# SEQUENCES FOR COMPLEXES

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## Introduction

Let $R$ be a commutative Noetherian ring and let $M \neq 0$ be a finite (that is, finitely generated) $R$–module. The concept of *$M$–sequences* is central for the study of $R$–modules by methods of homological algebra. Largely, the usefulness of these sequences is based on the following properties:

1° When $\mathfrak{a}$ is an ideal in $R$ and $M/\mathfrak{a}M \neq 0$, the number

$$\inf\{\ell \in \mathsf{Z} \mid \operatorname{Ext}_R^{\ell}(R/\mathfrak{a}, M) \neq 0\},$$

the so-called $\mathfrak{a}$–depth of $M$, is the maximal length of an $M$–sequence in $\mathfrak{a}$, and any maximal $M$–sequence in $\mathfrak{a}$ is of this finite length.

2° If $x_1, \ldots, x_n$ is an $M$–sequence contained in $\mathfrak{p} \in \operatorname{Supp}_R M$, then the sequence of fractions ${}^{x_1}/_1, \ldots, {}^{x_n}/_1$, in the maximal ideal of $R_{\mathfrak{p}}$, is an $M_{\mathfrak{p}}$–sequence.

In commutative algebra, a wave of work dealing with complexes of modules was started by A. Grothendieck, see [9]. The underlying idea is the following: Complexes (that is, complexes of modules) are tacitly involved whenever homological methods are applied, and since hyperhomological algebra, that is, homological algebra for complexes, is a very powerful tool, it is better to work consistently with complexes. Modules are also complexes, concentrated in degree zero, so results for complexes yield results for modules as special cases.

Like most concepts for modules that of $M$–sequences can be extended to complexes in several non-equivalent ways; this short paper explores two such possible extensions: (ordinary) *sequences* and *strong sequences* for complexes. Ordinary sequences have a property corresponding to 1°, at least over local rings where they coincide with the *regular sequences* suggested by H.–B. Foxby in [8, Sec. 12]. But ordinary sequences may fail to localize properly, whereas strong sequences not only enjoy the correspondent property of 2°, but also that of 1° in the special case where $R$ is local and $\mathfrak{a}$ the maximal ideal.

As a rule, the hyperhomological approach not only reproduces known results for modules, but also strengthens some of them. In this case we show, among other things, that also for a non-finite module $M$ the $\mathfrak{a}$–depth is an upper bound for the maximal length of an $M$–sequence in $\mathfrak{a}$, and the $\mathfrak{a}$–depth of such a module may be finite even if $M/\mathfrak{a}M = 0$.

## 1. Conventions, Notation, and Background

Throughout this paper $R$ is a non-trivial, commutative, Noetherian ring. We work in the derived category of the category of $R$–modules; this first section fixes the notation and sums up a few basic results.

**Notation 1.1.** As usual, the set of prime ideals containing an ideal $\mathfrak{a}$ in $R$ is written $\mathrm{V}(\mathfrak{a})$; when $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence in $R$ we write $\mathrm{V}(\boldsymbol{x})$ for the set of prime ideals containing $\boldsymbol{x}$. The set of *zero-divisors* for an $R$–module $M$ is denoted by $\mathrm{z}_R\, M$.

The ring $R$ is said to be *local* if it has a unique maximal ideal $\mathfrak{m}$, the residue field $R/\mathfrak{m}$ is then denoted by $k$. In general, for $\mathfrak{p} \in \operatorname{Spec} R$ the residue field of the local ring $R_\mathfrak{p}$ is denoted by $k(\mathfrak{p})$, that is, $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.

**Complexes 1.2.** An *$R$–complex* $X$ is a sequence of $R$–modules $X_\ell$ and $R$–linear maps, so-called *differentials*, $\partial_\ell^X : X_\ell \to X_{\ell-1}$, $\ell \in \mathsf{Z}$. Composition of two consecutive differentials always yields the zero map, i.e. $\partial_\ell^X \partial_{\ell+1}^X = 0$. If $X_\ell = 0$ for $\ell \neq 0$, we identify $X$ with the module in degree 0, and an $R$–module $M$ is considered as a complex $0 \to M \to 0$ with $M$ in degree 0.

A *morphism* $\alpha : X \to Y$ of $R$–complexes is a sequence of $R$–linear maps $\alpha_\ell : X_\ell \to Y_\ell$ satisfying $\partial_\ell^Y \alpha_\ell - \alpha_{\ell-1}\partial_\ell^X = 0$ for $\ell \in \mathsf{Z}$. We say that a morphism is a *quasi-isomorphism* if it induces an isomorphism in homology. The symbol $\simeq$ is used to indicate quasi-isomorphisms while $\cong$ indicates isomorphisms of complexes (and hence modules). For an element $r \in R$ the morphism $r_X : X \to X$ is given by multiplication by $r$.

The numbers *supremum*, *infimum*, and *amplitude*: $\sup X = \sup\{\ell \in \mathsf{Z} \mid \mathrm{H}_\ell(X) \neq 0\}$, $\inf X = \inf\{\ell \in \mathsf{Z} \mid \mathrm{H}_\ell(X) \neq 0\}$, and $\operatorname{amp} X = \sup X - \inf X$, capture the homological position and size of $X$. By convention, $\sup X = -\infty$ and $\inf X = \infty$ if $X \simeq 0$.

**Derived Functors 1.3.** The *derived category* of the category of $R$–modules is the category of $R$–complexes localized at the class of all quasi-isomorphisms (see [9] and [13]), we denote it by $\mathcal{D}(R)$. The symbol $\simeq$ is used for isomorphisms in $\mathcal{D}(R)$; a morphism of complexs is a quasi-isomorphism exactly if it represents an isomorphism in the derived category, so this is in agreement with the notation introduced above.

The full subcategories $\mathcal{D}_+(R)$, $\mathcal{D}_-(R)$, $\mathcal{D}_\mathrm{b}(R)$, and $\mathcal{D}_0(R)$ consist of complexes $X$ with $\mathrm{H}_\ell(X) = 0$ for, respectively, $\ell \ll 0$, $\ell \gg 0$, $|\ell| \gg 0$, and $\ell \neq 0$. By $\mathcal{D}^\mathrm{f}(R)$ we denote the full subcategory of $\mathcal{D}(R)$ consisting of complexes $X$ with $\mathrm{H}_\ell(X)$ a finite $R$–module for all $\ell \in \mathsf{Z}$. We also use combined notations: $\mathcal{D}^\mathrm{f}_-(R) = \mathcal{D}_-(R) \cap \mathcal{D}^\mathrm{f}(R)$, etc. The category of $R$–modules, respectively, finite $R$–modules, is naturally identified with $\mathcal{D}_0(R)$, respectively, $\mathcal{D}^\mathrm{f}_0(R)$.

The right derived functor of the homomorphism functor for $R$–complexes is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$, and $- \otimes^\mathbf{L}_R -$ is the left derived functor of the tensor product functor for $R$–complexes; by [2] and [12] no boundedness conditions are needed on the arguments. That is, for $X, Y \in \mathcal{D}(R)$ the complexes $\mathbf{R}\mathrm{Hom}_R(X,Y)$ and $X \otimes^\mathbf{L}_R Y$ are uniquely determined up to isomorphism in $\mathcal{D}(R)$, and they have the expected functorial properties. Note that $\mathrm{Tor}^R_\ell(M,N) = \mathrm{H}_\ell(M \otimes^\mathbf{L}_R N)$ and $\mathrm{Ext}^\ell_R(M,N) = \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(M,N))$ for $M, N \in \mathcal{D}_0(R)$ and $\ell \in \mathsf{Z}$.

Let $\mathfrak{p} \in \operatorname{Spec} R$; by [2, 5.2] there are isomorphisms: $(X \otimes^\mathbf{L}_R Y)_\mathfrak{p} \simeq X_\mathfrak{p} \otimes^\mathbf{L}_{R_\mathfrak{p}} Y_\mathfrak{p}$ and $\mathbf{R}\mathrm{Hom}_R(Z,Y)_\mathfrak{p} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(Z_\mathfrak{p}, Y_\mathfrak{p})$ in $\mathcal{D}(R_\mathfrak{p})$. The first one always holds, and the second holds when $Y \in \mathcal{D}_-(R)$ and $Z \in \mathcal{D}^\mathrm{f}_+(R)$.

The next results are standard, cf. [6, (2.1)]. Let $X \in \mathcal{D}_+(R)$ and $Y \in \mathcal{D}_-(R)$, then $\mathbf{R}\mathrm{Hom}_R(X,Y) \in \mathcal{D}_-(R)$ and there is an inequality:

$$\sup \mathbf{R}\mathrm{Hom}_R(X,Y) \leq \sup Y - \inf X. \tag{1.3.1}$$

Setting $i = \inf X$ and $s = \sup Y$ we have $\mathrm{H}_{s-i}(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y))$; in particular,

$$\sup \mathbf{R}\mathrm{Hom}_R(X,Y) = \sup Y - \inf X \iff \mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y)) \neq 0. \tag{1.3.2}$$

Let $X, Y \in \mathcal{D}_+(R)$, then $X \otimes^{\mathbf{L}}_R Y \in \mathcal{D}_+(R)$ and there is an inequality

$$\inf (X \otimes^{\mathbf{L}}_R Y) \geq \inf X + \inf Y; \tag{1.3.3}$$

furthermore, with $i = \inf X$ and $j = \inf Y$ we have

$$\mathrm{H}_{i+j}(X \otimes^{\mathbf{L}}_R Y) = \mathrm{H}_i(X) \otimes_R \mathrm{H}_j(Y). \tag{1.3.4}$$

**Depth over Local Rings 1.4.** Let $R$ be local; in [7, Sec. 3] the *depth* and (*Krull*) *dimension* of an $R$–complex $X$ are defined as follows:

$$\operatorname{depth}_R X = -\sup \mathbf{R}\mathrm{Hom}_R(k, X), \text{ for } X \in \mathcal{D}_-(R); \quad \text{and}$$
$$\dim_R X = \sup \{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Note that for modules these notions agree with the usual ones.

It follows immediately by 1.3.1 that $-\sup X \leq \operatorname{depth}_R X$ for $X \in \mathcal{D}_-(R)$, and if $s = \sup X > -\infty$ the next biconditional holds, cf. 1.3.2.

$$\operatorname{depth}_R X = -\sup X \iff \mathfrak{m} \in \operatorname{Ass}_R \mathrm{H}_s(X). \tag{1.4.1}$$

For $X \in \mathcal{D}_-(R)$ and $M \in \mathcal{D}^{\mathrm{f}}_0(R)$ the next equality holds, cf. [7, 3.4].

$$-\sup \mathbf{R}\mathrm{Hom}_R(M, X) = \inf \{\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}. \tag{1.4.2}$$

Let $X \in \mathcal{D}^{\mathrm{f}}_-(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$; a complex version of [3, (3.1)], cf. [5, (13.13)], accounts for the inequality

$$\operatorname{depth}_R X \leq \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p}. \tag{1.4.3}$$

Finally, let $X \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_-(R)$ and set $s = \sup X$; applying 1.4.3 to $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(X)$ with $\dim R/\mathfrak{p} = \dim_R \mathrm{H}_s(X)$ and using 1.4.1 we obtain the next inequalities.

$$\operatorname{depth}_R X + \sup X \leq \dim_R \mathrm{H}_s(X) \leq \dim R. \tag{1.4.4}$$

## 2. Ann, Supp, and Ass for Complexes

As for modules, regular elements for complexes are linked to concepts of *zero-divisors* and *associated prime ideals*. These are introduced below within the relevant setting of support and annihilators.

**Weak Notions 2.1.** Weak notions of *support* and *annihilators* for $X \in \mathcal{D}(R)$ are defined by uniting/intersecting the corresponding sets for the homology modules $\mathrm{H}_\ell(X)$, cf. [7, Sec. 2] and [1, Sec. 2]:

$$\operatorname{Supp}_R X = \bigcup_{\ell \in \mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_\ell(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \not\simeq 0\}; \quad \text{and}$$
$$\operatorname{Ann}_R X = \bigcap_{\ell \in \mathbb{Z}} \operatorname{Ann}_R \mathrm{H}_\ell(X) = \{r \in R \mid \mathrm{H}(r_X) = 0\}.$$

These are complemented by the next definitions. For $X \not\simeq 0$ in $\mathcal{D}_-(R)$ we set

$$\operatorname{ass}_R X = \operatorname{Ass}_R \mathrm{H}_{\sup X}(X) \quad \text{and} \quad \mathrm{z}_R X = \mathrm{z}_R \mathrm{H}_{\sup X}(X),$$

cf. [8, Sec. 12], and for $X \simeq 0$ we set $\operatorname{ass}_R X = \emptyset$ and $\mathrm{z}_R X = \emptyset$.

**The Small Support 2.2.** The *small*, or *homological*, *support* for $X \in \mathcal{D}_+(R)$ was introduced in [7, Sec. 2]:

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p}) \not\simeq 0\}.$$

Its principal properties developed ibid. are as follows:

Let $X \in \mathcal{D}_+(R)$. Then

$$X \not\simeq 0 \iff \operatorname{supp}_R X \neq \emptyset; \tag{2.2.1}$$

there is an inclusion

$$\operatorname{supp}_R X \subseteq \operatorname{Supp}_R X, \tag{2.2.2}$$

and equality holds when $X \in \mathcal{D}^{\mathrm{f}}_+(R)$. For $X, Y \in \mathcal{D}_+(R)$ the next equality holds.

$$\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{supp}_R X \cap \operatorname{supp}_R Y. \tag{2.2.3}$$

If $R$ is local, the next biconditional holds for $X \in \mathcal{D}_\mathrm{b}(R)$.

$$\mathfrak{m} \in \operatorname{supp}_R X \quad \iff \quad \operatorname{depth}_R X < \infty. \tag{2.2.4}$$

**Definitions 2.3.** Let $X \in \mathcal{D}_-(R)$; we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an *associated prime ideal* for $X$ if and only if $\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} = -\sup X_\mathfrak{p} < \infty$, that is,

$$\begin{aligned} \operatorname{Ass}_R X &= \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \sup X_\mathfrak{p} = 0\} \\ &= \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \mathfrak{p}_\mathfrak{p} \in \operatorname{ass}_{R_\mathfrak{p}} X_\mathfrak{p}\}, \end{aligned}$$

cf. 1.4.1. The union of the associated prime ideals forms the set of *zero-divisors* for $X$:

$$\mathrm{Z}_R X = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R X} \mathfrak{p}.$$

**Observations 2.4.** Let $X \in \mathcal{D}_-(R)$, $\mathfrak{p} \in \operatorname{Supp}_R X$, and set $s = \sup X_\mathfrak{p}$ $(\in \mathsf{Z})$; then

$$\mathfrak{p} \in \operatorname{Ass}_R X \quad \iff \quad \mathfrak{p}_\mathfrak{p} \in \operatorname{ass}_{R_\mathfrak{p}} X_\mathfrak{p} \quad \iff \quad \mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(X).$$

That is, $\mathfrak{p} \in \operatorname{Ass}_R X$ if and only if there exists an $m \in \mathsf{Z}$ such that $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_m(X)$ and $\mathfrak{p} \notin \operatorname{Supp}_R \mathrm{H}_\ell(X)$ for $\ell > m$. In particular there is an inclusion

$$\operatorname{ass}_R X \subseteq \operatorname{Ass}_R X; \tag{2.4.1}$$

and since $\mathrm{z}_R X = \cup_{\mathfrak{p} \in \operatorname{ass}_R X} \mathfrak{p}$, also the next inclusion holds.

$$\mathrm{z}_R X \subseteq \mathrm{Z}_R X. \tag{2.4.2}$$

We also note that $\operatorname{Ass}_R X$ is a finite set for $X$ in $\mathcal{D}^{\mathrm{f}}_\mathrm{b}(R)$.

**Modules 2.5.** For $M \in \mathcal{D}_0(R)$ the weak notions in 2.1 agree with the classical notions for modules; furthermore, $\operatorname{ass}_R M = \operatorname{Ass}_R M$ and $\mathrm{z}_R M = \mathrm{Z}_R M$, but $\operatorname{supp}_R M$ and $\operatorname{Supp}_R M$ may differ if $M$ is not finite.

**Proposition 2.6.** *Let $X \in \mathcal{D}_-(R)$; every minimal prime ideal in $\operatorname{Supp}_R X$ belongs to $\operatorname{Ass}_R X$, that is,*

$$\operatorname{Min}_R X \subseteq \operatorname{Ass}_R X;$$

*and for $X \in \mathcal{D}_\mathrm{b}(R)$ also the next inclusion holds.*

$$\operatorname{Ass}_R X \subseteq \operatorname{supp}_R X.$$

*Proof.* Let $X \in \mathcal{D}_-(R)$ and assume that $\mathfrak{p}$ is minimal in $\operatorname{Supp}_R X$. As $\operatorname{Supp}_{R_\mathfrak{p}} X_\mathfrak{p} = \{\mathfrak{p}_\mathfrak{p}\}$ it follows that $\mathfrak{p}_\mathfrak{p} \in \operatorname{ass}_{R_\mathfrak{p}} X_\mathfrak{p}$ and hence $\mathfrak{p} \in \operatorname{Ass}_R X$.

Let $X \in \mathcal{D}_\mathrm{b}(R)$; the first biconditional in the next chain is 2.2.4.

$$\begin{aligned} \mathfrak{p} \in \operatorname{Ass}_R X \quad &\Longrightarrow \quad \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty \\ &\Longleftrightarrow \quad \mathfrak{p}_\mathfrak{p} \in \operatorname{supp}_{R_\mathfrak{p}} X_\mathfrak{p} \\ &\Longleftrightarrow \quad X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p}) \not\simeq 0 \quad \Longleftrightarrow \quad \mathfrak{p} \in \operatorname{supp}_R X. \end{aligned}$$

□

**Lemma 2.7.** *Let $S$ be a multiplicative system in $R$; the following hold for $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$:*

(a) $\mathfrak{p} \in \operatorname{supp}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{supp}_{S^{-1}R} S^{-1}X$, *if* $X \in \mathcal{D}_+(R)$; *and*

(b) $\mathfrak{p} \in \operatorname{Ass}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{Ass}_{S^{-1}R} S^{-1}X$, *if* $X \in \mathcal{D}_-(R)$.

*Proof.* $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$ and

$$k(S^{-1}\mathfrak{p}) = (S^{-1}R/S^{-1}\mathfrak{p})_{S^{-1}\mathfrak{p}} \cong k(\mathfrak{p}),$$

so

$$\begin{gathered} (S^{-1}X)_{S^{-1}\mathfrak{p}} \otimes^{\mathbf{L}}_{(S^{-1}R)_{S^{-1}\mathfrak{p}}} k(S^{-1}\mathfrak{p}) \simeq X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p}); \quad \text{and} \\ \mathbf{R}\mathrm{Hom}_{(S^{-1}R)_{S^{-1}\mathfrak{p}}}(k(S^{-1}\mathfrak{p}), (S^{-1}X)_{S^{-1}\mathfrak{p}}) \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), X_\mathfrak{p}). \end{gathered}$$

(a) follows directly from the first isomorphism, and (b) follows from the second by the definition of depth. □

## 3. Three Types of Sequences

We are now ready to define sequences — and strong and weak ones — for complexes $Y \in \mathcal{D}_-(R)$. The main results of this section are that strong $Y$–sequences localize properly, and that for $M \in \mathcal{D}_0(R)$ the notions of $M$–sequences and strong $M$–sequences both agree with the classical notion for modules.

**Koszul Complexes 3.1.** For $x \in R$ the complex $\mathrm{K}(x) = 0 \to R \xrightarrow{x} R \to 0$, concentrated in degrees 1 and 0, is called the *Koszul complex* of $x$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in $R$, the Koszul complex $\mathrm{K}(\boldsymbol{x}) = \mathrm{K}(x_1, \dots, x_n)$ of $\boldsymbol{x}$ is the tensor product $\mathrm{K}(x_1) \otimes_R \cdots \otimes_R \mathrm{K}(x_n)$. The Koszul complex of the empty sequence is $R$.

For $Y \in \mathcal{D}(R)$ we set $\mathrm{K}(\boldsymbol{x}; Y) = Y \otimes_R \mathrm{K}(\boldsymbol{x})$, and for $m \in \{1, \dots, n\}$ we write $\mathrm{K}(\boldsymbol{x}_m; Y)$ for the complex $\mathrm{K}(x_1, \dots, x_m; Y)$. We also set $\mathrm{K}(\boldsymbol{x}_0; Y) = Y$, corresponding to the empty sequence.

**Observations 3.2.** In the following $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence in $R$ and $Y \in \mathcal{D}(R)$.

For $m \in \{0, \dots, n-1\}$ we have

$$\mathrm{K}(\boldsymbol{x}; Y) = \mathrm{K}(x_{m+1}, \dots, x_n; \mathrm{K}(\boldsymbol{x}_m; Y)), \tag{3.2.1}$$

by associativity of the tensor product. Let $\mathfrak{p} \in \operatorname{Spec} R$ and denote by ${}^{x_1}/_1, \dots, {}^{x_n}/_1$ the sequence of fractions in $R_\mathfrak{p}$ corresponding to $\boldsymbol{x}$. There is an isomorphism:

$$\mathrm{K}(x_1, \dots, x_n; Y)_\mathfrak{p} \cong \mathrm{K}({}^{x_1}/_1, \dots, {}^{x_n}/_1; Y_\mathfrak{p}). \tag{3.2.2}$$

For each $j$ the Koszul complex $\mathrm{K}(x_j)$ is a complex of finite free, in particular flat, modules, and hence so is $\mathrm{K}(\boldsymbol{x})$. Thus, we can identify $\mathrm{K}(\boldsymbol{x})$ with $\mathrm{K}(x_1) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathrm{K}(x_n)$ and $\mathrm{K}(\boldsymbol{x}; Y)$ with $Y \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x})$. It follows by 1.3.3 and 1.3.4 that

$$\inf \mathrm{K}(\boldsymbol{x}) \geq 0 \quad \text{and} \quad \mathrm{H}_0(\mathrm{K}(\boldsymbol{x})) = R/(\boldsymbol{x}). \tag{3.2.3}$$

It is well-known (see [1, Sec. 2] or [11, 16.4]) that

$$(x_1, \ldots, x_n) \subseteq \mathrm{Ann}_R \mathrm{K}(\boldsymbol{x}; Y). \tag{3.2.4}$$

It is easy to see that $\mathrm{Supp}_R \mathrm{K}(x_j) = \mathrm{V}(x_j)$, and it follows by 2.2.2 and 2.2.3 that $\mathrm{Supp}_R \mathrm{K}(\boldsymbol{x}) = \mathrm{supp}_R \mathrm{K}(\boldsymbol{x}) = \mathrm{V}(\boldsymbol{x})$. If $Y \in \mathcal{D}_+(R)$ it follows, also by 2.2.3, that

$$\mathrm{supp}_R \mathrm{K}(\boldsymbol{x}; Y) = \mathrm{supp}_R Y \cap \mathrm{V}(\boldsymbol{x}). \tag{3.2.5}$$

Finally, it follows by the definition of tensor product complexes that

(3.2.6) if $Y$ belongs $\mathcal{D}_-(R)$, respectively, $\mathcal{D}^{\mathrm{f}}_-(R)$ then also $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}_-(R)$, respectively, $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}^{\mathrm{f}}_-(R)$; and

(3.2.7) if $Y$ belongs $\mathcal{D}_{\mathrm{b}}(R)$, respectively, $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ then also $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}_{\mathrm{b}}(R)$, respectively, $\mathrm{K}(\boldsymbol{x}; Y) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

In view of 3.2.6 the next definitions make sense.

**Definitions 3.3.** Let $Y \in \mathcal{D}_-(R)$. An element $x \in R$ is said to be *regular* for $Y$ if and only if $x \notin \mathrm{z}_R Y$ and *strongly regular* for $Y$ if and only if $x \notin \mathrm{Z}_R Y$.

Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $R$. We say that

- $\boldsymbol{x}$ is a *weak $Y$–sequence* if and only if $x_j$ is regular for $\mathrm{K}(\boldsymbol{x}_{j-1}; Y)$ for each $j \in \{1, \ldots, n\}$;
- $\boldsymbol{x}$ is a *$Y$–sequence* if and only if $\boldsymbol{x}$ is a weak $Y$–sequence, and $\mathrm{K}(\boldsymbol{x}; Y) \not\simeq 0$ or $Y \simeq 0$; and
- $\boldsymbol{x}$ is a *strong $Y$–sequence* if and only if $x_j$ is strongly regular for $\mathrm{K}(\boldsymbol{x}_{j-1}; Y)$ for each $j \in \{1, \ldots, n\}$, and $\mathrm{K}(\boldsymbol{x}; Y) \not\simeq 0$ or $Y \simeq 0$.

**Remarks 3.4.** For $M \in \mathcal{D}_0(R)$ regular and strongly regular elements are the same, cf. 2.5, and the definition agrees with the usual definition of *$M$–regular elements*, cf. [11, Sec. 16]. In 3.8 we prove that also the definition of $M$–sequences agrees with the classical one.

Let $Y \in \mathcal{D}_-(R)$. By 2.4.2 a strongly regular element for $Y$ is also regular for $Y$; hence any strong $Y$–sequence is a $Y$–sequence and, thereby, a weak one.

The empty sequence is a strong $Y$–sequence for any complex $Y \in \mathcal{D}_-(R)$. A unit $u \in R$ is a strongly regular element for any complex $Y \in \mathcal{D}_-(R)$ and constitutes a weak $Y$–sequence, $u$ can, however, not be part of a $Y$–sequence if $Y \not\simeq 0$. On the other hand, if $Y \simeq 0$ then any sequence is a strong $Y$–sequence. Later we supply an example — 3.13 — to show that a $Y$–sequence need not be a strong one.

**Observation 3.5.** Let $Y \in \mathcal{D}_-(R)$, let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $R$, and let $m \in \{1, \ldots, n-1\}$. It follows by 3.2.1 that $\boldsymbol{x}$ is a $Y$–sequence, respectively, a weak or a strong one, if and only if $x_1, \ldots, x_m$ is a $Y$–sequence, respectively, a weak or strong one, and $x_{m+1}, \ldots, x_n$ is a $\mathrm{K}(\boldsymbol{x}_m; Y)$–sequence, respectively, a weak or a strong one.

**Lemma 3.6.** *The following hold for $x \in R$ and $Y \not\simeq 0$ in $\mathcal{D}_-(R)$:*

(a) $\sup \mathrm{K}(x; Y) \leq \sup Y + 1$*;*

(b) $\sup \mathrm{K}(x; Y) = \sup Y + 1$ *if and only if* $x \in \mathrm{z}_R Y$*; and*

(c) $\sup \mathrm{K}(x; Y) \geq \sup Y$ *if* $x\mathrm{H}_{\sup Y}(Y) \neq \mathrm{H}_{\sup Y}(Y)$.

*Proof.* It is easy to see that $\mathrm{K}(x;Y)$ is the mapping cone for the morphism $x_Y$, multiplication by $x$ on $Y$. Thus, $\mathrm{K}(x;Y)$ fits in the exact sequence of complexes

$$0 \to Y \to \mathrm{K}(x;Y) \to Y[1] \to 0,$$

where $Y[1]$ is a shift of $Y$: $Y[1]_\ell = Y_{\ell-1}$ and $\partial_\ell^{Y[1]} = -\partial_{\ell-1}^{Y}$. Now, set $s = \sup Y$ and examine the corresponding long exact sequence of homology modules:

$$0 \to \mathrm{H}_{s+1}(\mathrm{K}(x;Y)) \to \mathrm{H}_s(Y) \xrightarrow{x_{\mathrm{H}_s(Y)}} \mathrm{H}_s(Y) \to \mathrm{H}_s(\mathrm{K}(x;Y)) \to \cdots .$$

□

Parts (a) and (b) have the following immediate consequence:

**Corollary 3.7.** *Let $Y \in \mathcal{D}_-(R)$; a sequence $\boldsymbol{x} = x_1,\dots,x_n$ in $R$ is a weak $Y$–sequence if and only if* $\sup \mathrm{K}(\boldsymbol{x}_j;Y) \le \sup \mathrm{K}(\boldsymbol{x}_{j-1};Y)$ *for each $j \in \{1,\dots,n\}$.*

**Sequences for Modules 3.8.** *Let $M$ be an $R$–module; the following hold for a sequence $\boldsymbol{x} = x_1,\dots,x_n$ in $R$:*

(a) $\mathrm{H}_0(\mathrm{K}(\boldsymbol{x}_j;M)) = M/(x_1,\dots,x_j)M$ *for $j \in \{1,\dots,n\}$.*

(b) *The next three conditions are equivalent.*

- ($i$) *$\boldsymbol{x}$ is a weak $M$–sequence.*
- ($ii$) $\mathrm{K}(\boldsymbol{x}_j;M) \simeq M/(x_1,\dots,x_j)M$ *for each $j \in \{1,\dots,n\}$.*
- ($iii$) $x_j \notin \mathrm{z}_R\, M/(x_1,\dots,x_{j-1})M$ *for each $j \in \{1,\dots,n\}$*[1].

(c) *The next three conditions are equivalent.*

- ($i$) *$\boldsymbol{x}$ is a weak $M$–sequence, and $M/(x_1,\dots,x_n)M \neq 0$ or $M = 0$.*
- ($ii$) *$\boldsymbol{x}$ is an $M$–sequence.*
- ($iii$) *$\boldsymbol{x}$ is a strong $M$–sequence.*

*Proof.* All three assertions are trivial if $M = 0$, so we assume that $M$ is non-zero and let $\boldsymbol{x} = x_1,\dots,x_n$ be a sequence in $R$.

(a): Considering, as always, $M$ as a complex concentrated in degree 0, we see that

$$(*) \qquad \inf \mathrm{K}(\boldsymbol{x}_j;M) \ge 0 \quad \text{for} \quad j \in \{1,\dots,n\},$$

cf. 3.2.3 and 1.3.3, and $\mathrm{H}_0(\mathrm{K}(\boldsymbol{x}_j;M)) = M \otimes_R R/(x_1,\dots,x_j)$, cf. 1.3.4.

(b): For each $j \in \{1,\dots,n\}$ we have $\inf \mathrm{K}(\boldsymbol{x}_j;M) \ge 0$, cf. ($*$), so by 3.7 it follows that $\boldsymbol{x}$ is a weak $Y$–sequence if and only if $\mathrm{K}(\boldsymbol{x}_j;M) \in \mathcal{D}_0(R)$ for each $j$, that is (by (a)), if and only if $\mathrm{K}(\boldsymbol{x}_j;M) \simeq M/(x_1,\dots,x_j)M$ for each $j$. This proves the equivalence of ($i$) and ($ii$); that of ($ii$) and ($iii$) follows from (a), 3.2.1, and 3.6 by induction on $n$.

(c): First note that ($i$)$\Rightarrow$($ii$) by (a); it is then sufficient to prove that ($ii$) implies ($iii$): Suppose $\boldsymbol{x}$ is an $M$–sequence; for $j \in \{1,\dots,n\}$ we have $x_j \notin \mathrm{z}_R\,\mathrm{K}(\boldsymbol{x}_{j-1};M)$, and $\mathrm{K}(\boldsymbol{x}_{j-1};M) \in \mathcal{D}_0(R)$ by (b), so $\mathrm{z}_R\,\mathrm{K}(\boldsymbol{x}_{j-1};M) = \mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x}_{j-1};M)$, cf. 2.5, whence $\boldsymbol{x}$ is a strong $M$–sequence. □

**Remark 3.9.** Let $M$ be a non-zero $R$–module and let $\boldsymbol{x} = x_1,\dots,x_n$ be a sequence in $R$. Classically, cf. [11, Sec. 16], $\boldsymbol{x}$ is said to be an $M$–sequence if and only if (1) $x_j \notin \mathrm{z}_R\, M/(x_1,\dots,x_{j-1})M$ for $j \in \{1,\dots,n\}$, and (2) $M/(x_1,\dots,x_n)M \neq 0$. A sequence satisfying only the first condition is called a weak $M$–sequence, cf. [4,

[1] For $j = 1$ this means $x_1 \notin \mathrm{z}_R\, M$.

1.1.1]. It follows by (b) and (c) in 3.8 that the notions of (weak) $M$–sequences defined in 3.3 agree with the classical ones.

**Observation 3.10.** Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$; it follows by 3.6 that a sufficient condition for $x \in R$ to be a $Y$–sequence is that $x$ is a $\mathrm{H}_{\sup Y}(Y)$–sequence. This condition is, of course, not necessary, see 5.3 for an example.

**Theorem 3.11.** *Let $Y \in \mathcal{D}_\mathrm{b}(R)$ and $\mathfrak{p} \in \operatorname{supp}_R Y$; if $\boldsymbol{x} = x_1, \dots, x_n$ is a strong $Y$–sequence in $\mathfrak{p}$, then ${}^{x_1}/_1, \dots, {}^{x_n}/_1$ in the maximal ideal of $R_\mathfrak{p}$ is a strong $Y_\mathfrak{p}$–sequence.*

*Proof.* Let ${}^{\boldsymbol{x}}/_1 = {}^{x_1}/_1, \dots, {}^{x_n}/_1$ denote the sequence of fractions in $R_\mathfrak{p}$ corresponding to $\boldsymbol{x}$. Since $\mathfrak{p} \in \operatorname{supp}_R \mathrm{K}(\boldsymbol{x}; Y)$ by 3.2.5, it follows by 2.7(a) and 3.2.2 that $\mathfrak{p}_\mathfrak{p} \in \operatorname{supp}_{R_\mathfrak{p}} \mathrm{K}({}^{\boldsymbol{x}}/_1; Y_\mathfrak{p})$; in particular, $\mathrm{K}({}^{\boldsymbol{x}}/_1; Y_\mathfrak{p}) \not\simeq 0$. We are now required to prove that ${}^{x_j}/_1 \notin \mathrm{Z}_{R_\mathfrak{p}} \mathrm{K}({}^{x_1}/_1, \dots, {}^{x_{j-1}}/_1; Y_\mathfrak{p})$ for $j \in \{1, \dots, n\}$. This follows by the lemma below as $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{K}(\boldsymbol{x}_{j-1}; Y)$, $x_j \notin \mathrm{Z}_R \mathrm{K}(\boldsymbol{x}_{j-1}; Y)$, and $\mathrm{Z}_{R_\mathfrak{p}} \mathrm{K}(\boldsymbol{x}_{j-1}; Y)_\mathfrak{p} = \mathrm{Z}_{R_\mathfrak{p}} \mathrm{K}({}^{x_1}/_1, \dots, {}^{x_{j-1}}/_1; Y_\mathfrak{p})$. □

**Lemma 3.12.** *Let $Y$ belong to $\mathcal{D}_-(R)$ and $\mathfrak{p} \in \operatorname{Supp}_R Y$; if $x \in \mathfrak{p}$ and $x \notin \mathrm{Z}_R Y$ then ${}^{x}/_1 \notin \mathrm{Z}_{R_\mathfrak{p}} Y_\mathfrak{p}$.*

*Proof.* We assume that ${}^{x}/_1 \in \mathrm{Z}_{R_\mathfrak{p}} Y_\mathfrak{p}$ and want to prove that $x$ belongs to $\mathrm{Z}_R Y$. By assumption ${}^{x}/_1$ belongs to a prime ideal in $\operatorname{Ass}_{R_\mathfrak{p}} Y_\mathfrak{p}$, that is, ${}^{x}/_1 \in \mathfrak{q}_\mathfrak{p}$ for some $\mathfrak{q} \in \operatorname{Spec} R$ contained in $\mathfrak{p}$. Then $x \in \mathfrak{q}$, and $\mathfrak{q} \in \operatorname{Ass}_R Y$ by 2.7(b), so $x \in \mathrm{Z}_R Y$ as wanted. □

As the next example demonstrates, a $Y$–sequence does not necessarily localize properly, not even if $R$ is local and $Y \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$.

**Example 3.13.** Let $R$ be a local ring, assume that there exist $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{p} \not\subseteq \mathfrak{q}$ and $\mathfrak{q} \not\subseteq \mathfrak{p}$, and consider the complex $Y = 0 \to R/\mathfrak{q} \xrightarrow{0} R/\mathfrak{p} \to 0$. Let $x$ be an element in $\mathfrak{p}$ not in $\mathfrak{q}$; it follows by 3.6 that $x$ is a $Y$–sequence, but the localization of $Y$ at $\mathfrak{p}$ is the field $k(\mathfrak{p})$, and ${}^{x}/_1 \in R_\mathfrak{p}$ is certainly not a $k(\mathfrak{p})$–sequence.

Note that if $\mathfrak{p} \cap \mathfrak{q} = 0$, then there is no non-empty strong $Y$–sequence in $\mathfrak{p}$.

## 4. Length of Sequences and Depth of Complexes

In this section we prove that any (strong) sequence can be extended to a maximal (strong) sequence, and we discuss various upper bounds for the length of such sequences.

**Maximal Sequences 4.1.** Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$ and let $\mathfrak{a}$ be an ideal in $R$. A sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$ is said to be a *maximal* (*strong*) *$Y$–sequence* in $\mathfrak{a}$ if and only if it is a (strong) $Y$–sequence and not the first part of a longer (strong) $Y$–sequence in $\mathfrak{a}$.

**Lemma 4.2.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$; if $\boldsymbol{x} = x_1, \dots, x_n$ is a $Y$–sequence then $x_n \notin (x_1, \dots, x_{n-1})$.*

*Proof.* By 3.2.4 we have $(x_1, \dots, x_{n-1}) \subseteq \operatorname{Ann}_R \mathrm{K}(\boldsymbol{x}_{n-1}; Y)$, hence $(x_1, \dots, x_{n-1}) \subseteq \mathrm{z}_R \mathrm{K}(\boldsymbol{x}_{n-1}; Y)$ as $\mathrm{K}(\boldsymbol{x}_{n-1}; Y) \not\simeq 0$, and it follows that $x_n \notin (x_1, \dots, x_{n-1})$ as desired. □

**Corollary 4.3.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$ and let $\mathfrak{a}$ be an ideal in $R$. Any $Y$–sequence, respectively, strong $Y$–sequence in $\mathfrak{a}$ can be extended to a maximal $Y$–sequence, respectively, a maximal strong $Y$–sequence in $\mathfrak{a}$.*

*Proof.* The assertions follow immediately by 4.2 as $R$ is Noetherian. □

**Depth 4.4.** Let $\mathfrak{a}$ be an ideal in $R$ and let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite set of generators for $\mathfrak{a}$. By definition, cf. [10, Sec. 2], the $\mathfrak{a}$–*depth* of $Y \in \mathcal{D}(R)$ is the number

$$\operatorname{depth}_R(\mathfrak{a}, Y) = t - \sup \mathrm{K}(\boldsymbol{a}; Y);$$

it is, of course, independent of the choice of generating set $\boldsymbol{a}$.

We note that $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ if and only if $\mathrm{K}(\boldsymbol{a}; Y) \not\simeq 0$ for some, equivalently any, finite set of generators for $\mathfrak{a}$. Thus, by 2.2.1 and 3.2.5 we have

$$\operatorname{depth}_R(\mathfrak{a}, Y) < \infty \iff \operatorname{supp}_R Y \cap \mathrm{V}(\mathfrak{a}) \neq \emptyset, \tag{4.4.1}$$

for $Y \in \mathcal{D}_\mathrm{b}(R)$.

**Proposition 4.5.** *Let $Y \in \mathcal{D}_-(R)$, let $\mathfrak{a}$ be a proper ideal in $R$, and let $M$ belong to $\mathcal{D}_0^\mathrm{f}(R)$ with $\operatorname{Supp}_R M = \mathrm{V}(\mathfrak{a})$. The following equalities hold:*

$$\begin{aligned}\operatorname{depth}_R(\mathfrak{a}, Y) &= -\sup \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y)\\ &= \inf\{\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}\\ &= -\sup \mathbf{R}\mathrm{Hom}_R(M, Y).\end{aligned}$$

*Proof.* The first equality is [10, 6.1], the second and third both follow by 1.4.2. □

**Remark 4.6.** It follows from the first equality in 4.5 that the $\mathfrak{a}$–depth for complexes extends the usual concept of $\mathfrak{a}$–depth for modules, cf. [11, 16.7]; furthermore, it generalizes the concept of depth over local rings, that is, $\operatorname{depth}_R Y = \operatorname{depth}_R(\mathfrak{m}, Y)$ for $Y \in \mathcal{D}_-(R)$, when $R$ is local with maximal ideal $\mathfrak{m}$. By the second equality in 4.5 the next inequality holds for all $Y \in \mathcal{D}_-(R)$ and all $\mathfrak{p} \in \operatorname{Spec} R$.

$$\operatorname{depth}_R(\mathfrak{p}, Y) \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p}.$$

Part (a) of the next theorem is often referred to as the 'depth sensitivity of the Koszul complex'.

**Theorem 4.7.** *Let $Y \in \mathcal{D}(R)$ and let $\mathfrak{a}$ be an ideal in $R$. The following hold:*

(a) *For any sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$ there is an equality:*

$$\operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x}; Y)) = \operatorname{depth}_R(\mathfrak{a}, Y) - n.$$

(b) *For any ideal $\mathfrak{b} \subseteq \mathfrak{a}$ there is an inequality:*

$$\operatorname{depth}_R(\mathfrak{b}, Y) \leq \operatorname{depth}_R(\mathfrak{a}, Y).$$

*Proof.* Let $\boldsymbol{a} = a_1, \dots, a_t$ be a set of generators for $\mathfrak{a}$ and let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in $\mathfrak{a}$. Also $\boldsymbol{x}, \boldsymbol{a} = x_1, \dots, x_n, a_1, \dots, a_t$ is a generating set for $\mathfrak{a}$, and by (3.2.1) we have $\mathrm{K}(\boldsymbol{x}, \boldsymbol{a}; Y) = \mathrm{K}(\boldsymbol{a}; \mathrm{K}(\boldsymbol{x}; Y))$. Hence,

$$\begin{aligned}\operatorname{depth}_R(\mathfrak{a}, Y) &= n + t - \sup \mathrm{K}(\boldsymbol{x}, \boldsymbol{a}; Y)\\ &= n + t - \sup \mathrm{K}(\boldsymbol{a}; \mathrm{K}(\boldsymbol{x}; Y))\\ &= n + \operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x}; Y));\end{aligned}$$

and this proves (a).

To prove (b), let $\boldsymbol{b} = b_1, \dots, b_u$ be a generating set for $\mathfrak{b}$, then $\boldsymbol{b},\boldsymbol{a} = b_1, \dots, b_u, a_1, \dots, a_t$ is a generating set for $\mathfrak{a}$. If $\sup \mathrm{K}(\boldsymbol{b}; Y) = \infty$ the inequality is trivial, so we assume that $\mathrm{K}(\boldsymbol{b}; Y) \in \mathcal{D}_-(R)$. As above we have $\mathrm{K}(\boldsymbol{b},\boldsymbol{a}; Y) = \mathrm{K}(\boldsymbol{a}; \mathrm{K}(\boldsymbol{b}; Y))$, so it follows by 3.6(a) that $\sup \mathrm{K}(\boldsymbol{b},\boldsymbol{a}; Y) \leq \sup \mathrm{K}(\boldsymbol{b}; Y) + t$, whence

$$\begin{aligned} \operatorname{depth}_R(\mathfrak{a}, Y) &= u + t - \sup \mathrm{K}(\boldsymbol{b},\boldsymbol{a}; Y) \\ &\geq u + t - (\sup \mathrm{K}(\boldsymbol{b}; Y) + t) \\ &= \operatorname{depth}_R(\mathfrak{b}, Y), \end{aligned}$$

as desired. □

**Corollary 4.8.** *Let $Y \in \mathcal{D}_-(R)$ and let $\mathfrak{a}$ be a proper ideal in $R$. If $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ then the following hold for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$.*

(a) *If $\boldsymbol{x}$ is a weak $Y$–sequence then $\boldsymbol{x}$ is a $Y$–sequence.*
(b) *If $\boldsymbol{x}$ is a $Y$–sequence then $\boldsymbol{x}$ is maximal in $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathrm{z}_R \mathrm{K}(\boldsymbol{x}; Y)$.*
(c) *If $\boldsymbol{x}$ is a strong $Y$–sequence then $\boldsymbol{x}$ is maximal in $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathrm{Z}_R \mathrm{K}(\boldsymbol{x}; Y)$.*

*Proof.* Denote by $\mathfrak{b}$ the ideal generated by $\boldsymbol{x}$. It follows by 4.7(b) that $\operatorname{depth}_R(\mathfrak{b}, Y) < \infty$, in particular, $\mathrm{K}(\boldsymbol{x}; Y) \not\simeq 0$, cf. 4.4. The three assertions are now immediate by the definitions in 3.3. □

**Proposition 4.9.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_-(R)$ and let $\boldsymbol{x} = x_1, \dots, x_n$ be a weak $Y$–sequence. The next inequality holds for any ideal $\mathfrak{a}$ containing $\boldsymbol{x}$.*

$$n \leq \operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y.$$

*Proof.* Let $\mathfrak{b}$ be the ideal generated by the sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$. By 4.4, 4.7(b), and 3.7 we have

$$\begin{aligned} n &= \operatorname{depth}_R(\mathfrak{b}, Y) + \sup \mathrm{K}(\boldsymbol{x}; Y) \\ &\leq \operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y. \end{aligned}$$

□

**Corollary 4.10.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_\mathrm{b}(R)$ and let $\boldsymbol{x} = x_1, \dots, x_n$ be a strong $Y$–sequence. The following inequality holds:*

(a) $$n \leq \inf \{\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R Y \cap \mathrm{V}(\boldsymbol{x})\};$$

*and if $Y \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R)$, also the next inequality holds.*

(b) $$n \leq \inf \{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(\boldsymbol{x})\}.$$

*Proof.* Let $Y \in \mathcal{D}_\mathrm{b}(R)$ and assume that $\boldsymbol{x} = x_1, \dots, x_n$ is a strong $Y$–sequence in $\mathfrak{p} \in \operatorname{supp}_R Y$. By 3.11 the sequence ${}^{x_1}/_1, \dots, {}^{x_n}/_1$ in the maximal ideal of $R_\mathfrak{p}$ is a strong $Y_\mathfrak{p}$–sequence, so by 4.9 we have $n \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p}$, and this proves (a). If $Y \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R)$ then $\operatorname{supp}_R Y = \operatorname{Supp}_R Y$, cf. 2.2.2, and $Y_\mathfrak{p} \in \mathcal{D}^\mathrm{f}_\mathrm{b}(R_\mathfrak{p})$, so (b) follows from (a) as $\operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \sup Y_\mathfrak{p} \leq \dim R_\mathfrak{p}$ by 1.4.4. □

**Theorem 4.11.** *Let $Y \in \mathcal{D}^{\mathrm{f}}_{-}(R)$ and let $\mathfrak{a}$ be a proper ideal in $R$. If $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ then the following conditions are equivalent for a $Y$–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$:*

*(i)* $\boldsymbol{x}$ *is a maximal $Y$–sequence in $\mathfrak{a}$.*
*(ii)* $\mathfrak{a} \subseteq \mathrm{z}_R \operatorname{K}(\boldsymbol{x}; Y)$.
*(iii)* $\operatorname{depth}_R(\mathfrak{a}, \operatorname{K}(\boldsymbol{x}; Y)) + \sup \operatorname{K}(\boldsymbol{x}; Y) = 0$.
*(iv)* $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup \operatorname{K}(\boldsymbol{x}; Y) = n$.

*Proof.* We assume that $Y \in \mathcal{D}^{\mathrm{f}}_{-}(R)$ with $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$; the equivalence of (*i*) and (*ii*) is 4.8(b). From 4.7(a) it follows that (*iii*) $\Leftrightarrow$ (*iv*); this leaves us with one equivalence to prove:

Set $K = \operatorname{K}(\boldsymbol{x}; Y)$ and $s = \sup K$ $(\in \mathbb{Z})$; by 4.5 and 1.3.1 we have

$$-\operatorname{depth}_R(\mathfrak{a}, K) = \sup \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, K) \leq s,$$

and equality holds if and only if $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}_s(K)) \neq 0$, cf. 1.3.2. Since $\operatorname{H}_s(K)$ is a finite module, cf. 3.2.6, it is well-known that $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}_s(K)) \neq 0$ if and only if $\mathfrak{a} \subseteq \mathrm{z}_R K$, and this proves the equivalence of (*ii*) and (*iii*). □

**Remarks 4.12.** Let $Y$, $\mathfrak{a}$, and $\boldsymbol{x}$ be as in 4.11. Since $\operatorname{K}(\boldsymbol{x}; Y) \in \mathcal{D}^{\mathrm{f}}_{-}(R)$, cf. 3.2.6, it follows that

$$\mathfrak{a} \subseteq \mathrm{z}_R \operatorname{K}(\boldsymbol{x}; Y) \quad \Longleftrightarrow \quad \mathfrak{a} \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \operatorname{ass}_R \operatorname{K}(\boldsymbol{x}; Y).$$

This should be compared to (*ii*) and (*iii*) in 4.15.

For a finite $R$–module $M$ and an ideal $\mathfrak{a}$ in $R$ it follows by 4.4.1 and 2.2.2 that

$$\operatorname{depth}_R(\mathfrak{a}, M) < \infty \quad \Longleftrightarrow \quad M/\mathfrak{a}M \neq 0.$$

Spelling out 4.11 for modules — as done in 4.14 — we recover the property 1° advertised in the introduction. Thus, in a sense, 4.11 describes the corresponding property for complexes $Y \in \mathcal{D}^{\mathrm{f}}_{-}(R)$; but unless $R$ is local (see 5.4) the length of a maximal $Y$–sequence need not be a well-determined integer:

Let $Y$ and $\mathfrak{a}$ be as in 4.11. If $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y = 0$ then $\mathfrak{a} \subseteq \mathrm{z}_R Y$, so the empty sequence is the only $Y$–sequence in $\mathfrak{a}$. If $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y = 1$ then all maximal $Y$–sequences in $\mathfrak{a}$ are of length 1, but if $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y > 1$ there can be maximal $Y$–sequences in $\mathfrak{a}$ of different length. This is illustrated by the example below.

**Example 4.13.** Let $k$ be a field, set $R = k[U, V]$, and consider the $R$–complex $Y = 0 \to R/(U-1) \to 0 \to k \to 0$ concentrated in degrees 2, 1, and 0. Let $\mathfrak{a}$ be the maximal ideal $\mathfrak{a} = (U, V)$, then $\operatorname{depth}_R(\mathfrak{a}, Y) = 2 - 2 = 0$ and it is straightforward to check that $U$ as well as $V, U$ is a maximal $Y$–sequence in $\mathfrak{a}$.

**Corollary 4.14.** *Let $M$ be a finite $R$–module and let $\mathfrak{a}$ be a proper ideal in $R$. If $\operatorname{depth}_R(\mathfrak{a}, M) < \infty$ then the next four conditions are equivalent for an $M$–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$.*

*(i)* $\boldsymbol{x}$ *is a maximal $M$–sequence in $\mathfrak{a}$.*
*(ii)* $\mathfrak{a} \subseteq \mathrm{z}_R M/(x_1, \ldots, x_n)M$.
*(iii)* $\operatorname{depth}_R(\mathfrak{a}, M/(x_1, \ldots, x_n)M) = 0$.
*(iv)* $\operatorname{depth}_R(\mathfrak{a}, M) = n$.

*In particular, the maximal length of an $M$–sequence in $\mathfrak{a}$ is a well-determined integer:* $\operatorname{depth}_R(\mathfrak{a}, M) = \inf\{\ell \in \mathbb{Z} \mid \operatorname{Ext}^{\ell}_R(R/\mathfrak{a}, M) \neq 0\}$*, and all maximal $M$–sequences in $\mathfrak{a}$ have this length.*

*Proof.* By 3.8 we have $\mathrm{K}(\boldsymbol{x};M) \simeq M/(x_1,\dots,x_n)M \neq 0$, in particular, $\sup \mathrm{K}(\boldsymbol{x};M)=0$. The equivalence of the four conditions now follows from 4.11, and the last assertions are immediate, cf. 4.5. □

Other well-known characterizations of maximal sequences for finite modules are recovered by reading $M/(x_1,\dots,x_n)M$ for $\mathrm{K}(\boldsymbol{x};M)$ in the next theorem.

**Theorem 4.15.** *Let $Y \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and let $\mathfrak{a}$ be a proper ideal in $R$. If $\operatorname{depth}_R(\mathfrak{a},Y) < \infty$ then the next four conditions are equivalent for a strong $Y$–sequence $\boldsymbol{x} = x_1,\dots,x_n$ in $\mathfrak{a}$.*

(*i*) *$\boldsymbol{x}$ is a maximal strong $Y$–sequence in $\mathfrak{a}$.*
(*ii*) *$\mathfrak{a} \subseteq \mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x};Y)$.*
(*iii*) *$\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)$.*
(*iv*) *There is a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R Y$ containing $\mathfrak{a}$ such that the strong $Y_{\mathfrak{p}}$–sequence ${}^{x_1}/_1,\dots,{}^{x_n}/_1$ in $R_{\mathfrak{p}}$ is a maximal $Y_{\mathfrak{p}}$–sequence.*

*Proof.* The equivalence $(i) \Leftrightarrow (ii)$ is immediate as $\operatorname{depth}_R(\mathfrak{a},Y) < \infty$, cf. 4.8(c).

$(ii) \Leftrightarrow (iii)$: Clearly, $(iii)$ implies $(ii)$. On the other hand, $\mathrm{K}(\boldsymbol{x};Y) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ by 3.2.7, so $\mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x};Y) = \cup_{\mathfrak{p}\in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)}\,\mathfrak{p}$ is a finite union, cf. 2.4. Thus, if $\mathfrak{a} \subseteq \mathrm{Z}_R\,\mathrm{K}(\boldsymbol{x};Y)$ then $\mathfrak{a}$ must be contained in one of the prime ideals $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)$.

$(iii) \Leftrightarrow (iv)$: Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Supp}_R Y$ containing $\mathfrak{a}$, then $\operatorname{depth}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} < \infty$, cf. 2.2.2 and 2.2.4, and by 3.11 the sequence of fractions ${}^{\boldsymbol{x}}/_1 = {}^{x_1}/_1,\dots,{}^{x_n}/_1$ in $R_{\mathfrak{p}}$ is a strong $Y_{\mathfrak{p}}$–sequence. By 3.2.2 there is an equality:

$$\operatorname{depth}_{R_{\mathfrak{p}}} \mathrm{K}(\boldsymbol{x};Y)_{\mathfrak{p}} + \sup \mathrm{K}(\boldsymbol{x};Y)_{\mathfrak{p}} = \operatorname{depth}_{R_{\mathfrak{p}}} \mathrm{K}({}^{\boldsymbol{x}}/_1;Y_{\mathfrak{p}}) + \sup \mathrm{K}({}^{\boldsymbol{x}}/_1;Y_{\mathfrak{p}}).$$

By 4.11 and the definition of associated prime ideals it now follows that ${}^{\boldsymbol{x}}/_1$ is a maximal $Y_{\mathfrak{p}}$–sequence if and only if $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}(\boldsymbol{x};Y)$. □

## 5. Local Rings

In this section $R$ is local with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We focus on (strong) sequences for complexes in $\mathcal{D}^{\mathrm{f}}_{-}(R)$ and strengthen some of the results from the previous section. The results established here are essentially those lined out by H.–B. Foxby in [8, Sec. 12], exceptions are 5.7 and 5.9.

**Proposition 5.1.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}^{\mathrm{f}}_{-}(R)$; the following hold for a sequence $\boldsymbol{x} = x_1,\dots,x_n$ in $\mathfrak{m}$:*

(a) *There are inequalities*

$$\sup \mathrm{K}(\boldsymbol{x};Y) \geq \cdots \geq \sup \mathrm{K}(\boldsymbol{x}_j;Y) \geq \sup \mathrm{K}(\boldsymbol{x}_{j-1};Y) \geq \cdots \geq \sup Y;$$

*in particular,* $\mathrm{K}(\boldsymbol{x};Y) \not\simeq 0$.

(b) *The next three conditions are equivalent.*
   (*i*) *$\boldsymbol{x}$ is a weak $Y$–sequence.*
   (*ii*) *$\boldsymbol{x}$ is a $Y$–sequence.*
   (*iii*) $\sup \mathrm{K}(\boldsymbol{x};Y) = \sup Y$.

(c) *If $\boldsymbol{x}$ is a $Y$–sequence then so is any permutation of $\boldsymbol{x}$.*

*Proof.* (a): The inequalities hold by Nakayama's lemma and 3.6(c); in particular we have $\sup \mathrm{K}(\boldsymbol{x};Y) \geq \sup Y > -\infty$, so $\mathrm{K}(\boldsymbol{x};Y) \not\simeq 0$.

(b): It follows by 3.7 that $\boldsymbol{x}$ is a weak $Y$–sequence if and only if equality holds in each of the inequalities in (a). This proves the equivalence of $(i)$ and $(iii)$; also $(i) \Leftrightarrow (ii)$ is immediate by (a).

(c): By commutativity of the tensor product the number $\sup \mathrm{K}(\boldsymbol{x}; Y)$ is unaffected by permutations of $\boldsymbol{x}$, so the last assertion follows by (b). $\square$

The next corollary is an immediate consequence of 5.1(a). The example below shows that the equality $\sup \mathrm{K}(\boldsymbol{x}; Y) = \sup Y$ need not hold, not even for strong $Y$–sequence, if $Y$ does not have finite homology modules.

**Corollary 5.2.** *Let $\mathfrak{a}$ be a proper ideal in $R$. If $Y \in \mathcal{D}_{-}^{\mathrm{f}}(R)$ then*

$$\operatorname{depth}_R(\mathfrak{a}, Y) < \infty \quad \Longleftrightarrow \quad Y \not\simeq 0.$$

**Example 5.3.** Let $R$ be a local integral domain, not a field, and let $B = R_{(0)} \neq R$ be the field of fractions. Consider the complex $Y = 0 \to B \xrightarrow{0} R \to 0$. For any $\mathfrak{p} \in \operatorname{Spec} R$ we have $\sup Y_{\mathfrak{p}} = \sup Y$, so $\operatorname{Ass}_R Y = \operatorname{ass}_R Y = \operatorname{Ass}_R B = \{0\}$. Let $x \neq 0$ be an element in the maximal ideal of $R$, it follows that $x \notin \mathrm{Z}_R Y$ and $\mathrm{K}(x; Y) \simeq R/(x) \neq 0$, so $x$ is a strong $Y$–sequence, but $\sup \mathrm{K}(x; Y) < \sup Y$.

**Theorem 5.4.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_{-}^{\mathrm{f}}(R)$ and let $\mathfrak{a}$ be a proper ideal in $R$. The next four conditions are equivalent for a $Y$–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$.*

- $(i)$ *$\boldsymbol{x}$ is a maximal $Y$–sequence in $\mathfrak{a}$.*
- $(ii)$ $\mathfrak{a} \subseteq \mathrm{z}_R \mathrm{K}(\boldsymbol{x}; Y)$.
- $(iii)$ $\operatorname{depth}_R(\mathfrak{a}, \mathrm{K}(\boldsymbol{x}; Y)) + \sup Y = 0$.
- $(iv)$ $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y = n$.

*In particular, the maximal length of a $Y$–sequence in $\mathfrak{a}$ is a well-determined integer:* $\operatorname{depth}_R(\mathfrak{a}, Y) + \sup Y$*, and all maximal $Y$–sequences in $\mathfrak{a}$ have this length.*

*Proof.* By 5.2 and 5.1 we have $\operatorname{depth}_R(\mathfrak{a}, Y) < \infty$ and $\sup \mathrm{K}(\boldsymbol{x}; Y) = \sup Y$, so the equivalence is a special case of 4.11, and the last assertions follow. $\square$

**Corollary 5.5.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_{-}^{\mathrm{f}}(R)$; the integer*

$$\operatorname{depth}_R Y + \sup Y$$

*is the maximal length of a $Y$–sequence, and any maximal $Y$–sequence is of this length. Furthermore, the following inequalities hold:*

$$\operatorname{depth}_R Y + \sup Y \leq \dim_R \mathrm{H}_{\sup Y}(Y) \leq \dim R.$$

*Proof.* A $Y$–sequence must be contained in $\mathfrak{m}$, and the first part is 5.4 applied to $\mathfrak{a} = \mathfrak{m}$. The inequalities are 1.4.4. $\square$

**Corollary 5.6.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_{-}^{\mathrm{f}}(R)$ and $M \in \mathcal{D}_{0}^{\mathrm{f}}(R)$. The maximal length of a $Y$–sequence in $\operatorname{Ann}_R M$ is a well-determined integer $n$:*

$$\begin{aligned} n &= -\sup \mathbf{R}\mathrm{Hom}_R(M, Y) + \sup Y \\ &= \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{V}(\operatorname{Ann}_R M)\} + \sup Y; \end{aligned}$$

*and any maximal $Y$–sequence in $\operatorname{Ann}_R M$ is of this length.*

*Proof.* It follows by 5.4 that a $Y$–sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\operatorname{Ann}_R M$ is maximal if and only if $n = \operatorname{depth}_R(\operatorname{Ann}_R M, Y) + \sup Y$. As $M$ is finite $\operatorname{Supp}_R M = \mathrm{V}(\operatorname{Ann}_R M)$, and the desired equalities follow by 4.5. $\square$

It follows from the last remark in 3.13 that 5.6 has no counterpart for strong sequences, but 5.5 does have one:

**Corollary 5.7** (to 4.15)**.** *Let $Y \not\simeq 0$ belong to $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$. A maximal strong $Y$–sequence is a maximal $Y$–sequence; in particular, the maximal length of a strong $Y$–sequence is a well-determined integer $n$:*

$$n = \operatorname{depth}_R Y + \sup Y \leq \dim_R \mathrm{H}_{\sup Y}(Y) \leq \dim R;$$

*and any maximal strong $Y$–sequence is of this length.*

*Proof.* Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a maximal strong $Y$–sequence, that is, maximal in $\mathfrak{m}$. Since $\operatorname{depth}_R Y < \infty$ by 5.2 it follows by 4.15 that $\boldsymbol{x}$ is a maximal $Y$–sequence, and the desired equality and inequalities follow from 5.5. □

The number $\operatorname{depth}_R Y + \sup Y$ provides an upper bound for the length of a $Y$–sequence, even if $Y$ does not have finite homology modules, cf. 4.9. In view of 5.5 it is natural to ask if also $\dim R$ is a bound. If $\dim R = 0$ it obviously is, cf. 1.4.1, and so it is if $\dim R = 1$ and $\operatorname{depth}_R \mathrm{H}_{\sup Y}(Y) < \infty$ (this follows by [10, 2.3]); but the next example shows that the answer is negative. For bounded complexes, however, a bound involving $\dim R$ is available, see 5.9.

**Example 5.8.** Let $k$ be a field and consider the local ring $R = k[\![U,V]\!]/(UV)$ with $\dim R = 1$. The residue classes $u$ and $v$ of, respectively, $U$ and $V$ generate prime ideals in $R$; we set $Y = 0 \to R_{(v)} \xrightarrow{0} R/(u) \to 0$. Multiplication by $u$ on $R_{(v)}$ is an isomorphism, $v$ is a $R/(u)$–sequence, and it follows that $u, v$ is a $Y$–sequence.

**Corollary 5.9** (to 4.9)**.** *Let $Y \in \mathcal{D}_\mathrm{b}(R)$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a weak $Y$–sequence in $\mathfrak{m}$. If $\mathfrak{m} \in \operatorname{supp}_R Y$ then $\boldsymbol{x}$ is a $Y$–sequence, and*

$$n \leq \operatorname{depth}_R Y + \sup Y \leq \dim_R Y + \sup Y \leq \dim R + \operatorname{amp} Y.$$

*Proof.* It follows by 2.2.4 that $\operatorname{depth}_R Y < \infty$, so $\boldsymbol{x}$ is a $Y$–sequence by 4.8(a). The first inequality is a special case of 4.9. The inequality $\operatorname{depth}_R Y \leq \dim_R Y$ holds by [7, 3.9]; this gives the second inequality, and the third one follows as $\dim_R Y \leq \dim R - \inf Y$ by the definition of dimension. □

We close with an example, illustrating an application of sequences for complexes.

**Example 5.10** (Parameter Sequences)**.** In the following we assume that $R$ admits a dualizing complex $D$, cf. [9], and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $R$. For $Y \not\simeq 0$ in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ it follows by 3.2.7, 3.6, and well-known properties of dualizing complexes that

$$\dim_R \mathrm{K}(\boldsymbol{x}; Y) = \dim_R Y - n \quad \Longleftrightarrow \quad \boldsymbol{x} \text{ is a } \mathbf{R}\mathrm{Hom}_R(Y, D)\text{–sequence;}$$

and by [7, 3.12] there is an equality:

$$\dim_R \mathrm{K}(\boldsymbol{x}; Y) = \sup\{\dim_R(Y \otimes_R^{\mathbf{L}} \mathrm{H}_\ell(\mathrm{K}(\boldsymbol{x}))) - \ell \mid \ell \in \mathsf{Z}\}.$$

Let $M$ be a finite $R$–module; we say that $\boldsymbol{x}$ is an *$M$–parameter sequence* if and only if $\dim_R M/(x_1, \ldots, x_n)M = \dim_R M - n$, that is, if and only if $\boldsymbol{x}$ is part of a system of parameters for $M$. It follows by the definition of Krull dimension, Nakayama's lemma, and 3.2.3 that

$$\begin{aligned}
\dim_R \mathrm{K}(\boldsymbol{x}; M) &= \sup\{\dim_R(M \otimes_R^{\mathbf{L}} \mathrm{H}_\ell(\mathrm{K}(\boldsymbol{x}))) - \ell \mid \ell \in \mathsf{Z}\} \\
&= \sup\{\dim_R(M \otimes_R \mathrm{H}_\ell(\mathrm{K}(\boldsymbol{x}))) - \ell \mid \ell \in \mathsf{Z}\} \\
&= \dim_R M/(x_1, \ldots, x_n)M.
\end{aligned}$$

Thus, $\boldsymbol{x}$ is an $M$–parameter sequence if and only if $\boldsymbol{x}$ is a $\mathbf{R}\mathrm{Hom}_R(M, D)$–sequence. In particular, any $M$–sequence is a $\mathbf{R}\mathrm{Hom}_R(M, D)$–sequence. Only if $M$ is Cohen–Macaulay will $\mathbf{R}\mathrm{Hom}_R(M, D)$ have homology concentrated in one degree, that is, be equivalent to a module up to a shift.

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