PROGRAM FOR
MASTERCLASS ON DERIVED CATEGORY METHODS IN RING THEORY

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This document contains a plan for the lectures and problem sessions for the

Masterclass on derived category methods in ring theory

at Aarhus University 13–16 August 2024. The masterclass is based on the book manuscript

Derived category methods in commutative algebra

(to appear in the book series Springer Monographs in Mathematics) and all references in this document are to this manuscript.

FORMAT

As it appears from the following pages, the masterclass consists of eight lectures and eight problem sessions. Each lecture is divided into a list of bullet points of the form:

- *Some headline* [e.g. The category of chain complexes]
  A list of highlights from the book manuscript to be (partially) covered in the lecture [e.g. 2.1.22, ...]

In the lectures we plan to give a fairly representative cross section of Parts I and II of the book manuscript and, on the final day, some highlights from Part III. Few to no proofs will be given (they are left for self-study); instead we will try to define, explain, and exemplify the tools available in the derived category and demonstrate their usefulness in ring theory.

The exercises listed for Problems Class X generally revolve around to the material covered in Lecture X. In the problem session the participants will solve the listed exercises individually or in groups; a subset of the lecturers and/or the organizers will be present to offer help and guidance as needed.

This document will be updated up to and during the workshop; the present version was released on August 15, 2024.

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TUESDAY 13 AUGUST 2024

Lecture I The category of chain complexes **Lars, 9:30–10:30**

- The category of chain complexes
  2.1.22, 2.1.28, 2.1.37–38
- Shift, homology, acyclicity, and homotopy
  2.2.1, 2.2.7, 2.2.23, 2.2.26
- The (total) Hom complex
  2.3.1, 2.3.3, 2.3.5, 2.3.10–11, 2.3.14, 2.3.16
- The (total) tensor product complex
  2.4.1, 2.4.4, 2.4.9–10, 2.4.13–14
- Supremum, infimum, and truncations
  2.5.4, 2.5.20, 2.5.23–25

Lecture II Special morphisms and natural transformations **Henrik, 11:00–12:00**

- Mapping cones and quasi-isomorphisms
  4.1.1, 4.1.5, 4.1.7, 4.2.1–4, 4.2.16
- Homotopy equivalences and contractibility
  4.3.1, 4.3.12–13, 4.3.19–20, 4.3.22, 4.3.30
- Standard isomorphisms (commutativity, associativity, swap, adjunction)
  4.4.4, 4.4.7, 4.4.10, 4.4.12
- Evaluation maps (biduality, homothety formation, tensor evaluation, homomorphism evaluation)
  4.5.2–3, 4.5.5, 4.5.9–10, 4.5.12–13

Notes: We assume some knowledge of rings and modules; in particular, we assume familiarity with the notions and basic properties of projective, injective, and flat modules (see Chap. 1).

For a $\Bbbk$-algebra R the category of chain complexes of R-modules is a $\Bbbk$-linear abelian category with set-indexed limits and colimits, and these can be computed degreewise in the category of R-modules. Various facets of limits and colimits, like the Mittag-Leffler Condition, are treated in Chap. 3; while these are important for several results in the book, they play little role in this masterclass.

Problems Class I **14:00–15:00**

- E 2.2.2–3, E 2.2.17, E 2.2.20–21
- E 2.3.4, E 2.3.8
- E 2.4.1
- E 2.5.8, E 2.5.16

Problems Class II **15:30–16:30**

- E 4.1.2
- E 4.2.2, E 4.2.6
- E 4.3.1, E 4.3.11, E 4.3.19, E 4.3.21–22
- E 4.5.2, E 4.5.5

WEDNESDAY 14 AUGUST 2024

Lecture III Resolutions of complexes and the homotopy category Lars, 9:30–10:30

- Semi-projectivity
  5.2.5, 5.2.8, 5.2.10, 5.2.13–16, 5.2.19–21
- Semi-freeness
  5.1.1, 5.1.7–8
- Semi-injectivity
  5.3.9, 5.3.12–13, 5.3.16, 5.3.19, 5.3.22–24, 5.3.26
- Semi-flatness
  5.4.5, 5.4.8–10, 5.4.16
- The homotopy category
  6.1.1–3, 6.1.6–7, 6.2.2–4, 6.1.20, 6.3.11, 6.3.17

Lecture IV The derived category and derived functors Henrik, 11:00–12:00

- The derived category
  6.4.1, 6.4.4, 6.4.10, 6.4.12, 6.4.17, 6.4.24, 6.4.31, 6.5.5, 6.5.7
- Derived functors
  7.2.7–8, 7.3.1, 7.3.20, 7.4.1, 7.4.15
- Standard isomorphisms in the derived category (samples)
  7.5.15–16, 7.5.22–23, 7.5.28–29
- Boundedness and finiteness
  7.6.7–8, 7.6.16, 7.6.18

Notes: The homotopy category and the derived category of a $\Bbbk$-algebra R are $\Bbbk$-linear triangulated categories. We assume familiarity with basic definitions and properties of triangulated categories as described in Appn. E. Triangulated structures are certainly important for several results in the book; however, we do not cover them in the lectures. The definitions of the distinguished triangles in the homotopy category and the derived category and the verifications of the axioms required by a triangulated category can be found in 6.2.2–4 and 6.5.5–7.

Problems Class III 14:00–15:00

- E 5.2.3–4, E 5.2.6, E 5.2.10, 5.2.12, 5.2.15
- E 6.1.2, E 6.1.5
- E 6.3.2

Problems Class IV 15:30–16:30

- E 6.4.3, E 6.4.7–8, E 6.4.10
- E 7.3.1
- E 7.5.3
- E 7.6.4–5

THURSDAY 15 AUGUST 2024

Lecture V Homological dimensions **Henrik, 9:30–10:30**

- Projective dimension
  8.1.2–3, 8.1.8, 8.1.14
- Injective dimension
  8.2.2–3, 8.2.8, 8.2.15
- Flat dimension
  8.3.3–4, 8.3.6, 8.3.11, 8.3.17, 8.3.19
- Evaluation morphisms in the derived category (samples)
  8.4.1–2, 8.4.4–6, 8.4.13, 8.4.18–19, 8.4.25

Lecture VI Dualizing complexes and Iwanaga–Gorenstein rings **Lars, 11:00–12:00**

- Dualizing complexes and Grothendieck Duality
  10.1.12, 10.1.23
- Derived reflexive complexes
  10.2.1, 10.2.4, 10.2.8
- Auslander and Bass Categories; Foxby–Sharp Equivalence
  10.3.3–5, 10.3.7–8, 10.3.10, 10.3.13
- Iwanaga–Gorenstein rings
  8.5.29, 8.5.31, 10.1.24, 10.4.17

Notes: Theorem 8.2.15 (and several other results in the book) uses minimal semi-injective resolutions of complexes. We do not cover these in the lectures, but the relevant definitions and properties (including existence and uniqueness up to isomorphism) can be found in Appn. B.

Problems Class V **14:00–15:00**

- E 8.1.2–5, E 8.1.7–9
- E 8.2.19 (*Hint:* B.56)

Problems Class VI **15:30–16:30**

- E 10.1.4–5
- E 10.3.1–2, E 10.3.4
- E 8.5.12–13

FRIDAY 16 AUGUST 2024

In lectures VII and VIII all rings are assumed to be commutative and Noetherian.

Lecture VII Homological invariants in commutative algebra **Henrik, 9:30–10:30**

- Support and cosupport
  15.1.5, 15.1.15, 15.2.1, 15.2.8
- Depth and width
  11.4.1, 14.3.10, 14.3.21, 14.4.3, 14.4.8
- Depth of derived tensor product (the Auslander–Buchsbaum Formula)
  16.2.24, 16.3.1, 16.4.1, 16.4.2, 17.3.1, 17.3.4, 17.5.4
- Width of derived Hom (the Bass Formula)
  16.2.9, 16.3.9, 16.4.8, 16.4.11, 17.3.11, 17.3.14, 17.5.7
- Matlis Duality
  16.1.39, 18.1.6, 18.1.9, 18.2.40

Lecture VIII The derived category of a commutative Noetherian ring **Lars, 11:00–12:00**

- Derived torsion
  11.2.1, 11.3.20, 14.4.3, 15.3.23
- Local Duality
  18.2.21, 18.3.17–18
- Auslander/Bass categories and Foxby–Sharp Equivalence
  19.1.7, 19.4.15, 19.4.18–19.4.19
- Gorenstein rings
  19.5.3, 19.5.11–12, 19.5.14, 19.5.25
- Regular rings
  20.1.23, 20.2.1, 20.2.12, 20.2.14, 20.2.16

Problems Class VII **14:00–15:00**

- E 14.3.1
- E 14.4.1–2
- E 15.1.2, E 15.1.7, E 15.1.9
- E 15.2.1
- E 18.1.2

Problems Class VIII **15:30–16:30**

- E 11.2.3 (assume that $\mathfrak{a}$ is finitely generated)
- E 18.3.1
- E 19.5.3, E 19.5.6 (hint: 17.3.27)
- E 20.2.1, 20.2.2

Thank you for participating in the masterclass! We hope you enjoyed it.

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