert varieties and their unions behave in the optimal way described below. For each cofinal subset U of the partially ordered set \mathcal{PE}_n we consider the ideal J_U in $k[Y]$, the coordinate ring of Y , generated by the spinor coordinates from that subset. This set of ideals forms a distributive lattice L_1 with the join and meet operations given by $+$ and \cap . On the other hand we can form a lattice L_2 of the cofinal subsets in \mathcal{PE}_n with operations of join and meet given by \cup and \cap . The first part of the next statement follows from [18, Section 7.2]; the assertion about J_U being radical follows from Brion and Kumar [5, Corollary 2.3.3].

4.3 Proposition. *The lattices L_1 and L_2 are isomorphic. Moreover, the ideal J_U is the defining ideal of the union of the Schubert varieties it defines set-theoretically. Thus all ideals J_U are radical.*

Notice also that if we change the half-spinor representation $V(\omega_{n-1})$ to the other one, i.e. $V(\omega_n)$ then the lattice of Schubert varieties will change to the poset \mathcal{PO}_n of odd sized subsets of $\{1, \dots, n\}$. The action of the Weyl group $W(D_n)$ and the poset structure are similar to those on \mathcal{PE}_n . We refer the reader to [18, Section 7.2].

To give an interpretation of the Schubert varieties of codimension 3 in concrete terms, we adopt the notation from the notes by Coskun [12, Lecture 5]. Let

$\text{OGr}(n, 2n)$ be one of the two connected components of the orthogonal Grassmanian of n -dimensional isotropic subspaces in a $2n$ -dimensional vector space \mathbb{W} . As above denote by Q a non-degenerate quadratic form on \mathbb{W} that admits a hyperbolic basis. Fix an isotropic flag

$$F_\bullet : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F_n^\perp \subset F_{n-1}^\perp \subset \cdots \subset F_1^\perp \subset \mathbb{W}.$$

Here F_n is isotropic and F_i^\perp denotes the orthogonal complement of F_i . The Schubert varieties in $\text{OGr}(n, 2n)$ are parameterized by sequences λ ,

$$n-1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_s \geq 0,$$

of strictly decreasing integers where s has the same parity as n ; notice that $s \leq n$ holds. The sequence λ determines a unique sequence $\tilde{\lambda}$ of strictly decreasing integers

$$n-1 \geq \tilde{\lambda}_{s+1} > \tilde{\lambda}_{s+2} > \cdots > \tilde{\lambda}_n \geq 0$$

satisfying the condition that there is no i, j such that $\lambda_i + \tilde{\lambda}_j = n-1$. In other words, we obtain $\tilde{\lambda}$ by removing from the sequence $n-1, n-2, \dots, 0$ the numbers $n-1-\lambda_1, \dots, n-1-\lambda_s$.

The Schubert variety $\Omega_\lambda = \Omega_\lambda(F_\bullet)$ is defined as the closure of the locus

$$\Omega_\lambda^{(0)}(F_\bullet) = \left\{ V \in \text{OGr}(n, 2n) \left| \begin{array}{l} \dim_k(V \cap F_{n-\lambda_i}) = i, \text{ for } 1 \leq i \leq s \\ \dim_k(V \cap F_{\tilde{\lambda}_j}^\perp) = j, \text{ for } s < j \leq n \end{array} \right. \right\}.$$

The codimension of Ω_λ is $|\lambda| = \sum_i \lambda_i$. The cells $\Omega_\lambda^{(0)}(F_\bullet)$ are exactly the orbits of the Borel subgroup B of the spin group $\text{Spin}(2n)$ acting on the connected component of $\text{OGr}(n, 2n)$.

4.4 Remark. In order to connect with the previous description, let us indicate how the partitions λ translate to the subsets \mathcal{PE}_n and \mathcal{PO}_n . The partition $(\lambda_1, \dots, \lambda_s)$ such that

$$n-1 \geq \lambda_1 > \dots > \lambda_s \geq 0$$

corresponds to the set $w(\lambda)$ of s minuses in places $n-\lambda_1, n-\lambda_2, \dots, n-\lambda_s$. This set is either in \mathcal{PE}_n or \mathcal{PO}_n depending on parity of n .

The variety $\Omega_\lambda(F_\bullet)$ in Coskun's notation is then equal to the variety $\Omega_{w(\lambda)}$ in the notation of this section.

It is well-known—see for example the works [20, 21, 22, 23] by Mehta, Ramanan, Ramanathan, and Srinivas—that the Schubert varieties are defined over \mathbb{Z} and are normal and arithmetically Cohen-Macaulay and so are the affine varieties $Y_\lambda = \Omega_\lambda \cap Y$. Our goal in this section is to explicitly describe the varieties Y_λ of codimension 3 in the affine space Y , i.e. the subvarieties Y_λ such that $|\lambda| = 3$.

Spinor coordinates. Finally we describe the bijection between the spinor coordinates and the Pfaffians of the matrix X .

For n even and $I \in \mathcal{PE}_n$ the corresponding spinor coordinate q_I is a square root of the minor of the matrix M with columns corresponding to e_i with $i \in I$ and \bar{e}_I with $i \notin I$. This is the Pfaffian of the submatrix of X obtained by removing the rows and columns with indices $n+1-i$ for $i \in I$. The first spinor coordinate corresponds to the subset \emptyset , and it is the Pfaffian of X .

For n odd we consider the elements $I \in \mathcal{PO}_n$. The corresponding spinor coordinate q_I is the Pfaffian of the submatrix of X obtained by removing the rows and columns with indices $n+1-i$ for $i \in I$. The first spinor coordinate corresponds to

the subset $\{n\}$, and it is the Pfaffian of the submatrix obtained from X by removing the first row and column, i.e. the matrix $X[\bar{1}; \bar{1}]$ in the notation from 1.1.

The case of even n . Let us intersect our Schubert varieties with the open cell Y .

For $n = 2$ there are evidently no Schubert varieties of codimension 3, but for $n \geq 4$ there are precisely two of them, namely $\Omega_{(3,0)}$ and $\Omega_{(2,1)}$. For $n = 4$ it is easy to see they are both complete intersections, one given by $x_{12} = x_{13} = x_{14} = 0$ and the other by $x_{12} = x_{13} = x_{23} = 0$. We now assume $n \geq 6$.

We start with the intersection $Y_{(3,0)}$. The rank conditions related to the flags F_j^\perp are easily seen to be empty because our subspace is isotropic. The condition $\dim_k(V \cap F_{n-3}) \geq 1$ means exactly that the rank of the submatrix $X[1 \dots n; 4 \dots n]$ has to be less than $n - 3$. This is the matrix of the third differential of the almost complete intersection ideal of format $(1, 4, n, n - 3)$ described in Theorem 2.3. Note that the other condition $\dim_k(V \cap F_n) \geq 2$ holds as the matrix X has to be singular and therefore of rank at most $n - 2$. So the defining ideal of the Schubert variety $Y_{(3,0)}$ is almost complete intersection of odd type.

We turn to the intersection $Y_{(2,1)}$. Again the rank conditions related to the flags F_j^\perp are empty because our subspace is isotropic, so we get the conditions $\dim_k(V \cap F_{n-2}) \geq 1$ and $\dim_k(V \cap F_{n-1}) \geq 2$. The first condition is now redundant. The second condition means that the rank of the submatrix of $X[1 \dots n; 2 \dots n]$ has to be at most $n - 3$. This means that the submaximal Pfaffians $\text{Pf}_{\bar{i}}(X)$ of the matrix $X[\bar{1}; \bar{1}]$ vanish for $2 \leq i \leq n$. It follows that the rank of this matrix is at most $n - 4$, so the rank of $X[1 \dots n; 2 \dots n]$ is at most $n - 3$. We conclude that $Y_{(2,1)}$ is the subvariety given by vanishing of Pfaffians $\text{Pf}_{\bar{i}}(X)$; in other words, the defining ideal is a generic Gorenstein ideal of codimension 3.

The case of odd n . For $n \leq 3$ there are evidently no Schubert varieties of codimension 3, but for $n \geq 5$ there are precisely two of them, namely $\Omega_{(3)}$ and $\Omega_{(2,1,0)}$. Let us intersect them with the open cell Y .

We start with the intersection $Y_{(3)}$. The rank conditions related to the flags F_j^\perp are easily seen to be empty because our subspace is isotropic. The condition $\dim_k(V \cap F_{n-3}) \geq 1$ means that the rank of the submatrix $X[1 \dots n; 4 \dots n]$ has to be less than $n - 3$. But this is the matrix of third differential of the almost complete intersection ideal of format $(1, 4, n, n - 3)$ described in Theorem 2.9. So the defining ideal of the Schubert variety $Y_{(3)}$ is an almost complete intersection of even type.

We turn to the intersection $Y_{(2,1,0)}$. Again the rank conditions related to the flags F_j^\perp are empty because our subspace is isotropic. This leaves us with the conditions $\dim_k(V \cap F_{n-2}) \geq 1$ and $\dim_k(V \cap F_{n-1}) \geq 2$. This just means that submaximal Pfaffians of the matrix X are zero, so we get a generic Gorenstein ideal of codimension 3.

Minimal free resolutions. Let us look at the minimal free resolutions of the coordinate rings of the codimension 3 varieties Y_λ from the point of view of Schubert varieties. The defining equations of Schubert varieties have a general description in terms of ideals J_U ; we recall its meaning in our case. The Schubert varieties of codimension 3 correspond to elements

$$w' = s_n s_{n-2} s_{n-1} \quad \text{and} \quad w'' = s_{n-3} s_{n-2} s_{n-1},$$

as these are the only two elements of length 3 in $W(D_n)/W(A_{n-1})$. The generators of the corresponding ideals are:

$(q_{\text{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}}, q_{s_{n-3}s_{n-2}s_{n-1}}, \dots, q_{s_{n-i}\dots s_{n-3}s_{n-2}s_{n-1}}, \dots, q_{s_1\dots s_{n-3}s_{n-2}s_{n-1}})$,
where there are n generators in total, and

$$(q_{\text{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}}, q_{s_n s_{n-2}s_{n-1}}).$$

Identifying these generators with the corresponding Pfaffians, we see that for n odd the first ideal generated by the submaximal Pfaffians of X . The second ideal gives the almost complete intersection ideal described in Theorem 2.3. For even n , the first ideal gives the Pfaffian of X and submaximal Pfaffians of $X[\bar{1}; \bar{1}]$; in this case the first generator is redundant. The four generator ideal gives the almost complete intersection ideal described in Theorem 2.9.

There is one more statement one can make which plays an important role. It is proved in terms of commutative algebra Lemmas C.2 and C.6. Here we give a geometric reasoning proving the statement.

4.5 Proposition. *The ideal generated by the first three spinor coordinates,*

$$(q_{\text{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}}),$$

in the partial order on the homogeneous coordinate ring $k[\text{OGr}(n, 2n)]$ of the orthogonal Grassmannian is generated by a regular sequence. Therefore, these coordinates restricted to the open cell Y also generate an ideal generated by a regular sequence in $k[Y]$. Moreover, these elements generate a radical ideal.

Proof. Let B be a Borel subgroup of the group $\text{Spin}(2n)$. The almost complete intersection ideal $(q_{\text{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}})$ in $k[\text{OGr}(n, 2n)]$ is B -equivariant. This means that its vanishing locus is a union of Schubert cells. It follows that this vanishing set consists of the closure of the union of two Schubert varieties of codimension 3. It is therefore an ideal of depth three generated by three elements. Such ideal is then generated by a regular sequence, as the ring $k[\text{OGr}(n, 2n)]$ is Cohen-Macaulay. The ideal is radical by of [5, Corollary 2.3.3]. The result for $k[Y]$ follows by localization. \square

This result means we have an occurrence of the situation described by Ulrich [25]. The ideal $(q_{\text{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}})$ is the intersection $I_{w'} \cap I_{w''}$, and thus the ideals $I_{w'}$ and $I_{w''}$ are linked via the regular sequence $(q_{\text{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}})$. This is exactly the procedure described in [6]. By this argument, we can describe the format of the resolutions of our almost complete intersections.

Set $R = k[Y]$ and $n = 2m + 2$ for some natural number m . The resolution of the Gorenstein ideal $I_{w'}$ has format

$$0 \longrightarrow R(-2m-1) \longrightarrow R^{2m+1}(-m-1) \longrightarrow R^{2m+1}(-m) \longrightarrow R;$$

we link by a regular sequence of elements of degrees m , m , and $m+1$. Looking at the mapping cone

$$\begin{array}{ccccccc} 0 & \longrightarrow & R(-2m-1) & \longrightarrow & R^{2m+1}(-m-1) & \longrightarrow & R^{2m+1}(-m) & \longrightarrow & R \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & R(-3m-1) & \longrightarrow & R^2(-2m-1) \oplus R(-2m) & \longrightarrow & R(-m-1) \oplus R^2(-m) & \longrightarrow & R \end{array}$$

we deduce that the other ideal has a resolution with the format

$$0 \longrightarrow R^{2m-1}(-2m-1) \longrightarrow R^{2m+2}(-2m) \longrightarrow R(-m-1) \oplus R^3(-m) \longrightarrow R$$

which is exactly the format of the resolution from section 2.

Let us do this calculation for odd $n = 2m + 3$. The resolution of Gorenstein ideal of codimension 3 has format

$$0 \longrightarrow R(-2m-3) \longrightarrow R^{2m+3}(-m-2) \longrightarrow R^{2m+3}(-m-1) \longrightarrow R$$

and we link by a regular sequence with degrees $m+1, m+1, m+1$. Looking at the mapping cone

$$\begin{array}{ccccccc} 0 & \longrightarrow & R(-2m-3) & \longrightarrow & R^{2m+3}(-m-2) & \longrightarrow & R^{2m+3}(-m-1) & \longrightarrow & R \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & R(-3m-3) & \longrightarrow & R^3(-2m-2) & \longrightarrow & R^3(-m-1) & \longrightarrow & R \end{array}$$

we deduce that the other ideal has a resolution with the format

$$0 \longrightarrow R^{2m}(-2m-2) \longrightarrow R^{2m+3}(-2m-1) \longrightarrow R(-m) \oplus R^3(-m-1) \longrightarrow R$$

which is exactly the format of the resolution from section Section 2.

Next we interpret the matrices of the free resolutions of almost complete intersections in terms of spinor coordinates. Before we start, let us comment on the defining ideals of the coordinate rings $k[\text{OGr}(n, 2n)]$ thought of as factors of the symmetric algebra on the half-spinor representation. By Kostant's Theorem [14] these ideals are defined by quadratic equations, therefore, they are generated by the kernel of the map

$$S_2(V(\omega_{n-1})) \longrightarrow V(2\omega_{n-1}).$$

One has the following formula, see Adams [1, p. 25],

$$S_2(V(\omega_{n-1})) = V(2\omega_{n-1}) \oplus \bigoplus_{i \geq 1} V(\omega_{n-4i}).$$

We use the notation from Section 3 for the differentials in our complexes.

Let us start with the case of odd $n = 2m + 3$. The generators of our ideal in terms of Plücker coordinates are

$$\begin{aligned} x_1 &= p_{\bar{1}, \bar{2}, \dots, \bar{2m}, 2m+1, 2m+2, 2m+3} \\ x_2 &= p_{\bar{1}, \bar{2}, \dots, \bar{2m}, 2m+1, \bar{2m+2}, \bar{2m+3}} \\ x_3 &= p_{\bar{1}, \bar{2}, \dots, \bar{2m}, \bar{2m+1}, 2m+2, \bar{2m+3}} \\ x_4 &= p_{\bar{1}, \bar{2}, \dots, \bar{2m}, \bar{2m+1}, \bar{2m+2}, 2m+3}. \end{aligned}$$

The entries of the second differential ∂_2 are as follows. The element w_i is the Plücker coordinate with $2m+2$ bars, the only number without bar is $2m+1-i$. The element $v_{\{\alpha, \beta\}i}$ is a Plücker coordinate with $2m$ bars. The numbers without bars are $2m+1-i, 2m+\alpha, 2m+\beta$.

The entries of the matrix ∂_3 are also Plücker coordinates. The element $u_{\alpha i}$ is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+4-\alpha$. The element c_{ij} is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+1-j$.

The gradings of the basis vectors in the modules of the complex are: The basis element in $F_0 = R$ has weight (0^{2m+3}) . In the following we use 1^m to denote $(1, 1, \dots, 1)$ with m coordinates. The basis elements in $F_1 = R \oplus R^3$ have weights $((\frac{1}{2})^{2m}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $((\frac{1}{2})^{2m}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $((\frac{1}{2})^{2m}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $((\frac{1}{2})^{2m}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

The basis elements in $F_2 = R^{2m} \oplus R^3$ have weights

$$(1^{2m-1}, 0, 0^3), (1^{2m-2}, 0, 1, 0^3), \dots, (0, 1^{2m-1}, 0^3), \\ (1^{2m}, 0, 0, -1), (1^{2m}, 0, -1, 0), (1^{2m}, -1, 0, 0).$$

Finally the basis vectors in $F_3 = R^{2m}$ have weights $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, (-\frac{1}{2})^3)$.

The composite $\partial_1 \partial_2$ is easily explained. It is a $1 \times (2m+3)$ matrix. Its entry in the i^{th} row is the Plücker coordinate with the weight with $2m-1$ entries of -1 's and 4 zeros in positions $2m+1-i, 2m+1, 2m+2, 2m+3$. This entry is zero because it corresponds to the extremal weight vector in the representation $V(\omega_{2m-1})$ which occurs in the 2^{nd} symmetric power of $V(\omega_{2m+2})$.

The composite $\partial_2 \partial_3$ is a $4 \times 2m$ matrix with the weights in the first row being $(0, 0, \dots, 0, -1, 0, \dots, 0, 0, 0, 0)$, where -1 appears in positions $1, \dots, 2m$; the entries in the second row are $(0, 0, \dots, 0, -1, 0, \dots, 0, 0, 1, 1)$, where -1 appears in positions $1, \dots, 2m$; the entries in the third row are $(0, 0, \dots, 0, -1, 0, \dots, 0, 1, 0, 1)$, where -1 appears in positions $1, \dots, 2m$, finally the entries in the fourth row are $(0, 0, \dots, 0, -1, 0, \dots, 0, 1, 1, 0)$, where -1 appears in positions $1, \dots, 2m$.

The interpretation of the identity $\partial_2 \partial_3 = 0$ from this point of view requires further analysis.

Similarly we treat the even case $n = 2m+4$. The generators of the ideal $I_{w''}$ in terms of Plücker coordinates are

$$x_1 = p_{\overline{1}, \overline{2}, \dots, \overline{2m+4}} \\ x_2 = p_{\overline{1}, \overline{2}, \dots, \overline{2m}, \overline{2m+1}, \overline{2m+2}, 2m+3, 2m+4} \\ x_3 = p_{\overline{1}, \overline{2}, \dots, \overline{2m}, \overline{2m+1}, 2m+2, \overline{2m+3}, 2m+4} \\ x_4 = p_{\overline{1}, \overline{2}, \dots, \overline{2m}, \overline{2m+1}, 2m+2, 2m+3, \overline{2m+4}}.$$

The entries of the second differential ∂_2 are as follows. The element w_i is the Plücker coordinate with $2m$ bars, the only numbers without bar are $2m+2-i$ and $2m+2, 2m+3, 2m+4$. The element $v_{\alpha i}$ is a Plücker coordinate with $2m+2$ bars. The numbers without bars are $2m+2-i, 2m+1+\alpha$.

The entries of the matrix ∂_3 are also Plücker coordinates. The element $u_{\alpha i}$ is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+4-\alpha$. The element c_{ij} is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+1-j$.

The gradings of the basis vectors in the modules of the complex are: The basis element in $F_0 = R$ has weight (0^{2m+4}) . The basis elements in $F_1 = R \oplus R^3$ have weights

$$((\frac{1}{2})^{2m+4}), ((\frac{1}{2})^{2m+1}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}), ((\frac{1}{2})^{2m+1}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), ((\frac{1}{2})^{2m+1}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}).$$

The basis elements in $F_2 = R^{2m} \oplus R^3$ have weights

$$(1^{2m}, 0, 0^3), (1^{2m-1}, 0, 1, 0^3), \dots, (0, 1^{2m}, 0^3), \\ (1^{2m+1}, 0, 0, -1), (1^{2m+1}, 0, -1, 0), (1^{2m+1}, -1, 0, 0).$$

Finally the basis vectors in $F_3 = R^{2m}$ have weights $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, (-\frac{1}{2})^3)$.

The composite $\partial_1\partial_2$ is easily explained. It is a $1 \times (2m+4)$ matrix. Its entry in the i^{th} row is the Plücker coordinate with the weight with $2m$ entries of -1 's and 4 zeros in positions $2m+2-i, 2m+2, 2m+3, 2m+4$. This entry is zero because it corresponds to the extremal weight vector in the representation $V(\omega_{2m})$ which occurs in the 2^{nd} symmetric power of $V(\omega_{2m+3})$.

The composition $\partial_2\partial_3$ is a $4 \times (2m+1)$ matrix with the weights in the first row being $(0, 0, \dots, 0, -1, 0, \dots, 0, 1, 1, 1)$, where -1 appears in positions $1, \dots, 2m+1$; the entries in the second row are $(0, 0, \dots, 0, -1, 0, \dots, 0, 1, 0, 0)$, where -1 appears in positions $1, \dots, 2m+1$; the third row entries are $(0, 0, \dots, 0, -1, 0, \dots, 0, 0, 1, 0)$, where -1 appears in positions $1, \dots, 2m+1$; finally the entries in the fourth row are $(0, 0, \dots, 0, -1, 0, \dots, 0, 0, 0, 1)$, where -1 appears in positions $1, \dots, 2m+1$.

The interpretation of the identity $\partial_2\partial_3 = 0$ from this point of view requires further analysis.

APPENDIX A. PFAFFIAN IDENTITIES FOLLOWING KNUTH

For the benefit of the reader, we quote from [11] a short introduction to Knuth's [17] combinatorial approach to Pfaffians.

Let $T = (t_{ij})$ be an $n \times n$ skew symmetric matrix with entries in a commutative ring. Assume that T has zeros on the diagonal; this is, of course, automatic if the characteristic of the ring is not 2. Set $\mathcal{P}[ij] = t_{ij}$ for $i, j \in \{1, \dots, n\}$ and extend \mathcal{P} to a function on words in letters from $\{1, \dots, n\}$ as follows:

$$\mathcal{P}[i_1 \dots i_m] = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \sum \text{sgn} \binom{i_1 \dots i_{2k}}{j_1 \dots j_{2k}} \mathcal{P}[j_1 j_2] \cdots \mathcal{P}[j_{2k-1} j_{2k}] & \text{if } m = 2k \text{ is even} \end{cases}$$

where the sum is over all partitions of $\{i_1, \dots, i_{2k}\}$ in k subsets of cardinality 2. The order of elements in each subset is irrelevant as the difference in sign $\mathcal{P}[jj'] = -\mathcal{P}[j'j]$ is offset by a change of sign of the permutation; see [17, Section 0]. The value of \mathcal{P} on the empty word is by convention 1, and the value of \mathcal{P} on a word with a repeated letter is 0. The latter is a convention in characteristic 2 and otherwise automatic.

The function \mathcal{P} computes the Pfaffians of submatrices of T . Indeed, for a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with elements $i_1 < \dots < i_k$ one has

$$\text{pf}_T(i_1 \dots i_k) = \mathcal{P}[i_1 \dots i_k],$$

in the notation introduced in 1.1 and (2.0.1).

A.1 Overlapping Pfaffians. Let α, β , and γ be disjoint words in letters from $\{1, \dots, n\}$. For b a letter in β , the formula [17, (5.0)] reads

$$(A.1.1) \quad \begin{aligned} \mathcal{P}[\alpha\beta] \mathcal{P}[\alpha\gamma] &= \sum_{i \in \beta} \text{sgn} \binom{i}{b_i(\beta \setminus b_i)} \mathcal{P}[\alpha\beta \setminus bi] \mathcal{P}[\alpha\gamma bi] \\ &+ \sum_{j \in \gamma} \text{sgn} \binom{j}{b(\beta \setminus b)} \text{sgn} \binom{j}{j(\gamma \setminus j)} \mathcal{P}[\alpha j \beta \setminus b] \mathcal{P}[\alpha b \gamma \setminus j]. \end{aligned}$$

We record a number of special cases of this formula.

For $\beta = b$ the formula (A.1.1) reduces to

$$(A.1.2) \quad \mathcal{P}[\alpha b] \mathcal{P}[\alpha\gamma] = \sum_{j \in \gamma} \text{sgn} \binom{j}{j(\gamma \setminus j)} \mathcal{P}[\alpha j] \mathcal{P}[\alpha b \gamma \setminus j].$$

For $\gamma = c$ the formula (A.1.1) reduces to

$$(A.1.3) \quad \begin{aligned} \mathcal{P}[\alpha\beta] \mathcal{P}[\alpha c] &= \sum_{i \in \beta} \operatorname{sgn} \left({}_{bi(\beta \setminus bi)}^\beta \right) \mathcal{P}[\alpha\beta \setminus bi] \mathcal{P}[\alpha cbi] \\ &\quad + \operatorname{sgn} \left({}_{b(\beta \setminus b)}^\beta \right) \mathcal{P}[\alpha c\beta \setminus b] \mathcal{P}[\alpha b]. \end{aligned}$$

With γ empty the formula (A.1.1) reduces to

$$(A.1.4) \quad \mathcal{P}[\alpha\beta] \mathcal{P}[\alpha] = \sum_{i \in \beta} \operatorname{sgn} \left({}_{bi(\beta \setminus bi)}^\beta \right) \mathcal{P}[\alpha\beta \setminus bi] \mathcal{P}[\alpha bi].$$

With α and γ empty the formula (A.1.1) reduces to

$$(A.1.5) \quad \mathcal{P}[\beta] = \sum_{i \in \beta} \operatorname{sgn} \left({}_{bi(\beta \setminus bi)}^\beta \right) \mathcal{P}[\beta \setminus bi] \mathcal{P}[bi].$$

In this first appendix we derive some consequences of (A.1.1) that facilitate the computations in Appendices B and C. The first lemma is just the classic Laplacian expansion of the Pfaffian of a skew symmetric submatrix of T .

A.2 Lemma. *For integers $1 \leq u_1 < \dots < u_k \leq n$ and every integer ℓ with $1 \leq \ell \leq k$ one has*

$$\begin{aligned} (-1)^{\ell-1} \operatorname{pf}_T(u_1 \dots u_k) &= \sum_{i=1}^{\ell-1} (-1)^i t_{u_i u_\ell} \operatorname{pf}_T(u_1 \dots u_k \setminus u_i u_\ell) \\ &\quad + \sum_{i=\ell+1}^k (-1)^i t_{u_\ell u_i} \operatorname{pf}_T(u_1 \dots u_k \setminus u_i u_\ell). \end{aligned}$$

Proof. With $\beta = u_1 \dots u_k$ and $b = u_\ell$ the formula (A.1.5) yields

$$\begin{aligned} \mathcal{P}[\beta] &= \sum_{i=1}^{\ell-1} (-1)^{i+\ell} \mathcal{P}[\beta \setminus u_\ell u_i] \mathcal{P}[u_\ell u_i] + \sum_{i=\ell+1}^k (-1)^{i+\ell-1} \mathcal{P}[\beta \setminus u_\ell u_i] \mathcal{P}[u_\ell u_i] \\ &= \sum_{i=1}^{\ell-1} (-1)^{i+\ell-1} \mathcal{P}[\beta \setminus u_\ell u_i] \mathcal{P}[u_i u_\ell] + \sum_{i=\ell+1}^k (-1)^{i+\ell-1} \mathcal{P}[\beta \setminus u_\ell u_i] \mathcal{P}[u_\ell u_i]. \quad \square \end{aligned}$$

A.3 Lemma. *For integers $1 \leq u_1 < \dots < u_k \leq n$ and for every integer ℓ with $1 \leq \ell \leq k$ one has*

$$\begin{aligned} &\sum_{i=1}^{\ell-1} (-1)^i t_{u_i u_\ell} \operatorname{pf}_T(u_1 \dots u_{i-1} u_{i+1} \dots u_k) \\ &= \sum_{i=\ell+1}^n (-1)^i t_{u_\ell u_i} \operatorname{pf}_T(u_1 \dots u_{i-1} u_{i+1} \dots u_k). \end{aligned}$$

Proof. First assume that $\ell \geq 2$ holds. With $\alpha = u_\ell$, $b = u_1$, and $\gamma = u_2 \dots u_k \setminus u_\ell$ the equation (A.1.2) yields

$$\begin{aligned} &\mathcal{P}[u_\ell u_1] \mathcal{P}[u_\ell \gamma] \\ &= \sum_{j=2}^{\ell-1} (-1)^j \mathcal{P}[u_\ell u_j] \mathcal{P}[u_\ell u_1 \gamma \setminus u_j] + \sum_{j=\ell+1}^k (-1)^{j-1} \mathcal{P}[u_\ell u_j] \mathcal{P}[u_\ell u_1 \gamma \setminus u_j], \end{aligned}$$

which after reordering and multiplication by a sign becomes

$$\begin{aligned} & \mathcal{P}[u_1 u_\ell] \mathcal{P}[u_2 \dots u_k] \\ &= \sum_{j=2}^{\ell-1} (-1)^j \mathcal{P}[u_j u_\ell] \mathcal{P}[u_1 \dots u_k \setminus u_j] + \sum_{j=\ell+1}^n (-1)^{j-1} \mathcal{P}[u_\ell u_j] \mathcal{P}[u_1 \dots u_k \setminus u_j], \end{aligned}$$

and that can be rewritten as

$$\sum_{j=1}^{\ell-1} (-1)^j \mathcal{P}[u_j u_\ell] \mathcal{P}[u_1 \dots u_k \setminus u_j] = \sum_{j=\ell+1}^k (-1)^j \mathcal{P}[u_\ell u_j] \mathcal{P}[u_1 \dots u_k \setminus u_j].$$

Next assume that $\ell = 1$ holds. With $\alpha = u_1$, $b = u_2$, and $\gamma = u_3 \dots u_k$ the equation (A.1.2) yields

$$\mathcal{P}[u_1 u_2] \mathcal{P}[u_1 \gamma] = \sum_{j=3}^k (-1)^{j-1} \mathcal{P}[u_1 u_j] \mathcal{P}[u_1 u_2 \gamma \setminus u_j]$$

which can be rewritten as

$$\sum_{j=2}^k (-1)^j \mathcal{P}[u_1 u_j] \mathcal{P}[u_1 \dots u_k \setminus u_j] = 0. \quad \square$$

A.4 Lemma. For integers $1 \leq u_1 < \dots < u_k \leq n$ and every integer ℓ with $1 \leq \ell \leq k$ one has,

$$(-1)^{\ell-1} \text{pf}_T \text{pf}_T(\overline{u_1 \dots u_k}) = \sum_{i=1}^k (-1)^i \text{pf}_T(\overline{u_i u_\ell}) \text{pf}_T(\overline{u_1 \dots u_k \setminus u_i u_\ell}).$$

Proof. With $\alpha = 1 \dots n \setminus u_1 \dots u_k$, $\beta = u_1 \dots u_k$, and $b = u_\ell$ the formula (A.1.4) yields

$$\begin{aligned} \mathcal{P}[\alpha \beta] \mathcal{P}[\alpha] &= \sum_{i=1}^{\ell-1} (-1)^{i+\ell} \mathcal{P}[\alpha \beta \setminus u_\ell u_i] \mathcal{P}[\alpha u_\ell u_i] \\ &\quad + \sum_{i=\ell+1}^k (-1)^{i+\ell-1} \mathcal{P}[\alpha \beta \setminus u_\ell u_i] \mathcal{P}[\alpha u_\ell u_i], \end{aligned}$$

which after reordering and multiplying by a sign becomes

$$\begin{aligned} & \mathcal{P}[1 \dots n] \mathcal{P}[1 \dots n \setminus u_1 \dots u_k] \\ &= \sum_{i=1}^k (-1)^{i+\ell-1} \mathcal{P}[1 \dots n \setminus u_\ell u_i] \mathcal{P}[1 \dots n \setminus (u_1 \dots u_k \setminus u_\ell u_i)]. \quad \square \end{aligned}$$

A.5 Lemma. For integers $1 \leq u_1 < \dots < u_k \leq n$ and every integer ℓ with $1 \leq \ell \leq k-1$ one has

$$\sum_{i=1}^{\ell-1} (-1)^i \text{pf}_T(\overline{u_1 \dots u_k \setminus u_i}) \text{pf}_T(\overline{u_i u_\ell}) = \sum_{i=\ell+1}^k (-1)^i \text{pf}_T(\overline{u_1 \dots u_k \setminus u_i}) \text{pf}_T(\overline{u_\ell u_i}).$$

Proof. With $\alpha = 1 \dots n \setminus u_1 \dots u_k$, $b = u_k$, and $\gamma = u_1 \dots u_{k-1} \setminus u_\ell$ formula (A.1.2) yields

$$\begin{aligned} \mathcal{P}[\alpha u_k] \mathcal{P}[\alpha \gamma] &= \sum_{j=1}^{\ell-1} (-1)^{j-1} \mathcal{P}[\alpha u_j] \mathcal{P}[\alpha u_k \gamma \setminus u_j] \\ &\quad + \sum_{j=\ell+1}^{k-1} (-1)^j \mathcal{P}[\alpha u_j] \mathcal{P}[\alpha u_k \gamma \setminus u_j], \end{aligned}$$

which after reordering and multiplication by a sign can be rewritten as

$$\begin{aligned} &(-1)^{k-1} \mathcal{P}[1 \dots n \setminus u_1 \dots u_{k-1}] \mathcal{P}[1 \dots n \setminus u_\ell u_k] \\ &= \sum_{j=1}^{\ell-1} (-1)^{j-1} \mathcal{P}[1 \dots n \setminus (u_1 \dots u_k \setminus u_j)] \mathcal{P}[1 \dots n \setminus u_j u_\ell] \\ &\quad + \sum_{j=\ell+1}^{k-1} (-1)^j \mathcal{P}[1 \dots n \setminus (u_1 \dots u_k \setminus u_j)] \mathcal{P}[1 \dots n \setminus u_\ell u_j]. \end{aligned}$$

This can also be written

$$\begin{aligned} &\sum_{j=1}^{\ell-1} (-1)^j \mathcal{P}[1 \dots n \setminus (u_1 \dots u_k \setminus u_j)] \mathcal{P}[1 \dots n \setminus u_j u_\ell] \\ &= \sum_{j=\ell+1}^k (-1)^j \mathcal{P}[1 \dots n \setminus (u_1 \dots u_k \setminus u_j)] \mathcal{P}[1 \dots n \setminus u_\ell u_j]. \quad \square \end{aligned}$$

A.6 Lemma. For integers $1 \leq u_1 < \dots < u_k \leq n$ one has

$$\sum_{i=1}^k (-1)^i \text{pf}_T(\overline{u_i}) \text{pf}_T(\overline{u_1 \dots u_{i-1} u_{i+1} \dots u_k}) = 0.$$

Proof. With $\alpha = 1 \dots n \setminus u_1 \dots u_k$, $b = u_1$, and $\gamma = u_2 \dots u_k$ equation (A.1.2) yields

$$\mathcal{P}[\alpha u_1] \mathcal{P}[\alpha \gamma] = \sum_{j=2}^k (-1)^j \mathcal{P}[\alpha u_j] \mathcal{P}[\alpha u_1 \gamma \setminus u_j],$$

which after reordering and multiplication by a sign becomes

$$\sum_{j=1}^k (-1)^j \mathcal{P}[1 \dots n \setminus u_1 \dots u_{j-1} u_{j+1} \dots u_k] \mathcal{P}[1 \dots n \setminus u_j] = 0. \quad \square$$

A.7 Lemma. For integers $1 \leq u < v < w < x < y < z \leq n$ one has

$$\begin{aligned} &\text{pf}_T(\overline{y}) \text{pf}_T(\overline{uvw x z}) - \text{pf}_T(\overline{z}) \text{pf}_T(\overline{uvw x y}) \\ &= \text{pf}_T(\overline{u y z}) \text{pf}_T(\overline{v w x}) - \text{pf}_T(\overline{v y z}) \text{pf}_T(\overline{u w x}) \\ &\quad + \text{pf}_T(\overline{w y z}) \text{pf}_T(\overline{u v x}) - \text{pf}_T(\overline{x y z}) \text{pf}_T(\overline{u v w}). \end{aligned}$$

Proof. With $\alpha = 1 \dots n \setminus uvwxyz$, $\beta = uvwxy$, $b = y$, and $c = z$ equation (A.1.3) yields

$$\mathcal{P}[\alpha \beta] \mathcal{P}[\alpha z] - \mathcal{P}[\alpha z \beta \setminus y] \mathcal{P}[\alpha y] = \sum_{i \in \beta} \text{sgn} \left({}_{y i} \beta_{y i} \right) \mathcal{P}[\alpha \beta \setminus y i] \mathcal{P}[\alpha z y i],$$

which expands into

$$\begin{aligned} \mathcal{P}[\alpha uvwxy] \mathcal{P}[\alpha z] - \mathcal{P}[\alpha zuvwx] \mathcal{P}[\alpha y] &= \mathcal{P}[\alpha vwx] \mathcal{P}[\alpha zyu] - \mathcal{P}[\alpha uwx] \mathcal{P}[\alpha zyv] \\ &\quad + \mathcal{P}[\alpha uvx] \mathcal{P}[\alpha zyw] - \mathcal{P}[\alpha uvw] \mathcal{P}[\alpha zyx]. \end{aligned}$$

After reordering and multiplication by $(-1)^{u+v+w+x+y+z}$ it becomes

$$\begin{aligned} -\mathcal{P}[1 \dots n \setminus z] \mathcal{P}[1 \dots n \setminus uvwxy] + \mathcal{P}[1 \dots n \setminus y] \mathcal{P}[1 \dots n \setminus uvwxz] \\ = \mathcal{P}[1 \dots n \setminus uyz] \mathcal{P}[1 \dots n \setminus vwx] \\ - \mathcal{P}[1 \dots n \setminus vyz] \mathcal{P}[1 \dots n \setminus uwx] \\ + \mathcal{P}[1 \dots n \setminus wyz] \mathcal{P}[1 \dots n \setminus uvx] \\ + \mathcal{P}[1 \dots n \setminus xyz] \mathcal{P}[1 \dots n \setminus uvw]. \quad \square \end{aligned}$$

A.8 Lemma. For integers $1 \leq u < v < w < x < y < z \leq n$ one has

$$\begin{aligned} \text{pf}_T(\overline{xy}) \text{pf}_T(\overline{uvwz}) - \text{pf}_T(\overline{xz}) \text{pf}_T(\overline{uvwxy}) + \text{pf}_T(\overline{yz}) \text{pf}_T(\overline{uvwxx}) \\ = \text{pf}_T(\overline{uv}) \text{pf}_T(\overline{wxyz}) - \text{pf}_T(\overline{uv}) \text{pf}_T(\overline{vxyz}) + \text{pf}_T(\overline{vw}) \text{pf}_T(\overline{uxyz}). \end{aligned}$$

Proof. With $\alpha = 1 \dots n \setminus uvwxyz$, $\beta = uvwx$, $b = x$, and $\gamma = yz$ equation (A.1.1) yields

$$\begin{aligned} \mathcal{P}[\alpha uvwx] \mathcal{P}[\alpha yz] &= -\mathcal{P}[\alpha vw] \mathcal{P}[\alpha yz xu] + \mathcal{P}[\alpha uw] \mathcal{P}[\alpha yz xv] - \mathcal{P}[\alpha vw] \mathcal{P}[\alpha yz xw] \\ &\quad - \mathcal{P}[\alpha yuvw] \mathcal{P}[\alpha xz] + \mathcal{P}[\alpha zuvw] \mathcal{P}[\alpha xy] \\ &= \mathcal{P}[\alpha vw] \mathcal{P}[\alpha uxyz] - \mathcal{P}[\alpha uw] \mathcal{P}[\alpha vxyz] + \mathcal{P}[\alpha vw] \mathcal{P}[\alpha wxyz] \\ &\quad + \mathcal{P}[\alpha uvwy] \mathcal{P}[\alpha xz] - \mathcal{P}[\alpha uvwz] \mathcal{P}[\alpha xy], \end{aligned}$$

which after reordering and multiplication by a sign becomes

$$\begin{aligned} \mathcal{P}[1 \dots n \setminus yz] \mathcal{P}[1 \dots n \setminus uvwx] - \mathcal{P}[1 \dots n \setminus xz] \mathcal{P}[1 \dots n \setminus uvwy] \\ + \mathcal{P}[1 \dots n \setminus xy] \mathcal{P}[1 \dots n \setminus uvwz] \\ = \mathcal{P}[1 \dots n \setminus vw] \mathcal{P}[1 \dots n \setminus uxyz] - \mathcal{P}[1 \dots n \setminus uw] \mathcal{P}[1 \dots n \setminus vxyz] \\ + \mathcal{P}[1 \dots n \setminus uv] \mathcal{P}[1 \dots n \setminus wxyz]. \quad \square \end{aligned}$$

A.9 Lemma. For integers $1 \leq u < x < y \leq n$ and $1 \leq v < w < x$ one has

$$\begin{aligned} \text{pf}_T(\overline{uxy}) \text{pf}_T(\overline{uvw}) - \text{pf}_T(\overline{u}) \text{pf}_T(\overline{uvwxy}) \\ = \text{pf}_T(\overline{uvx}) \text{pf}_T(\overline{uwy}) - \text{pf}_T(\overline{uvx}) \text{pf}_T(\overline{uwy}). \end{aligned}$$

Proof. With $\alpha = 1 \dots n \setminus uvwxy$, $\beta = vwxy$, and $b = x$ equation (A.1.4) yields

$$\mathcal{P}[\alpha \beta] \mathcal{P}[\alpha] = \sum_{i \in \beta} \text{sgn} \left(\begin{matrix} \beta \\ xi(\beta \setminus xi) \end{matrix} \right) \mathcal{P}[\alpha \beta \setminus xi] \mathcal{P}[\alpha xi],$$

which expands into

$$\mathcal{P}[\alpha uvwxy] \mathcal{P}[\alpha] = \mathcal{P}[\alpha wy] \mathcal{P}[\alpha xv] - \mathcal{P}[\alpha vy] \mathcal{P}[\alpha xw] + \mathcal{P}[\alpha vw] \mathcal{P}[\alpha xy].$$

After reordering and multiplication by $(-1)^{v+w+x+y}$ this expression becomes

$$\begin{aligned} \mathcal{P}[1 \dots n \setminus u] \mathcal{P}[\alpha] &= -\mathcal{P}[1 \dots n \setminus uvx] \mathcal{P}[1 \dots n \setminus uvw] \\ &\quad + \mathcal{P}[1 \dots n \setminus uvx] \mathcal{P}[1 \dots n \setminus uvy] \\ &\quad + \mathcal{P}[1 \dots n \setminus uxy] \mathcal{P}[1 \dots n \setminus uvw]. \quad \square \end{aligned}$$

APPENDIX B. MINORS VIA PFAFFIANS FOLLOWING BRILL

The formula in the next theorem was first discovered by Brill [4]; the theorem stated here is [11, Theorem 2.1].

B.1 Theorem. *Let T be an $n \times n$ skew symmetric matrix. Let $\{i_1, \dots, i_m\}$ and $\{j_1, \dots, j_m\}$ be subsets of $\{1, \dots, n\}$ with $i_1 < \dots < i_m$ and $j_1 < \dots < j_m$, and set $\rho = i_1 \dots i_m$ and $\sigma = j_1 \dots j_m$. The following equality holds:*

$$\begin{aligned} \det(T[i_1 \dots i_m; j_1 \dots j_m]) \\ = (-1)^{\lfloor \frac{m}{2} \rfloor} \sum_{0 \leq k \leq \lfloor \frac{m}{2} \rfloor} (-1)^k \sum_{\substack{|\omega|=2k \\ \omega \subseteq \rho}} \operatorname{sgn}(\omega_{(\rho \setminus \omega)}) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma]. \end{aligned}$$

Notice that only subwords ω of ρ that contain $\rho \cap \sigma$ contribute to the sum above.

The two lemmas proved below are applied in Appendix C to calculate the maximal minors of the matrices ∂_3 from Theorems 2.3 and 2.9.

B.2 Lemma. *Let $n \geq 5$ be an odd number. For integers $1 \leq r_1 < r_2 < r_3 \leq n$ one has*

$$\begin{aligned} \det(T[\overline{r_1 r_2 r_3}; \overline{123}]) \\ = \begin{cases} \operatorname{pf}_T(\overline{r_1 r_2 r_3}) \operatorname{pf}_T(\overline{123}) & \text{if } r_2 \leq 3 \\ \operatorname{pf}_T(\overline{r_1 r_2 r_3}) \operatorname{pf}_T(\overline{123}) - \operatorname{pf}_T(\overline{123 r_2 r_3}) \operatorname{pf}_T(\overline{r_1}) & \text{if } r_1 \leq 3 < r_2 \\ \operatorname{pf}_T(\overline{r_1 r_2 r_3}) \operatorname{pf}_T(\overline{123}) - \operatorname{pf}_T(\overline{23 r_1 r_2 r_3}) \operatorname{pf}_T(\overline{1}) \\ \quad + \operatorname{pf}_T(\overline{13 r_1 r_2 r_3}) \operatorname{pf}_T(\overline{2}) - \operatorname{pf}_T(\overline{12 r_1 r_2 r_3}) \operatorname{pf}_T(\overline{3}) & \text{if } 3 < r_1. \end{cases} \end{aligned}$$

Proof. Consider the words

$$\rho = 1 \dots n \setminus r_1 r_2 r_3 \quad \text{and} \quad \sigma = 4 \dots n$$

of length $n - 3$. One has $\rho \cap \sigma = \sigma \setminus r_1 r_2 r_3$, and Theorem B.1 yields

$$(1) \quad \det(T[\overline{r_1 r_2 r_3}; \overline{123}]) = (-1)^{\frac{n-3}{2}} \sum_{k=\lceil \frac{|\sigma \setminus r_1 r_2 r_3|}{2} \rceil}^{\frac{n-3}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma \setminus r_1 r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega_{(\rho \setminus \omega)}) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma].$$

If $r_2 \leq 3$ holds, then one has $|\rho \cap \sigma| = n - 3$ or $|\rho \cap \sigma| = n - 4$. In either case the shortest word ω contributing to the sum (1) has length $n - 3$. Thus, $\omega = \rho$ is the only contributing word and one gets

$$\det(T[\overline{r_1 r_2 r_3}; \overline{123}]) = \mathcal{P}[\rho] \mathcal{P}[\sigma] = \operatorname{pf}_T(\overline{r_1 r_2 r_3}) \operatorname{pf}_T(\overline{123}).$$

If $r_1 \leq 3 < r_2$ holds, then one has $|\rho \cap \sigma| = n - 5$. As $n - 5$ is even, the shortest subwords ω of

$$\rho = (123 \setminus r_1)(\sigma \setminus r_2 r_3)$$

that contribute to the sum (1) have length $n - 5$, so $\omega = \sigma \setminus r_2 r_3$ is the only one. Now one has

$$\begin{aligned} \det(T[\overline{r_1 r_2 r_3}; \overline{123}]) \\ = (-1)^{\frac{n-3}{2}} \sum_{k=\frac{n-5}{2}}^{\frac{n-3}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma \setminus r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega_{(\rho \setminus \omega)}) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma] \end{aligned}$$

$$\begin{aligned}
&= (-1)^{\frac{n-3}{2}} \left((-1)^{\frac{n-5}{2}} \operatorname{sgn}(\sigma \setminus r_2 r_3 \overset{\rho}{(123 \setminus r_1)}) \mathcal{P}[\sigma \setminus r_2 r_3] \mathcal{P}[(123 \setminus r_1)\sigma] \right. \\
&\quad \left. + (-1)^{\frac{n-3}{2}} \mathcal{P}[\rho] \mathcal{P}[\sigma] \right) \\
&= -\operatorname{sgn}(\sigma \setminus r_2 r_3 \overset{\rho}{(123 \setminus r_1)}) \mathcal{P}[\sigma \setminus r_2 r_3] \mathcal{P}[(123 \setminus r_1)\sigma] + \mathcal{P}[\rho] \mathcal{P}[\sigma] \\
&= -\mathcal{P}[\sigma \setminus r_2 r_3] \mathcal{P}[(123 \setminus r_1)\sigma] + \mathcal{P}[\rho] \mathcal{P}[\sigma] \\
&= -\operatorname{pf}_T(\overline{123r_2r_3}) \operatorname{pf}_T(\overline{r_1}) + \operatorname{pf}_T(\overline{r_1r_2r_3}) \operatorname{pf}_T(\overline{123}).
\end{aligned}$$

If $3 < r_1$ holds, then one has $|\rho \cap \sigma| = n - 6$. As $n - 6$ is odd, the shortest subwords ω of

$$\rho = 123(\sigma \setminus r_1 r_2 r_3)$$

that contribute to the sum (1) have length $n - 5$. Now one has

$$\begin{aligned}
&\det(T[\overline{r_1 r_2 r_3}; \overline{123}]) \\
&= (-1)^{\frac{n-3}{2}} \sum_{k=\frac{n-5}{2}}^{\frac{n-3}{2}} (-1)^k \sum_{\substack{|\omega| = 2k \\ \sigma \setminus r_1 r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega \overset{\rho}{(\rho \setminus \omega)}) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma] \\
&= (-1)^{\frac{n-3}{2}} \left((-1)^{\frac{n-5}{2}} \sum_{\substack{|\omega| = n-5 \\ \sigma \setminus r_1 r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega \overset{\rho}{(\rho \setminus \omega)}) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma] \right. \\
&\quad \left. + (-1)^{\frac{n-3}{2}} \mathcal{P}[\rho] \mathcal{P}[\sigma] \right) \\
&= -\left(\operatorname{sgn}(1(\sigma \setminus r_1 r_2 r_3)23) \mathcal{P}[1\sigma \setminus r_1 r_2 r_3] \mathcal{P}[23\sigma] \right. \\
&\quad \left. + \operatorname{sgn}(2(\sigma \setminus r_1 r_2 r_3)13) \mathcal{P}[2\sigma \setminus r_1 r_2 r_3] \mathcal{P}[13\sigma] \right. \\
&\quad \left. + \operatorname{sgn}(3(\sigma \setminus r_1 r_2 r_3)12) \mathcal{P}[3\sigma \setminus r_1 r_2 r_3] \mathcal{P}[12\sigma] \right) + \mathcal{P}[\rho] \mathcal{P}[\sigma] \\
&= \mathcal{P}[1\sigma \setminus r_1 r_2 r_3] \mathcal{P}[23\sigma] - \mathcal{P}[2\sigma \setminus r_1 r_2 r_3] \mathcal{P}[13\sigma] + \mathcal{P}[3\sigma \setminus r_1 r_2 r_3] \mathcal{P}[12\sigma] \\
&\quad + \mathcal{P}[\rho] \mathcal{P}[\sigma] \\
&= -\operatorname{pf}_T(\overline{23r_1r_2r_3}) \operatorname{pf}_T(\overline{1}) + \operatorname{pf}_T(\overline{13r_1r_2r_3}) \operatorname{pf}_T(\overline{2}) \\
&\quad - \operatorname{pf}_T(\overline{12r_1r_2r_3}) \operatorname{pf}_T(\overline{3}) + \operatorname{pf}_T(\overline{r_1r_2r_3}) \operatorname{pf}_T(\overline{123}). \quad \square
\end{aligned}$$

B.3 Lemma. Let $n \geq 6$ be an even number. For integers $1 \leq r_1 < r_2 < r_3 \leq n$ one has

$$\det(T[\overline{r_1 r_2 r_3}; \overline{123}]) = \begin{cases} 0 & \text{if } r_3 = 3 \\ \operatorname{pf}_T(\overline{123r_3}) \operatorname{pf}_T(\overline{r_1 r_2}) & \text{if } r_2 \leq 3 < r_3 \\ \operatorname{pf}_T(\overline{12r_2r_3}) \operatorname{pf}_T(\overline{13}) - \operatorname{pf}_T(\overline{13r_2r_3}) \operatorname{pf}_T(\overline{12}) & \text{if } 1 = r_1 \leq 3 < r_2 \\ \operatorname{pf}_T(\overline{12r_2r_3}) \operatorname{pf}_T(\overline{23}) - \operatorname{pf}_T(\overline{23r_2r_3}) \operatorname{pf}_T(\overline{12}) & \text{if } 2 = r_1 \leq 3 < r_2 \\ \operatorname{pf}_T(\overline{13r_2r_3}) \operatorname{pf}_T(\overline{23}) - \operatorname{pf}_T(\overline{23r_2r_3}) \operatorname{pf}_T(\overline{13}) & \text{if } 3 = r_1 < r_2 \\ \operatorname{pf}_T(\overline{1r_1r_2r_3}) \operatorname{pf}_T(\overline{23}) - \operatorname{pf}_T(\overline{2r_1r_2r_3}) \operatorname{pf}_T(\overline{13}) \\ \quad + \operatorname{pf}_T(\overline{3r_1r_2r_3}) \operatorname{pf}_T(\overline{12}) - \operatorname{pf}_T(\overline{123r_1r_2r_3}) \operatorname{pf}_T(\overline{123}) & \text{if } 3 < r_1. \end{cases}$$

Proof. Consider the words

$$\rho = 1 \dots n \setminus r_1 r_2 r_3 \quad \text{and} \quad \sigma = 4 \dots n$$

of length $n - 3$. One has $\rho \cap \sigma = \sigma \setminus r_1 r_2 r_3$ and Theorem B.1 yields

$$(1) \quad \det(T[\overline{r_1 r_2 r_3}; \overline{123}]) = (-1)^{\frac{n-4}{2}} \sum_{k=\lceil \frac{|\sigma \setminus r_1 r_2 r_3|}{2} \rceil}^{\frac{n-4}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma \setminus r_1 r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega_{(\rho \setminus \omega)}) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma].$$

If $r_3 = 3$ holds, then one has $|\rho \cap \sigma| = |\sigma| = n - 3$, so the sum (1) is empty. i.e.

$$\det(T[\overline{123}; \overline{123}]) = 0.$$

If $r_2 \leq 3 < r_3$ hold, then one has $|\rho \cap \sigma| = n - 4$, so the shortest subwords ω of

$$\rho = (123 \setminus r_1 r_2)(\sigma \setminus r_3)$$

contributing to the sum (1) have length $n - 4$. Thus, $\omega = \sigma \setminus r_3$ is the only contributing word, and one gets

$$\det(T[\overline{r_1 r_2 r_3}; \overline{123}]) = \mathcal{P}[\sigma \setminus r_3] \mathcal{P}[(123 \setminus r_1 r_2)\sigma] = \operatorname{pf}_T(\overline{123 r_3}) \operatorname{pf}_T(\overline{r_1 r_2}).$$

If $r_1 \leq 3 < r_2$ holds, then one has $|\rho \cap \sigma| = n - 5$, which is odd. Therefore, the shortest subwords ω of

$$\rho = (123 \setminus r_1)(\sigma \setminus r_2 r_3)$$

contributing to the sum (1) have length $n - 4$. Hence, one gets

$$(2) \quad \det(T[\overline{r_1 r_2 r_3}; \overline{123}]) = \sum_{r \in 123 \setminus r_1} \operatorname{sgn}((r\sigma \setminus r_2 r_3)_{(123 \setminus r r_1)}^\rho) \mathcal{P}[r\sigma \setminus r_2 r_3] \mathcal{P}[(123 \setminus r_1 r)\sigma].$$

For $r_1 = 1$ this specializes to

$$\begin{aligned} \det(T[\overline{1 r_2 r_3}; \overline{123}]) &= \sum_{r \in 23} \operatorname{sgn}((r\sigma \setminus r_2 r_3)_{(23 \setminus r)}^\rho) \mathcal{P}[r\sigma \setminus r_2 r_3] \mathcal{P}[(23 \setminus r)\sigma] \\ &= -\operatorname{pf}_T(\overline{13 r_2 r_3}) \operatorname{pf}_T(\overline{12}) + \operatorname{pf}_T(\overline{12 r_2 r_3}) \operatorname{pf}_T(\overline{13}). \end{aligned}$$

The specialization of (2) with $r_1 = 2$ is

$$\begin{aligned} \det(T[\overline{2 r_2 r_3}; \overline{123}]) &= \sum_{r \in 13} \operatorname{sgn}((r\sigma \setminus r_2 r_3)_{(13 \setminus r)}^\rho) \mathcal{P}[r\sigma \setminus r_2 r_3] \mathcal{P}[(13 \setminus r)\sigma] \\ &= -\operatorname{pf}_T(\overline{23 r_2 r_3}) \operatorname{pf}_T(\overline{12}) + \operatorname{pf}_T(\overline{12 r_2 r_3}) \operatorname{pf}_T(\overline{23}). \end{aligned}$$

The specialization of (2) with $r_1 = 3$ is

$$\begin{aligned} \det(T[\overline{3 r_2 r_3}; \overline{123}]) &= \sum_{r \in 12} \operatorname{sgn}((r\sigma \setminus r_2 r_3)_{(12 \setminus r)}^\rho) \mathcal{P}[r\sigma \setminus r_2 r_3] \mathcal{P}[(12 \setminus r)\sigma] \\ &= -\operatorname{pf}_T(\overline{23 r_2 r_3}) \operatorname{pf}_T(\overline{13}) + \operatorname{pf}_T(\overline{13 r_2 r_3}) \operatorname{pf}_T(\overline{23}). \end{aligned}$$

If $3 < r_1$ holds, then one has $|\rho \cap \sigma| = n - 6$, which is even. Therefore, the shortest subwords ω of

$$\rho = 123(\sigma \setminus r_1 r_2 r_3)$$

that contribute to the sum (1) have length $n - 6$, which means that $\omega = \sigma \setminus r_1 r_2 r_3$ is the only one. Thus one has

$$\det(T[\overline{r_1 r_2 r_3}; \overline{123}])$$

$$\begin{aligned}
&= (-1)^{\frac{n-4}{2}} \sum_{k=\frac{n-6}{2}}^{\frac{n-4}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma \setminus r_1 r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega(\rho \setminus \omega)) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma] \\
&= -\mathcal{P}[\sigma \setminus r_1 r_2 r_3] \mathcal{P}[123\sigma] + \sum_{\substack{|\omega|=n-4 \\ \sigma \setminus r_1 r_2 r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}(\omega(\rho \setminus \omega)) \mathcal{P}[\omega] \mathcal{P}[(\rho \setminus \omega)\sigma] \\
&= -\mathcal{P}[\sigma \setminus r_1 r_2 r_3] \mathcal{P}[123\sigma] + \mathcal{P}[23\sigma \setminus r_1 r_2 r_3] \mathcal{P}[1\sigma] \\
&\quad - \mathcal{P}[13\sigma \setminus r_1 r_2 r_3] \mathcal{P}[2\sigma] + \mathcal{P}[12\sigma \setminus r_1 r_2 r_3] \mathcal{P}[3\sigma] \\
&= -\operatorname{pf}_T(\overline{123r_1 r_2 r_3}) \operatorname{pf}_T + \operatorname{pf}_T(\overline{1r_1 r_2 r_3}) \operatorname{pf}_T(\overline{23}) \\
&\quad - \operatorname{pf}_T(\overline{2r_1 r_2 r_3}) \operatorname{pf}_T(\overline{13}) + \operatorname{pf}_T(\overline{3r_1 r_2 r_3}) \operatorname{pf}_T(\overline{12}). \quad \square
\end{aligned}$$

APPENDIX C. GENERIC ALMOST COMPLETE INTERSECTIONS: THE PROOFS

In this final appendix we provide the detailed computations that underpin the theorems in Section 2.

Quotients of even type.

C.1 Lemma. *Let $n \geq 5$ be an odd number and adopt the setup from 2.3. The sequence $0 \rightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R} \rightarrow 0$ is a complex.*

Proof. The product $\partial_1 \partial_2$ is a $1 \times n$ matrix; the first three entries are evidently 0. For $i \in \{4, \dots, n\}$ the i^{th} entry is

$$\pm(-\operatorname{pf}_T(\overline{1}) \operatorname{pf}_T(\overline{23i}) + \operatorname{pf}_T(\overline{2}) \operatorname{pf}_T(\overline{13i}) - \operatorname{pf}_T(\overline{3}) \operatorname{pf}_T(\overline{12i}) + \operatorname{pf}_T(\overline{123}) \operatorname{pf}_T(\overline{i})),$$

which is zero by Lemma A.6 applied with $u_1 \dots u_k = 123i$.

The product $\partial_2 \partial_3$ is a $4 \times (n-3)$ matrix. Let $i \in \{4, \dots, n\}$; the entry in position $(1, i-3)$ is

$$\tau_{1i} \operatorname{pf}_T(\overline{123}) + \sum_{j=4}^{i-1} (-1)^{j-1} \tau_{ji} \operatorname{pf}_T(\overline{23j}) - \sum_{j=i+1}^n (-1)^{j-1} \tau_{ij} \operatorname{pf}_T(\overline{23j}).$$

Applied with $u_1 \dots u_k = 14 \dots n$ and $u_\ell = i$, Lemma A.3 shows that this quantity is zero. Similarly, Lemma A.3 applied with $u_1 \dots u_k = 24 \dots n$ and $u_\ell = i$ shows that the entry in position $(2, i-3)$ is zero, and an application with $u_1 \dots u_k = 3 \dots n$ and $u_\ell = i$ shows that the entry in position $(3, i-3)$ is zero. The entry in position $(4, i-3)$ is

$$\sum_{j=1}^{i-1} (-1)^{j-1} \tau_{ji} \operatorname{pf}_T(\overline{j}) - \sum_{j=i+1}^n (-1)^{j-1} \tau_{ij} \operatorname{pf}_T(\overline{j}).$$

Applied with $u_1 \dots u_k = 1 \dots n$ and $u_\ell = i$, Lemma A.3 shows that also this quantity is zero. \square

Józefiak and Pragacz [16] calculate the grade of ideals generated by Pfaffians; we combine this with a classic result of Eagon and Northcott [13] to obtain the next lemma and Lemma C.6, which deals with the case of even n .

C.2 Lemma. *Let $n \geq 5$ be an odd number and adopt the setup from 2.3. The Pfaffians $\operatorname{pf}_T(\overline{1})$, $\operatorname{pf}_T(\overline{2})$, and $\operatorname{pf}_T(\overline{123})$ form a regular sequence in \mathcal{R} .*

Proof. The $(n-3) \times (n-3)$ Pfaffians of the matrix $\mathcal{T}[3 \dots n; 3 \dots n]$ generate by [16, Corollary 2.5] an ideal of grade 3 in the subring $\mathcal{R}' = \mathbb{Z}[\tau_{ij} \mid 3 \leq i < j \leq n]$ of \mathcal{R} ; they are the Pfaffians $\text{pf}_{\mathcal{T}}(\overline{12i})$ for $3 \leq i \leq n$. As $\text{pf}_{\mathcal{T}}(\overline{123})$ is a regular element in the domain \mathcal{R}' , the Pfaffians $\text{pf}_{\mathcal{T}}(\overline{12i})$ for $4 \leq i \leq n$ generate an ideal of grade 2 in $\mathcal{S}' = \mathcal{R}' / \text{pf}_{\mathcal{T}}(\overline{123})$. In $\mathcal{S} = \mathcal{R} / \text{pf}_{\mathcal{T}}(\overline{123})$ one has,

$$\text{pf}_{\mathcal{T}}(\overline{1}) = \sum_{i=4}^n (-1)^{i-1} \tau_{2i} \text{pf}_{\mathcal{T}}(\overline{12i}) \quad \text{and} \quad \text{pf}_{\mathcal{T}}(\overline{2}) = \sum_{i=4}^n (-1)^{i-1} \tau_{1i} \text{pf}_{\mathcal{T}}(\overline{12i}).$$

Indeed, the first equality follows from Lemma A.2 applied with $u_1 \dots u_k = 2 \dots n$ and $\ell = 1$; the same lemma applied with $u_1 \dots u_k = 13 \dots n$ and $\ell = 1$ yields the second equality. Now it follows from [13, Lemma 6] that $\text{pf}_{\mathcal{T}}(\overline{1})$ and $\text{pf}_{\mathcal{T}}(\overline{2})$ form a regular sequence in \mathcal{B} . \square

C.3 Lemma. *Let $n \geq 5$ be an odd number and adopt the setup from 2.3. The ideal generated by the $(n-3) \times (n-3)$ minors of the matrix ∂_3 contains the elements*

$$(\text{pf}_{\mathcal{T}}(\overline{1}))^2, \quad (\text{pf}_{\mathcal{T}}(\overline{2}))^2, \quad (\text{pf}_{\mathcal{T}}(\overline{3}))^2, \quad \text{and} \quad (\text{pf}_{\mathcal{T}}(\overline{123}))^2.$$

Proof. One has $(\text{pf}_{\mathcal{T}}(\overline{1}))^2 = \det(\mathcal{T}[2 \dots n; 2 \dots n])$ and expansion of this determinant along the first two columns, see Horn and Johnson [15, 0.8.9], yields:

$$\begin{aligned} \det(\mathcal{T}[2 \dots n; 2 \dots n]) &= \sum_{2 \leq i < j \leq n} \pm \det(\mathcal{T}[ij; 23]) \det(\mathcal{T}[\overline{1ij}; \overline{123}]) \\ &= \sum_{2 \leq i < j \leq n} \pm \det(\mathcal{T}[ij; 23]) \det(\partial_3[\overline{1ij}; 1 \dots n-3]). \end{aligned}$$

Similarly, one gets

$$\begin{aligned} (\text{pf}_{\mathcal{T}}(\overline{2}))^2 &= \det(\mathcal{T}[13 \dots n; 13 \dots n]) \\ &= \sum_{3 \leq j \leq n} \pm \det(\mathcal{T}[1j; 13]) \det(\partial_3[\overline{12j}; 1 \dots n-3]) \\ &\quad + \sum_{3 \leq i < j \leq n} \pm \det(\mathcal{T}[ij; 13]) \det(\partial_3[\overline{1ij}; 1 \dots n-3]) \end{aligned}$$

and

$$\begin{aligned} (\text{pf}_{\mathcal{T}}(\overline{3}))^2 &= \det(\mathcal{T}[124 \dots n; 124 \dots n]) \\ &= \sum_{\substack{1 \leq i < j \leq n \\ i \neq 3 \neq j}} \pm \det(\mathcal{T}[ij; 12]) \det(\partial_3[\overline{1ij}; 1 \dots n-3]). \end{aligned}$$

Finally, one trivially has

$$(\text{pf}_{\mathcal{T}}(\overline{123}))^2 = \det(\mathcal{T}[\overline{123}; \overline{123}]) = \det(\partial_3[\overline{123}; 1 \dots n-3]). \quad \square$$

C.4 Proposition. *Let $n \geq 5$ be an odd number and adopt the setup from 2.3. For integers $1 \leq r_1 < r_2 < r_3 \leq n$ and $1 \leq s_1 < s_2 < s_3 \leq 4$ one has*

$$\det(\partial_3[\overline{r_1 r_2 r_3}; 1 \dots n-3]) \det(\partial_1[1; \overline{s_1 s_2 s_3}]) = \pm \det(\partial_2[s_1 s_2 s_3; r_1 r_2 r_3]).$$

Proof. First notice that one has $\det(\partial_3[\overline{r_1 r_2 r_3}; 1 \dots n-3]) = \det(\mathcal{T}[\overline{r_1 r_2 r_3}; \overline{123}])$. With the notation

$$\begin{aligned} \text{LHS} &= \det(\mathcal{T}[\overline{r_1 r_2 r_3}; \overline{123}]) \det(\partial_1[1; \overline{s_1 s_2 s_3}]) \quad \text{and} \\ \text{RHS} &= \det(\partial_2[s_1 s_2 s_3; r_1 r_2 r_3]) \end{aligned}$$

the goal is to prove that $\text{LHS} = \pm\text{RHS}$ holds. Set

$$\rho = 1 \dots n \setminus r_1 r_2 r_3 \quad \text{and} \quad \{s\} = \overline{\{s_1, s_2, s_3\}}.$$

The possible values of s_3 are 3 and 4, and we treat these cases separately.

Case I. Assuming that $s_3 = 3$ holds one has $s = 4$ and, therefore,

$$(1) \quad \det(\partial_1[1; \overline{123}]) = \text{pf}_{\mathcal{T}}(\overline{123}).$$

Because the first three columns of the matrix ∂_2 are special, our argument depends on the size of the intersection $\{1, 2, 3\} \cap \{r_1, r_2, r_3\}$. We therefore consider four subcases determined by the (in)equalities

$$(2) \quad r_3 = 3, \quad r_2 \leq 3 < r_3, \quad r_1 \leq 3 < r_2, \quad \text{and} \quad r_1 < 3.$$

Subcase I.a. If $r_3 = 3$ holds, then (1) and Lemma B.2 yield

$$\text{LHS} = (\text{pf}_{\mathcal{T}}(\overline{123}))^2 \text{pf}_{\mathcal{T}}(\overline{123}),$$

and evidently one has $\text{RHS} = (\text{pf}_{\mathcal{T}}(\overline{123}))^3$.

Subcase I.b. If $r_2 \leq 3 < r_3$ hold, then (1) and Lemma B.2 yield

$$\text{LHS} = \text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) (\text{pf}_{\mathcal{T}}(\overline{123}))^2.$$

Expanding the determinant along the first column one has

$$\begin{aligned} \pm\text{RHS} &= \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & 0 & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ 0 & \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{12r_3}) \end{pmatrix} \\ &= \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{12r_3}) - \text{pf}_{\mathcal{T}}(\overline{13r_3}) \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{123})) \\ &\quad + \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ &= (\text{pf}_{\mathcal{T}}(\overline{123}))^2 \\ &\quad \cdot (\delta_{1r_1} \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{12r_3}) - \delta_{1r_1} \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{13r_3}) + \delta_{2r_1} \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{23r_3})). \end{aligned}$$

For all three choices of $r_1 < r_2$ in $\{1, 2, 3\}$ one gets $\text{RHS} = \pm(\text{pf}_{\mathcal{T}}(\overline{123}))^2 \text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3})$ as desired.

Subcase I.c. If $r_1 \leq 3 < r_2$ hold, then (1) and Lemma B.2 yield

$$\text{LHS} = (\text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1})) \text{pf}_{\mathcal{T}}(\overline{123}).$$

In view of Lemma A.9 this can be rewritten as

$$\begin{aligned} \text{LHS} &= \delta_{1r_1} (\text{pf}_{\mathcal{T}}(\overline{12r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{12r_3})) \text{pf}_{\mathcal{T}}(\overline{123}) \\ &\quad + \delta_{2r_1} (\text{pf}_{\mathcal{T}}(\overline{12r_2}) \text{pf}_{\mathcal{T}}(\overline{23r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{12r_3})) \text{pf}_{\mathcal{T}}(\overline{123}) \\ &\quad + \delta_{3r_1} (\text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{23r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3})) \text{pf}_{\mathcal{T}}(\overline{123}). \end{aligned}$$

Expansion of the determinant along the first column yields the matching expression

$$\begin{aligned} \pm\text{RHS} &= \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{23r_2}) & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{13r_2}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{12r_2}) & \text{pf}_{\mathcal{T}}(\overline{12r_3}) \end{pmatrix} \\ &= \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{12r_3}) - \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{12r_2})) \\ &\quad - \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{12r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{12r_2})) \\ &\quad + \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{13r_2})). \end{aligned}$$

Subcase I.d. If $3 < r_1$ holds, then (1) and Lemma B.2 yield

$$\begin{aligned} \text{LHS} &= (\text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{23 r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{1}) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{13 r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{12 r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{3})) \text{pf}_{\mathcal{T}}(\overline{123}). \end{aligned}$$

Expansion of the determinant along the first column yields the second equality in the computation below. The third equality follows from Lemma A.9 while the fifth follows from Lemmas A.6 and A.7. Finally, the last equality follows from another application of Lemma A.6.

$$\begin{aligned} \pm \text{RHS} &= \det \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{23 r_1}) & \text{pf}_{\mathcal{T}}(\overline{23 r_2}) & \text{pf}_{\mathcal{T}}(\overline{23 r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{13 r_1}) & \text{pf}_{\mathcal{T}}(\overline{13 r_2}) & \text{pf}_{\mathcal{T}}(\overline{13 r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{12 r_1}) & \text{pf}_{\mathcal{T}}(\overline{12 r_2}) & \text{pf}_{\mathcal{T}}(\overline{12 r_3}) \end{pmatrix} \\ &= \text{pf}_{\mathcal{T}}(\overline{23 r_1}) (\text{pf}_{\mathcal{T}}(\overline{13 r_2}) \text{pf}_{\mathcal{T}}(\overline{12 r_3}) - \text{pf}_{\mathcal{T}}(\overline{13 r_3}) \text{pf}_{\mathcal{T}}(\overline{12 r_2})) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{13 r_1}) (\text{pf}_{\mathcal{T}}(\overline{23 r_2}) \text{pf}_{\mathcal{T}}(\overline{12 r_3}) - \text{pf}_{\mathcal{T}}(\overline{23 r_3}) \text{pf}_{\mathcal{T}}(\overline{12 r_2})) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{12 r_1}) (\text{pf}_{\mathcal{T}}(\overline{23 r_2}) \text{pf}_{\mathcal{T}}(\overline{13 r_3}) - \text{pf}_{\mathcal{T}}(\overline{23 r_3}) \text{pf}_{\mathcal{T}}(\overline{13 r_2})) \\ &= \text{pf}_{\mathcal{T}}(\overline{23 r_1}) (\text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{1}) - \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{1 r_2 r_3})) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{13 r_1}) (\text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{2 r_2 r_3})) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{12 r_1}) (\text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{3 r_2 r_3})) \\ &= \text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) (\text{pf}_{\mathcal{T}}(\overline{23 r_1}) \text{pf}_{\mathcal{T}}(\overline{1}) - \text{pf}_{\mathcal{T}}(\overline{13 r_1}) \text{pf}_{\mathcal{T}}(\overline{2}) + \text{pf}_{\mathcal{T}}(\overline{12 r_1}) \text{pf}_{\mathcal{T}}(\overline{3})) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{23 r_1}) \text{pf}_{\mathcal{T}}(\overline{1 r_2 r_3}) \\ &\quad \quad - \text{pf}_{\mathcal{T}}(\overline{13 r_1}) \text{pf}_{\mathcal{T}}(\overline{2 r_2 r_3}) + \text{pf}_{\mathcal{T}}(\overline{12 r_1}) \text{pf}_{\mathcal{T}}(\overline{3 r_2 r_3})) \\ &= \text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{r_1}) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{123 r_1 r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2}) \\ &\quad \quad - \text{pf}_{\mathcal{T}}(\overline{123 r_1 r_2}) \text{pf}_{\mathcal{T}}(\overline{r_3}) + \text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123})) \\ &= -\text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1}) \\ &\quad \quad + \text{pf}_{\mathcal{T}}(\overline{123 r_1 r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2}) - \text{pf}_{\mathcal{T}}(\overline{123 r_1 r_2}) \text{pf}_{\mathcal{T}}(\overline{r_3})) \\ &= -\text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{23 r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{1}) \\ &\quad \quad + \text{pf}_{\mathcal{T}}(\overline{13 r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{12 r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{3})). \end{aligned}$$

Thus $\text{LHS} = \pm \text{RHS}$ holds, also in this subcase.

Case II. Assuming now that $s_3 = 4$ holds, one has $s \in \{1, 2, 3\}$ and hence

$$(3) \quad \det(\partial_1[1; \overline{s_1 s_2 s_3}]) = (-1)^s \text{pf}_{\mathcal{T}}(\overline{s}).$$

As in Case I the argument is broken into subcases following the (in)equalities (2).

Subcase II.a. If $r_3 = 3$ holds, then (3) and Lemma B.2 yield

$$\text{LHS} = \pm (\text{pf}_{\mathcal{T}}(\overline{123}))^2 \text{pf}_{\mathcal{T}}(\overline{s}),$$

and evidently one has $\text{RHS} = \pm (\text{pf}_{\mathcal{T}}(\overline{123}))^2 \text{pf}_{\mathcal{T}}(\overline{s})$.

Subcase II.b. If $r_2 \leq 3 < r_3$ hold, then (3) and Lemma B.2 again yield

$$\text{LHS} = \pm \text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{s}).$$

This has to be compared to

$$\text{RHS} = \pm \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & 0 & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ 0 & \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{12r_3}) \\ (-1)^{r_1-1} \text{pf}_{\mathcal{T}}(\overline{r_1}) & (-1)^{r_2-1} \text{pf}_{\mathcal{T}}(\overline{r_2}) & \text{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} [s_1 s_2 4; 123].$$

Notice that the zeros in the matrix stand for $\delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{123})$ and $\delta_{1r_2} \text{pf}_{\mathcal{T}}(\overline{123})$; the determinant is thus symmetric in the three possible choices of $\{s_1, s_2\} \subset \{1, 2, 3\}$. By this symmetry it is sufficient to treat the choice $\{s_1, s_2\} = \{1, 2\}$. In this case one has $s = 3$ and, therefore,

$$(4) \quad \text{LHS} = \pm \text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{3}).$$

Expansion of the determinant along the first column yields

$$\begin{aligned} \pm \text{RHS} &= \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & 0 & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ (-1)^{r_1-1} \text{pf}_{\mathcal{T}}(\overline{r_1}) & (-1)^{r_2-1} \text{pf}_{\mathcal{T}}(\overline{r_2}) & \text{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} \\ (5) \quad &= \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{r_3}) + (-1)^{r_2} \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2})) \\ &\quad + (-1)^{r_2} \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2}) \\ &\quad + (-1)^{r_1} \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1}). \end{aligned}$$

For $\{r_1, r_2\} = \{1, 2\}$ one has $\text{LHS} = \pm \text{pf}_{\mathcal{T}}(\overline{12r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{3})$. In the next computation, which shows that this agrees with $\pm \text{RHS}$, the last equality follows from Lemma A.6.

$$\begin{aligned} \pm \text{RHS} &= \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{r_3}) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{1})) \\ &= \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{1}) + \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{2})) \\ &= \text{pf}_{\mathcal{T}}(\overline{123}) (-\text{pf}_{\mathcal{T}}(\overline{3}) \text{pf}_{\mathcal{T}}(\overline{12r_3})). \end{aligned}$$

For $\{r_1, r_2\} = \{1, 3\}$ one has $\text{LHS} = \pm \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{3})$, see (4), and (5) specializes to the same expression. Similarly, for $\{r_1, r_2\} = \{2, 3\}$ one has $\text{LHS} = \pm \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{3})$ and (5) specializes to the same expression.

Subcase II.c. If $r_1 \leq 3 < r_2$ hold, then (3) and Lemma B.2 yield

$$\text{LHS} = \pm (\text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1})) \text{pf}_{\mathcal{T}}(\overline{s}).$$

This has to be compared to

$$\text{RHS} = \pm \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{23r_2}) & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{13r_2}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{12r_2}) & \text{pf}_{\mathcal{T}}(\overline{12r_3}) \\ (-1)^{r_1-1} \text{pf}_{\mathcal{T}}(\overline{r_1}) & \text{pf}_{\mathcal{T}}(\overline{r_2}) & \text{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} [s_1 s_2 4; 123].$$

This determinant is symmetric in the three possible choices of $\{s_1, s_2\} \subset \{1, 2, 3\}$. It suffices to treat the case $\{s_1, s_2\} = \{1, 2\}$, where one has $s = 3$ and, therefore,

$$(6) \quad \text{LHS} = \pm (\text{pf}_{\mathcal{T}}(\overline{r_1 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{123 r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1})) \text{pf}_{\mathcal{T}}(\overline{3}).$$

Expanding the determinant along the first column one gets

$$\pm \text{RHS} = \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{23r_2}) & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) & \text{pf}_{\mathcal{T}}(\overline{13r_2}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ (-1)^{r_1-1} \text{pf}_{\mathcal{T}}(\overline{r_1}) & \text{pf}_{\mathcal{T}}(\overline{r_2}) & \text{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix}$$

$$\begin{aligned}
&= \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{r_3}) - \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2})) \\
&\quad - \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2})) \\
&\quad + (-1)^{r_1-1} \text{pf}_{\mathcal{T}}(\overline{r_1}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{13r_2})).
\end{aligned}$$

For $r_1 = 1$ this expression specializes to

$$\begin{aligned}
\pm\text{RHS} &= \text{pf}_{\mathcal{T}}(\overline{123}) (\text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{r_3}) - \text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{r_2})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{1}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{13r_2})) \\
&= \text{pf}_{\mathcal{T}}(\overline{13r_2}) (\text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{1})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{13r_3}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{1}) - \text{pf}_{\mathcal{T}}(\overline{123}) \text{pf}_{\mathcal{T}}(\overline{r_2})) \\
&= \text{pf}_{\mathcal{T}}(\overline{13r_2}) (\text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{12r_3}) \text{pf}_{\mathcal{T}}(\overline{3})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{13r_3}) (\text{pf}_{\mathcal{T}}(\overline{12r_2}) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{2})) \\
&= (\text{pf}_{\mathcal{T}}(\overline{13r_3}) \text{pf}_{\mathcal{T}}(\overline{12r_2}) - \text{pf}_{\mathcal{T}}(\overline{13r_2}) \text{pf}_{\mathcal{T}}(\overline{12r_3})) \text{pf}_{\mathcal{T}}(\overline{3}) \\
&= (\text{pf}_{\mathcal{T}}(\overline{1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{123r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{1})) \text{pf}_{\mathcal{T}}(\overline{3}),
\end{aligned}$$

where the third equality follows from Lemma A.6 and the last equality holds by Lemma A.9. This matches (6).

For $r_1 = 2$ a parallel computation using the same lemmas yields

$$\text{RHS} = \pm (\text{pf}_{\mathcal{T}}(\overline{123r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{2r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123})) \text{pf}_{\mathcal{T}}(\overline{3}),$$

which again matches (6).

For $r_1 = 3$ the RHS expression specializes to

$$\begin{aligned}
\pm\text{RHS} &= \text{pf}_{\mathcal{T}}(\overline{3}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{13r_2})) \\
&= (\text{pf}_{\mathcal{T}}(\overline{123r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{3r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123})) \text{pf}_{\mathcal{T}}(\overline{3}),
\end{aligned}$$

where the second equality holds by Lemma A.9. This matches (6).

Subcase II.d. If $3 < r_1$ holds, then (3) and Lemma B.2 yield

$$\begin{aligned}
\text{LHS} &= \pm (\text{pf}_{\mathcal{T}}(\overline{r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{1})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{3}) \text{pf}_{\mathcal{T}}(\overline{s}).
\end{aligned}$$

This has to be compared to

$$\text{RHS} = \pm \det \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{23r_1}) & \text{pf}_{\mathcal{T}}(\overline{23r_2}) & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{13r_1}) & \text{pf}_{\mathcal{T}}(\overline{13r_2}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{12r_1}) & \text{pf}_{\mathcal{T}}(\overline{12r_2}) & \text{pf}_{\mathcal{T}}(\overline{12r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{r_1}) & \text{pf}_{\mathcal{T}}(\overline{r_2}) & \text{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} [s_1 s_2 4; 123].$$

This determinant is symmetric in the three possible choices of $\{s_1, s_2\} \subset \{1, 2, 3\}$. It is sufficient to treat the case $\{s_1, s_2\} = \{1, 2\}$, where one has $s = 3$ and, therefore,

$$\begin{aligned}
(7) \quad \text{LHS} &= \pm (\text{pf}_{\mathcal{T}}(\overline{r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123}) - \text{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{1})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{2}) - \text{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{3}) \text{pf}_{\mathcal{T}}(\overline{3}).
\end{aligned}$$

Expansion along the third row yields

$$\begin{aligned}
\pm\text{RHS} &= \det \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{23r_1}) & \text{pf}_{\mathcal{T}}(\overline{23r_2}) & \text{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{13r_1}) & \text{pf}_{\mathcal{T}}(\overline{13r_2}) & \text{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{r_1}) & \text{pf}_{\mathcal{T}}(\overline{r_2}) & \text{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} \\
&= \text{pf}_{\mathcal{T}}(\overline{r_1}) (\text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{13r_2}))
\end{aligned}$$

$$\begin{aligned}
& - \text{pf}_{\mathcal{T}}(\overline{r_2})(\text{pf}_{\mathcal{T}}(\overline{23r_1}) \text{pf}_{\mathcal{T}}(\overline{13r_3}) - \text{pf}_{\mathcal{T}}(\overline{23r_3}) \text{pf}_{\mathcal{T}}(\overline{13r_1})) \\
& \quad + \text{pf}_{\mathcal{T}}(\overline{r_3})(\text{pf}_{\mathcal{T}}(\overline{23r_1}) \text{pf}_{\mathcal{T}}(\overline{13r_2}) - \text{pf}_{\mathcal{T}}(\overline{23r_2}) \text{pf}_{\mathcal{T}}(\overline{13r_1})) \\
= & \text{pf}_{\mathcal{T}}(\overline{r_1})(\text{pf}_{\mathcal{T}}(\overline{123r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{3r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123})) \\
& - \text{pf}_{\mathcal{T}}(\overline{r_2})(\text{pf}_{\mathcal{T}}(\overline{123r_1r_3}) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{3r_1r_3}) \text{pf}_{\mathcal{T}}(\overline{123})) \\
& \quad + \text{pf}_{\mathcal{T}}(\overline{r_3})(\text{pf}_{\mathcal{T}}(\overline{123r_1r_2}) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{3r_1r_2}) \text{pf}_{\mathcal{T}}(\overline{123})) \\
= & (\text{pf}_{\mathcal{T}}(\overline{r_1}) \text{pf}_{\mathcal{T}}(\overline{123r_2r_3}) \\
& \quad - \text{pf}_{\mathcal{T}}(\overline{r_2}) \text{pf}_{\mathcal{T}}(\overline{123r_1r_3}) + \text{pf}_{\mathcal{T}}(\overline{r_3}) \text{pf}_{\mathcal{T}}(\overline{123r_1r_2})) \text{pf}_{\mathcal{T}}(\overline{3}) \\
& - (\text{pf}_{\mathcal{T}}(\overline{r_1}) \text{pf}_{\mathcal{T}}(\overline{3r_2r_3}) \\
& \quad - \text{pf}_{\mathcal{T}}(\overline{r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_1r_3}) + \text{pf}_{\mathcal{T}}(\overline{r_3}) \text{pf}_{\mathcal{T}}(\overline{3r_1r_2})) \text{pf}_{\mathcal{T}}(\overline{123}) \\
= & (\text{pf}_{\mathcal{T}}(\overline{1}) \text{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3}) - \text{pf}_{\mathcal{T}}(\overline{2}) \text{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3}) \\
& \quad + \text{pf}_{\mathcal{T}}(\overline{3}) \text{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3})) \text{pf}_{\mathcal{T}}(\overline{3}) - \text{pf}_{\mathcal{T}}(\overline{3}) \text{pf}_{\mathcal{T}}(\overline{r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{123})
\end{aligned}$$

where the last two equalities follow from Lemmas A.6 and A.9. \square

Quotients of odd type. The proofs of C.5–C.8 below are, if anything, slightly simpler than the proofs of C.1–C.4.

C.5 Lemma. *Let $n \geq 6$ be an even number and adopt the setup from 2.9. The sequence $0 \rightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}$ is a complex.*

Proof. The product $\partial_1\partial_2$ is a $1 \times n$ matrix; the first three entries are evidently 0. For $i \in \{4, \dots, n\}$ the i^{th} entry is

$$\pm(\text{pf}_{\mathcal{T}} \text{pf}_{\mathcal{T}}(\overline{123i}) - \text{pf}_{\mathcal{T}}(\overline{12}) \text{pf}_{\mathcal{T}}(\overline{3i}) + \text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{2i}) - \text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{1i})),$$

which is zero by Lemma A.4 applied with $u_1 \dots u_k = 123i$ and $u_\ell = i$.

The product $\partial_2\partial_3$ is a $4 \times (n-3)$ matrix. Let $i \in \{4, \dots, n\}$; the entry in position $(1, i-3)$ is

$$\sum_{j=4}^{i-1} (-1)^{j-1} \tau_{ji} \text{pf}_{\mathcal{T}}(\overline{123j}) - \sum_{j=i+1}^n (-1)^{j-1} \tau_{ij} \text{pf}_{\mathcal{T}}(\overline{123j}).$$

Applied with $u_1 \dots u_k = 4 \dots n$ and $u_\ell = i$, Lemma A.3 shows that this quantity is zero. The entry in position $(2, i-3)$ is

$$\tau_{1i} \text{pf}_{\mathcal{T}}(\overline{13}) - \tau_{2i} \text{pf}_{\mathcal{T}}(\overline{23}) + \sum_{j=4}^{i-1} (-1)^j \tau_{ji} \text{pf}_{\mathcal{T}}(\overline{3j}) - \sum_{j=i+1}^n (-1)^j \tau_{ij} \text{pf}_{\mathcal{T}}(\overline{3j}).$$

Applied with $u_1 \dots u_k = 124 \dots n$ and $u_\ell = i$, Lemma A.3 shows that this quantity is zero. Similarly, Lemma A.3 applied with $u_1 \dots u_k = 134 \dots n$ and $u_\ell = i$ shows that the entry in position $(3, i-3)$ is zero, and an application with $u_1 \dots u_k = 2 \dots n$ and $u_\ell = i$ shows that the entry in position $(4, i-3)$ is zero. \square

C.6 Lemma. *Let $n \geq 6$ be an even number and adopt the setup from 2.9. The Pfaffians $\text{pf}_{\mathcal{T}}(\overline{12})$, $\text{pf}_{\mathcal{T}}(\overline{13})$, and $\text{pf}_{\mathcal{T}}(\overline{23})$ form a regular sequence in \mathcal{R} .*

Proof. The $(n-4) \times (n-4)$ Pfaffians of the matrix $\mathcal{T}[4 \dots n; 4 \dots n]$ generate by [16, Corollary 2.5] an ideal of grade 3 in the subring $\mathcal{R}' = \mathbb{Z}[\tau_{ij} \mid 4 \leq i < j \leq n]$ of

\mathcal{R} ; they are the Pfaffians $\text{pf}_{\mathcal{T}}(\overline{123i})$ for $4 \leq i \leq n$. Applied with $u_1 \dots u_k = 3 \dots n$ and $\ell = 1$, Lemma A.2 yields

$$\text{pf}_{\mathcal{T}}(\overline{12}) = \sum_{i=4}^n (-1)^i \tau_{3i} \text{pf}_{\mathcal{T}}(\overline{123i}).$$

Similarly, applied with $u_1 \dots u_k = 24 \dots n$ and $\ell = 2$ the same lemma yields

$$\text{pf}_{\mathcal{T}}(\overline{13}) = \sum_{i=4}^n (-1)^i \tau_{2i} \text{pf}_{\mathcal{T}}(\overline{123i}).$$

Finally, with $u_1 \dots u_k = 14 \dots n$ and $\ell = 1$ one gets

$$\text{pf}_{\mathcal{T}}(\overline{23}) = \sum_{i=4}^n (-1)^i \tau_{1i} \text{pf}_{\mathcal{T}}(\overline{123i}).$$

Now it follows from [13, Lemma 6] that $\text{pf}_{\mathcal{T}}(\overline{12})$, $\text{pf}_{\mathcal{T}}(\overline{13})$, and $\text{pf}_{\mathcal{T}}(\overline{23})$ form a regular sequence in \mathcal{R} . \square

C.7 Lemma. *Let $n \geq 6$ be an even number and adopt the setup from 2.9. The ideal generated by the $(n-3) \times (n-3)$ minors of the matrix ∂_3 contains the elements*

$$(\text{pf}_{\mathcal{T}})^2, \quad (\text{pf}_{\mathcal{T}}(\overline{12}))^2, \quad (\text{pf}_{\mathcal{T}}(\overline{13}))^2, \quad \text{and} \quad (\text{pf}_{\mathcal{T}}(\overline{23}))^2.$$

Proof. One has $(\text{pf}_{\mathcal{T}})^2 = \det(\mathcal{T})$ and expansion of this determinant along the first three columns, see [15, 0.8.9], yields:

$$\begin{aligned} \det(\mathcal{T}) &= \sum_{1 \leq i < j < k \leq n} \pm \det(\mathcal{T}[ijk; 123]) \det(\mathcal{T}[\overline{ijk}; \overline{123}]) \\ &= \sum_{1 \leq i < j < k \leq n} \pm \det(\mathcal{T}[ijk; 123]) \det(\partial_3[\overline{ijk}; 1 \dots n-3]). \end{aligned}$$

Similarly, expanding along the first column one gets

$$\begin{aligned} (\text{pf}_{\mathcal{T}}(\overline{12}))^2 &= \det(\mathcal{T}[3 \dots n; 3 \dots n]) = \sum_{i=3}^n \pm \mathcal{T}[i; 3] \det(\partial_3[\overline{12i}; 1 \dots n-3]), \\ (\text{pf}_{\mathcal{T}}(\overline{13}))^2 &= \det(\mathcal{T}[24 \dots n; 24 \dots n]) = \sum_{\substack{2 \leq i \leq n \\ i \neq 3}} \pm \mathcal{T}[i; 2] \det(\partial_3[\overline{13i}; 1 \dots n-3]), \end{aligned}$$

and

$$(\text{pf}_{\mathcal{T}}(\overline{23}))^2 = \det(\mathcal{T}[14 \dots n; 14 \dots n]) = \sum_{\substack{1 \leq i \leq n \\ i \neq 2,3}} \pm \mathcal{T}[i; 1] \det(\partial_3[\overline{23i}; 1 \dots n-3]). \quad \square$$

C.8 Proposition. *Let $n \geq 6$ be an even number and adopt the setup from 2.9. For integers $1 \leq r_1 < r_2 < r_3 \leq n$ and $1 \leq s_1 < s_2 < s_3 \leq 4$ one has*

$$\det(\partial_3[\overline{r_1 r_2 r_3}; 1 \dots n-3]) \det(\partial_1[1; \overline{s_1 s_2 s_3}]) = \pm \det(\partial_2[s_1 s_2 s_3; r_1 r_2 r_3]).$$

Proof. Notice that $\det(\partial_3[\overline{r_1 r_2 r_3}; 1 \dots n-3]) = \det(\mathcal{T}[\overline{r_1 r_2 r_3}; \overline{123}])$ holds and set

$$\begin{aligned} \text{LHS} &= \det(\mathcal{T}[\overline{r_1 r_2 r_3}; \overline{123}]) \det(\partial_1[1; \overline{s_1 s_2 s_3}]) \quad \text{and} \\ \text{RHS} &= \det(\partial_2[s_1 s_2 s_3; r_1 r_2 r_3]). \end{aligned}$$

The goal is now to prove that $\text{LHS} = \pm \text{RHS}$ holds. Set

$$\rho = 1 \dots n \setminus r_1 r_2 r_3 \quad \text{and} \quad \{s\} = \overline{\{s_1, s_2, s_3\}}.$$

The possible values of s_1 are 1 and 2, and we treat these cases separately.

Case I. Assuming that $s_1 = 1$ holds, one has $s \in \{2, 3, 4\}$. By symmetry it suffices to treat the case $s = 4$. In this case one has

$$(1) \quad \det(\partial_1[1; \overline{s_1 s_2 s_3}]) = \text{pf}_{\mathcal{T}}(\overline{23}).$$

Because the first three columns of the matrix ∂_2 are special, our argument depends on the size of the intersection $\{1, 2, 3\} \cap \{r_1, r_2, r_3\}$. We therefore consider four subcases determined by the (in)equalities

$$(2) \quad r_3 = 3, \quad r_2 \leq 3 < r_3, \quad r_1 \leq 3 < r_2, \quad \text{and} \quad r_1 < 3.$$

Subcase I.a. If $r_3 = 3$ holds, then Lemma B.3 yields LHS = 0, and ∂_2 has a zero row, so RHS = 0 holds as well.

Subcase I.b. If $r_2 \leq 3 < r_3$ hold, then (1) and Lemma B.3 yield

$$\text{LHS} = \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1 r_2}) \text{pf}_{\mathcal{T}}(\overline{23}).$$

Expansion of the determinant along the first row yields

$$\begin{aligned} \pm \text{RHS} &= \det \begin{pmatrix} 0 & 0 & \text{pf}_{\mathcal{T}}(\overline{123r_3}) \\ \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{23}) & -\delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{23}) & -\text{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{12}) & \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{23}) & \text{pf}_{\mathcal{T}}(\overline{2r_3}) \end{pmatrix} \\ &= \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) (\delta_{1r_1} (\delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{12})) - \delta_{2r_1} \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{23})) \\ &= \pm \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{r_1 r_2}). \end{aligned}$$

Subcase I.c. If $r_1 \leq 3 < r_2$ hold, then (1) and Lemma B.3 yield

$$\text{LHS} = \text{pf}_{\mathcal{T}}(\overline{23}) \cdot \begin{cases} \text{pf}_{\mathcal{T}}(\overline{12r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) - \text{pf}_{\mathcal{T}}(\overline{13r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 1 \\ \text{pf}_{\mathcal{T}}(\overline{12r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{23r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 2 \\ \text{pf}_{\mathcal{T}}(\overline{13r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{23r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) & \text{if } r_1 = 3. \end{cases}$$

This has to be compared to

$$\begin{aligned} &\pm \text{RHS} \\ &= \det \begin{pmatrix} 0 & \text{pf}_{\mathcal{T}}(\overline{123r_2}) & \text{pf}_{\mathcal{T}}(\overline{123r_3}) \\ \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{23}) & -\text{pf}_{\mathcal{T}}(\overline{3r_2}) & -\text{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{12}) + \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{23}) & \text{pf}_{\mathcal{T}}(\overline{2r_2}) & \text{pf}_{\mathcal{T}}(\overline{2r_3}) \end{pmatrix} \\ &= -\delta_{1r_1} (\text{pf}_{\mathcal{T}}(\overline{13}) (\text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) - \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_2})) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{12}) (-\text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}) + \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{3r_2}))) \\ &\quad + \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{23}) (\text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) - \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_2})) \\ &\quad + \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{23}) (-\text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}) + \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{3r_2})). \end{aligned}$$

Lemma A.5 applied with $u_1 \dots u_k = 123r_2 r_3$ and $\ell = 2$ and $\ell = 3$ yields

$$\begin{aligned} &-\text{pf}_{\mathcal{T}}(\overline{23r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) + \text{pf}_{\mathcal{T}}(\overline{12r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) \\ &= \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_2}) - \text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) \end{aligned}$$

and

$$\begin{aligned} &-\text{pf}_{\mathcal{T}}(\overline{23r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) + \text{pf}_{\mathcal{T}}(\overline{13r_2 r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) \\ &= \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{3r_2}) - \text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}). \end{aligned}$$

The first of these identities immediately yields $\text{LHS} = \pm\text{RHS}$ in case $r_1 = 2$, and for $r_1 = 3$ the second identity yields the same conclusion. In case $r_1 = 1$ one applies both identities to see that $\text{LHS} = \pm\text{RHS}$ holds.

Subcase I.d. If $3 < r_1$ holds, then (1) and Lemma B.3 yield

$$\begin{aligned} \text{LHS} &= (\text{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{13})) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) - \text{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}). \end{aligned}$$

Expansion of the determinant along the third row yields the second equality in the computation below. The third equality follows from three applications of Lemma A.5 with $u_1 \dots u_k = 123r_2r_3/123r_1r_3/123r_2r_2$ and $\ell = 3$. The fifth follows from Lemma A.5 applied with $u_1 \dots u_k = 23r_1r_2r_3$ and $\ell = 1$ and Lemma A.4 applied with $u_1 \dots u_k = 123r_1r_2r_3$ and $\ell = 2$.

$$\begin{aligned} \pm\text{RHS} &= \det \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{123r_1}) & \text{pf}_{\mathcal{T}}(\overline{123r_2}) & \text{pf}_{\mathcal{T}}(\overline{123r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{3r_1}) & \text{pf}_{\mathcal{T}}(\overline{3r_2}) & \text{pf}_{\mathcal{T}}(\overline{3r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{2r_1}) & \text{pf}_{\mathcal{T}}(\overline{2r_2}) & \text{pf}_{\mathcal{T}}(\overline{2r_3}) \end{pmatrix} \\ &= \text{pf}_{\mathcal{T}}(\overline{2r_1}) (\text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}) - \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{3r_2})) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{2r_2}) (\text{pf}_{\mathcal{T}}(\overline{123r_1}) \text{pf}_{\mathcal{T}}(\overline{3r_3}) - \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{3r_1})) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{2r_3}) (\text{pf}_{\mathcal{T}}(\overline{123r_1}) \text{pf}_{\mathcal{T}}(\overline{3r_2}) - \text{pf}_{\mathcal{T}}(\overline{123r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_1})) \\ &= \text{pf}_{\mathcal{T}}(\overline{2r_1}) (\text{pf}_{\mathcal{T}}(\overline{23r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) - \text{pf}_{\mathcal{T}}(\overline{13r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23})) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{2r_2}) (\text{pf}_{\mathcal{T}}(\overline{23r_1r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) - \text{pf}_{\mathcal{T}}(\overline{13r_1r_3}) \text{pf}_{\mathcal{T}}(\overline{23})) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{2r_3}) (\text{pf}_{\mathcal{T}}(\overline{23r_1r_2}) \text{pf}_{\mathcal{T}}(\overline{13}) - \text{pf}_{\mathcal{T}}(\overline{13r_1r_2}) \text{pf}_{\mathcal{T}}(\overline{23})) \\ &= \text{pf}_{\mathcal{T}}(\overline{13}) (\text{pf}_{\mathcal{T}}(\overline{23r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_1}) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{23r_1r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_2}) + \text{pf}_{\mathcal{T}}(\overline{23r_1r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3})) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{23}) (\text{pf}_{\mathcal{T}}(\overline{13r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_1}) \\ &\quad - \text{pf}_{\mathcal{T}}(\overline{13r_1r_3}) \text{pf}_{\mathcal{T}}(\overline{2r_2}) + \text{pf}_{\mathcal{T}}(\overline{13r_1r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3})) \\ &= \text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{23}) (\text{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) \\ &\quad + \text{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23})). \end{aligned}$$

Up to a sign, this is LHS.

Case II. Assuming that $s_1 = 2$ holds one has $s = 1$ and, therefore,

$$(3) \quad \det(\partial_1[1; \overline{234}]) = \text{pf}_{\mathcal{T}}.$$

As in Case I the argument is broken into subcases following the (in)equalities (2).

Subcase II.a. If $r_3 = 3$, then (3) and Lemma B.3 yield $\text{LHS} = 0$, and one has

$$\begin{aligned} \text{RHS} &= \det \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{13}) & -\text{pf}_{\mathcal{T}}(\overline{23}) & 0 \\ -\text{pf}_{\mathcal{T}}(\overline{12}) & 0 & \text{pf}_{\mathcal{T}}(\overline{23}) \\ 0 & \text{pf}_{\mathcal{T}}(\overline{12}) & -\text{pf}_{\mathcal{T}}(\overline{13}) \end{pmatrix} \\ &= \text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{12}) - \text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{12}) \text{pf}_{\mathcal{T}}(\overline{13}) = 0. \end{aligned}$$

Subcase II.b. If $r_2 \leq 3 < r_3$ hold, then (1) and Lemma B.3 yield

$$\text{LHS} = \text{pf}_{\mathcal{T}}(\overline{123r_3}) \text{pf}_{\mathcal{T}}(\overline{r_1r_2}) \text{pf}_{\mathcal{T}}.$$

In the computation below, the last equality follows from Lemma A.4 applied with $u_1 \dots u_k = 123r_3$ and $\ell = 4$; it shows that LHS and RHS agree up to a sign.

\pm RHS

$$\begin{aligned}
&= \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{23}) & -\delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{23}) & \text{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{12}) & \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{23}) & -\text{pf}_{\mathcal{T}}(\overline{2r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{12}) & \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{12}) - \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{13}) & \text{pf}_{\mathcal{T}}(\overline{1r_3}) \end{pmatrix} \\
&= \delta_{1r_1} \delta_{2r_2} \text{pf}_{\mathcal{T}}(\overline{12}) (\text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) - \text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{12}) \text{pf}_{\mathcal{T}}(\overline{3r_3})) \\
&\quad + \delta_{1r_1} \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{13}) (\text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) + \text{pf}_{\mathcal{T}}(\overline{12}) \text{pf}_{\mathcal{T}}(\overline{3r_3})) \\
&\quad - \delta_{2r_1} \delta_{3r_2} \text{pf}_{\mathcal{T}}(\overline{23}) (\text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) + \text{pf}_{\mathcal{T}}(\overline{12}) \text{pf}_{\mathcal{T}}(\overline{3r_3})) \\
&= \pm \text{pf}_{\mathcal{T}}(\overline{r_1 r_2}) (\text{pf}_{\mathcal{T}}(\overline{23}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{13}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) + \text{pf}_{\mathcal{T}}(\overline{12}) \text{pf}_{\mathcal{T}}(\overline{3r_3})) \\
&= \text{pf}_{\mathcal{T}}(\overline{r_1 r_2}) \text{pf}_{\mathcal{T}} \text{pf}_{\mathcal{T}}(\overline{123r_3}).
\end{aligned}$$

Subcase II.c. If $r_1 \leq 3 < r_2$ hold, then (3) and Lemma B.3 yield

$$\text{LHS} = \text{pf}_{\mathcal{T}} \cdot \begin{cases} \text{pf}_{\mathcal{T}}(\overline{12r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) - \text{pf}_{\mathcal{T}}(\overline{13r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 1 \\ \text{pf}_{\mathcal{T}}(\overline{12r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{23r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 2 \\ \text{pf}_{\mathcal{T}}(\overline{13r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{23r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) & \text{if } r_1 = 3. \end{cases}$$

This has to be compared to

$$\begin{aligned}
\pm \text{RHS} &= \det \begin{pmatrix} \delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{23}) & \text{pf}_{\mathcal{T}}(\overline{3r_2}) & \text{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1} \text{pf}_{\mathcal{T}}(\overline{12}) + \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{23}) & -\text{pf}_{\mathcal{T}}(\overline{2r_2}) & -\text{pf}_{\mathcal{T}}(\overline{2r_3}) \\ \delta_{2r_1} \text{pf}_{\mathcal{T}}(\overline{12}) - \delta_{3r_1} \text{pf}_{\mathcal{T}}(\overline{13}) & \text{pf}_{\mathcal{T}}(\overline{1r_2}) & \text{pf}_{\mathcal{T}}(\overline{1r_3}) \end{pmatrix} \\
&= \delta_{1r_1} (\text{pf}_{\mathcal{T}}(\overline{13}) (-\text{pf}_{\mathcal{T}}(\overline{2r_2}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) + \text{pf}_{\mathcal{T}}(\overline{1r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{12}) (\text{pf}_{\mathcal{T}}(\overline{3r_2}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{1r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}))) \\
&- \delta_{2r_1} (\text{pf}_{\mathcal{T}}(\overline{23}) (-\text{pf}_{\mathcal{T}}(\overline{2r_2}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) + \text{pf}_{\mathcal{T}}(\overline{1r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3})) \\
&\quad - \text{pf}_{\mathcal{T}}(\overline{12}) (-\text{pf}_{\mathcal{T}}(\overline{3r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) + \text{pf}_{\mathcal{T}}(\overline{2r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}))) \\
&- \delta_{3r_1} (\text{pf}_{\mathcal{T}}(\overline{23}) (\text{pf}_{\mathcal{T}}(\overline{3r_2}) \text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{1r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3})) \\
&\quad - \text{pf}_{\mathcal{T}}(\overline{12}) (-\text{pf}_{\mathcal{T}}(\overline{3r_2}) \text{pf}_{\mathcal{T}}(\overline{2r_3}) + \text{pf}_{\mathcal{T}}(\overline{2r_2}) \text{pf}_{\mathcal{T}}(\overline{3r_3}))).
\end{aligned}$$

For $r_1 = 1$ it follows from two applications of Lemma A.4, namely with $u_1 \dots u_k = 12r_2r_3/13r_2r_3$ and $\ell = 1$, that LHS and RHS agree up to a sign. For $r_1 = 2$ one gets the same conclusion by applying Lemma A.4 with $u_1 \dots u_k = 12r_2r_3/23r_2r_3$ and $\ell = 1$. For $r_1 = 3$ one gets the desired conclusion from Lemma A.4 applied with $u_1 \dots u_k = 13r_2r_3/23r_2r_3$ and $\ell = 1$.

Subcase II.d. If $3 < r_1$ holds, then (3) and Lemma B.3 yield

$$\begin{aligned}
\text{LHS} &= (\text{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{23}) - \text{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{13}) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3}) \text{pf}_{\mathcal{T}}(\overline{12}) - \text{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3}) \text{pf}_{\mathcal{T}}) \text{pf}_{\mathcal{T}}.
\end{aligned}$$

Expansion of the determinant along the first column yields the second equality in the computation below. The third equality follows from three applications of Lemma A.4. The fifth follows from two applications of Lemma A.4 with $u_1 \dots u_k = 123r_1r_2r_3/123r_1$ and $\ell = 4$. The last equality follows from Lemma A.8.

\pm RHS

$$\begin{aligned}
&= \det \begin{pmatrix} \text{pf}_{\mathcal{T}}(\overline{3r_1}) & \text{pf}_{\mathcal{T}}(\overline{3r_2}) & \text{pf}_{\mathcal{T}}(\overline{3r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{2r_1}) & \text{pf}_{\mathcal{T}}(\overline{2r_2}) & \text{pf}_{\mathcal{T}}(\overline{2r_3}) \\ \text{pf}_{\mathcal{T}}(\overline{1r_1}) & \text{pf}_{\mathcal{T}}(\overline{1r_2}) & \text{pf}_{\mathcal{T}}(\overline{1r_3}) \end{pmatrix} \\
&= \text{pf}_{\mathcal{T}}(\overline{3r_1})(\text{pf}_{\mathcal{T}}(\overline{2r_2})\text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{2r_3})\text{pf}_{\mathcal{T}}(\overline{1r_2})) \\
&\quad - \text{pf}_{\mathcal{T}}(\overline{2r_1})(\text{pf}_{\mathcal{T}}(\overline{3r_2})\text{pf}_{\mathcal{T}}(\overline{1r_3}) - \text{pf}_{\mathcal{T}}(\overline{3r_3})\text{pf}_{\mathcal{T}}(\overline{1r_2})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{1r_1})(\text{pf}_{\mathcal{T}}(\overline{3r_2})\text{pf}_{\mathcal{T}}(\overline{2r_3}) - \text{pf}_{\mathcal{T}}(\overline{3r_3})\text{pf}_{\mathcal{T}}(\overline{2r_2})) \\
&= \text{pf}_{\mathcal{T}}(\overline{3r_1})(\text{pf}_{\mathcal{T}}(\overline{12r_2r_3})\text{pf}_{\mathcal{T}} - \text{pf}_{\mathcal{T}}(\overline{12})\text{pf}_{\mathcal{T}}(\overline{r_2r_3})) \\
&\quad - \text{pf}_{\mathcal{T}}(\overline{2r_1})(\text{pf}_{\mathcal{T}}(\overline{13r_2r_3})\text{pf}_{\mathcal{T}} - \text{pf}_{\mathcal{T}}(\overline{13})\text{pf}_{\mathcal{T}}(\overline{r_2r_3})) \\
&\quad + \text{pf}_{\mathcal{T}}(\overline{1r_1})(\text{pf}_{\mathcal{T}}(\overline{23r_2r_3})\text{pf}_{\mathcal{T}} - \text{pf}_{\mathcal{T}}(\overline{23})\text{pf}_{\mathcal{T}}(\overline{r_2r_3})) \\
&= (\text{pf}_{\mathcal{T}}(\overline{1r_1})\text{pf}_{\mathcal{T}}(\overline{23r_2r_3}) - \text{pf}_{\mathcal{T}}(\overline{2r_1})\text{pf}_{\mathcal{T}}(\overline{13r_2r_3}) + \text{pf}_{\mathcal{T}}(\overline{3r_1})\text{pf}_{\mathcal{T}}(\overline{12r_2r_3}))\text{pf}_{\mathcal{T}} \\
&\quad + (-\text{pf}_{\mathcal{T}}(\overline{1r_1})\text{pf}_{\mathcal{T}}(\overline{23}) + \text{pf}_{\mathcal{T}}(\overline{2r_1})\text{pf}_{\mathcal{T}}(\overline{13}) - \text{pf}_{\mathcal{T}}(\overline{3r_1})\text{pf}_{\mathcal{T}}(\overline{12}))\text{pf}_{\mathcal{T}}(\overline{r_2r_3}) \\
&= (\text{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\text{pf}_{\mathcal{T}} - \text{pf}_{\mathcal{T}}(\overline{r_1r_2})\text{pf}_{\mathcal{T}}(\overline{123r_3}) + \text{pf}_{\mathcal{T}}(\overline{r_1r_3})\text{pf}_{\mathcal{T}}(\overline{123r_2}) \\
&\quad - \text{pf}_{\mathcal{T}}(\overline{r_2r_3})\text{pf}_{\mathcal{T}}(\overline{123r_1}))\text{pf}_{\mathcal{T}} \\
&= (\text{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\text{pf}_{\mathcal{T}} - \text{pf}_{\mathcal{T}}(\overline{12})\text{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3}) + \text{pf}_{\mathcal{T}}(\overline{13})\text{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3}) \\
&\quad - \text{pf}_{\mathcal{T}}(\overline{23})\text{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3}))\text{pf}_{\mathcal{T}}.
\end{aligned}$$

Up to a sign, this is LHS. \square

ACKNOWLEDGMENTS

We thank Ela Celikbas, Luigi Ferraro, and Jai Laxmi for helpful comments on earlier versions of this paper.

REFERENCES

- [1] J. F. Adams, *Lectures on exceptional Lie groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996, With a foreword by J. Peter May, Edited by Zafer Mahmud and Mamoru Mimura. MR1428422
- [2] Luchezar L. Avramov, *Poincaré series of almost complete intersections of embedding dimension three*, PLISKA Stud. Math. Bulgar. **2** (1981), 167–172. MR0633871
- [3] Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204. MR0961334
- [4] J. Brill, *On the Minors of a Skew-Symmetrical Determinant*, Proc. London Math. Soc. (2) **1** (1904), 103–111. MR1576761
- [5] Michel Brion and Shrawan Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston, Inc., Boston, MA, 2005. MR2107324
- [6] Anne E. Brown, *A structure theorem for a class of grade three perfect ideals*, J. Algebra **105** (1987), no. 2, 308–327. MR0873666
- [7] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988. MR0953963
- [8] David A. Buchsbaum and David Eisenbud, *What makes a complex exact?*, J. Algebra **25** (1973), 259–268. MR0314819
- [9] David A. Buchsbaum and David Eisenbud, *Some structure theorems for finite free resolutions*, Advances in Math. **12** (1974), 84–139. MR0340240
- [10] David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), no. 3, 447–485. MR0453723

- [11] Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *Minors of a skew symmetric matrix: a combinatorial approach*, Electron. J. Linear Algebra **36** (2020), 658–663. MR4165809
- [12] Izzet Coskun, *Lectures in Warsaw Poland on homogeneous varieties, December 2013*, <http://homepages.math.uic.edu/~coskun/poland.html>.
- [13] Jack A. Eagon and Douglas G. Northcott, *Generically acyclic complexes and generically perfect ideals*, Proc. Roy. Soc. Ser. A **299** (1967), 147–172. MR0214586
- [14] Devra Garfinkle, *A NEW CONSTRUCTION OF THE JOSEPH IDEAL*, ProQuest LLC, Ann Arbor, MI, 1982, Thesis (Ph.D.)—Massachusetts Institute of Technology. MR2941017
- [15] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013. MR2978290
- [16] Tadeusz Józefiak and Piotr Pragacz, *Ideals generated by Pfaffians*, J. Algebra **61** (1979), no. 1, 189–198. MR0554859
- [17] Donald E. Knuth, *Overlapping Pfaffians*, Electron. J. Combin. **3** (1996), no. 2, Research Paper 5, approx. 13 pp., The Foata Festschrift. MR1392490
- [18] Venkatramani Lakshmibai and Komaranapuram N. Raghavan, *Standard monomial theory*, Encyclopaedia of Mathematical Sciences, vol. 137, Springer-Verlag, Berlin, 2008, Invariant theoretic approach, Invariant Theory and Algebraic Transformation Groups, 8. MR2388163
- [19] Laurent Manivel, *On spinor varieties and their secants*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 078, 22 pp. MR2529169
- [20] V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. (2) **122** (1985), no. 1, 27–40. MR0799251
- [21] V. B. Mehta and V. Srinivas, *Normality of Schubert varieties*, Amer. J. Math. **109** (1987), no. 5, 987–989. MR0910360
- [22] S. Ramanan and A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*, Invent. Math. **79** (1985), no. 2, 217–224. MR0778124
- [23] A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Invent. Math. **80** (1985), no. 2, 283–294. MR0788411
- [24] Steven V. Sam and Jerzy Weyman, *Schubert varieties and finite free resolutions of length three*, Proc. Amer. Math. Soc. **149** (2021), no. 5, 1943–1955. MR4232188
- [25] Bernd Ulrich, *Sums of linked ideals*, Trans. Amer. Math. Soc. **318** (1990), no. 1, 1–42. MR0964902
- [26] Jerzy Weyman, *On the structure of free resolutions of length 3*, J. Algebra **126** (1989), no. 1, 1–33. MR1023284
- [27] Jerzy Weyman, *Generic free resolutions and root systems*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 3, 1241–1296. MR3805772

TEXAS TECH UNIVERSITY, LUBBOCK, TX 79409, U.S.A.

Email address: `lars.w.christensen@ttu.edu`

URL: <http://www.math.ttu.edu/~lchrste>

NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, U.S.A.

Email address: `o.veliche@northeastern.edu`

URL: <https://web.northeastern.edu/oveliche>

JAGIELLONIAN UNIVERSITY, 30-348 KRAKÓW, POLAND AND

UNIVERSITY OF CONNECTICUT, STORRS, CT 06269, U.S.A.

Email address: `jerzy.weyman@uj.edu.pl`

URL: <http://www.math.uconn.edu/~weyman>