Appendix A
Triangulated Categories

In this appendix, $\mathcal{T}$ is an additive category equipped with an invertible additive endofunctor Σ. We describe the conditions for $(\mathcal{T}, \Sigma)$ to form a triangulated category; the first step is to settle on a collection of triangles.

A.1 Definition. A *candidate triangle* in $\mathcal{T}$ is a diagram

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M ,$$

such that the composites $\beta\alpha$, $\gamma\beta$, and $(\Sigma\alpha)\gamma$ are all zero. An *morphism* (φ, ψ, χ) of candidate triangles is a commutative diagram in $\mathcal{T}$,

$$\begin{array}{ccccccc} M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & X & \xrightarrow{\gamma} & \Sigma M \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\Sigma\varphi} \\ M' & \xrightarrow{\alpha'} & N' & \xrightarrow{\beta'} & X' & \xrightarrow{\gamma'} & \Sigma M' \end{array};$$

it is called an *isomorphism* if φ, ψ, and χ are isomorphisms in $\mathcal{T}$.

A.2. For a collection $\triangle$ of candidate triangles in $\mathcal{T}$, consider the next conditions.

(TR0) For every M in $\mathcal{T}$, the candidate triangle

$$M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma M$$

is in $\triangle$. Every candidate triangle that is isomorphic to one from $\triangle$ is in $\triangle$.

(TR1) Every morphism $\alpha \colon M \to N$ in $\mathcal{T}$ fits in a candidate triangle from $\triangle$,

$$M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M .$$

(TR2) For every candidate triangle $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\triangle$, the following two candidate triangles belong to $\triangle$ as well,

$$N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N \qquad \text{and} \qquad \Sigma^{-1} X \xrightarrow{-\Sigma^{-1}\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X .$$

(TR2') Consider two candidate triangles,

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \qquad \text{and} \qquad N \xrightarrow{-\beta} X \xrightarrow{-\gamma} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N\,.$$

If one belongs to $\triangle$ then so does the other.

(TR3) For every commutative diagram

$$\text{(A.2.1)} \qquad \begin{array}{ccccccc} M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & X & \xrightarrow{\gamma} & \Sigma M \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & & & \\ M' & \xrightarrow{\alpha'} & N' & \xrightarrow{\beta'} & X' & \xrightarrow{\gamma'} & \Sigma M' \end{array},$$

where the rows are candidate triangles in $\triangle$, there exists a (not necessarily unique) morphism $\chi\colon X \to X'$, such that (φ,ψ,χ) is a morphism of candidate triangles.

(TR4') For every commutative diagram (A.2.1), where the rows are candidate triangles in $\triangle$, there exists a (not necessarily unique) morphism $\chi\colon X \to X'$ such that (φ,ψ,χ) is a morphism of candidate triangles, and such that the following candidate triangle belongs to $\triangle$,

$$\text{(A.2.2)} \qquad \begin{array}{c} M' \\ \oplus \\ N \end{array} \xrightarrow{\begin{pmatrix} \alpha' & \psi \\ 0 & -\beta \end{pmatrix}} \begin{array}{c} N' \\ \oplus \\ X \end{array} \xrightarrow{\begin{pmatrix} \beta' & \chi \\ 0 & -\gamma \end{pmatrix}} \begin{array}{c} X' \\ \oplus \\ \Sigma M \end{array} \xrightarrow{\begin{pmatrix} \gamma' & \Sigma\varphi \\ 0 & -\Sigma\alpha \end{pmatrix}} \begin{array}{c} \Sigma M' \\ \oplus \\ \Sigma N \end{array}.$$

The candidate triangle (A.2.2) is called the *mapping cone* of (φ,ψ,χ).

Condition (TR4') is evidently stronger than (TR3), and it is proved in A.4 below that (TR2) and (TR2') are equivalent under assumption of (TR0). The conditions in A.2 supply the axioms for a triangulated category.

A.3 Definition. A *triangulated category* is an additive category $\mathcal{T}$ equipped with an invertible endofunctor Σ and a collection $\triangle$ of candidate triangles, called *distinguished triangles*, such that (TR0), (TR1), (TR2'), and (TR4') are satisfied.

REMARK. If the collection $\triangle$ in $(\mathcal{T},\Sigma)$ satisfies only (TR0), (TR1), (TR2'), and (TR3), then $\mathcal{T}$ is called *pretriangulated*. It can be proved that for a pretriangulated category the so-called *octahedral axiom*, which is usually denoted (TR4), is equivalent to (TR4'); see Neeman [38]. That is, a triangulated category is a pretriangulated category that satisfies the octahedral axiom. This perspective goes back to Verdier's thesis on derived categories [49] from the mid 1960s—it was published 30 years late and only after Verdier's death. Indeed, triangulated categories in algebra were originally defined through axiomatization of the properties of derived categories; the axioms being (TR0), (TR1), (TR2), (TR3), and (TR4); usually with (TR0) included in (TR1). The contemporaty formulation of the definition in A.3 follows Neeman's monograph [39].

A.4 Lemma. *Let $\triangle$ be a collection of candidate triangles in $(\mathcal{T},\Sigma)$ such that* (TR0) *is satisfied. Condition* (TR2) *is then satisfied if and only if* (TR2') *is satisfied.*

PROOF. Assume that (TR2') is satisfied. Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ be a candidate triangle in $\triangle$. Consider the following isomorphism of candidate triangles,

$$(\star) \qquad \begin{array}{ccccccc} N & \xrightarrow{\beta} & X & \xrightarrow{\gamma} & \Sigma M & \xrightarrow{-\Sigma\alpha} & \Sigma N \\ \| & & \cong\downarrow -1^X & & \| & & \| \\ N & \xrightarrow{-\beta} & X & \xrightarrow{-\gamma} & \Sigma M & \xrightarrow{-\Sigma\alpha} & \Sigma N\,. \end{array}$$

By (TR2') the lower row in $(\star)$ belongs to $\triangle$, and hence so does the upper row by (TR0). To show that the candidate triangle

$$\Sigma^{-1} X \xrightarrow{-\Sigma^{-1}\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X$$

is in $\triangle$, is by (TR2') equivalent to showing that $M \xrightarrow{-\alpha} N \xrightarrow{-\beta} X \xrightarrow{\gamma} \Sigma M$ is in $\triangle$; and that follows from (TR0) and the next isomorphism of candidate triangles,

$$\begin{array}{ccccccc} M & \xrightarrow{-\alpha} & N & \xrightarrow{-\beta} & X & \xrightarrow{\gamma} & \Sigma M \\ \| & & \cong\downarrow -1^N & & \| & & \| \\ M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & X & \xrightarrow{\gamma} & \Sigma M\,. \end{array}$$

Similar arguments show that (TR2) implies (TR2'). □

A.5 Proposition. *Let $(\mathcal{T},\Sigma)$ be a triangulated category. The opposite category $(\mathcal{T}^{\mathrm{op}},\Sigma^{-1})$ is triangulated in the following canonical way: A candidate triangle $M \to N \to X \to \Sigma^{-1} M$ in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the corresponding diagram $\Sigma^{-1} M \to X \to N \to M$ is a distinguished triangle in $\mathcal{T}$.*

PROOF. It is evident that a diagram in $\mathcal{T}^{\mathrm{op}}$ is a candidate triangle if and only if the corresponding diagram in $\mathcal{T}$ is a candidate triangle. Let $\triangle$ be the collection of distinguished triangles in $\mathcal{T}$. It is elementary to verify that the collection of diagrams $M \to N \to X \to \Sigma^{-1} M$ in $\mathcal{T}^{\mathrm{op}}$ such that the corresponding diagram in $\mathcal{T}$ belongs to $\triangle$ satisfies the axioms in A.3. As an example, we provide the details for (TR0).

Let M be an object in $\mathcal{T}^{\mathrm{op}}$, and hence in $\mathcal{T}$. The candidate triangle

$$(\star) \qquad M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma^{-1} M$$

in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the corresponding candidate triangle in $\mathcal{T}$,

$$(\ddagger) \qquad \Sigma^{-1} M \longrightarrow 0 \longrightarrow M \xrightarrow{1^M} M\,,$$

belongs to $\triangle$. By (TR2'), applied twice, $(\ddagger)$ is in $\triangle$ if and only if the followiing candidate triangle is in $\triangle$,

$$(\diamond) \qquad M \xrightarrow{1^M} M \longrightarrow \Sigma 0 \longrightarrow \Sigma M\,.$$

There is an isomorphism $\Sigma 0 \cong 0$ in $\mathcal{T}$, so by (TR0) the triangle ($\diamond$) is in $\triangle$, whence ($\star$) is distinguished in $\mathcal{T}^{\mathrm{op}}$. Next, let

$$(\S) \qquad \begin{array}{ccccccc} M' & \xrightarrow{\alpha'} & N' & \xrightarrow{\beta'} & X' & \xrightarrow{\gamma'} & \Sigma^{-1} M' \\ \cong\big\downarrow \varphi & & \cong\big\downarrow \psi & & \cong\big\downarrow \chi & & \cong\big\downarrow \Sigma^{-1}\varphi \\ M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & X & \xrightarrow{\gamma} & \Sigma^{-1} M \end{array}$$

be an isomorphism of candidate triangles in $\mathcal{T}^{\mathrm{op}}$, and assume that the bottom row is distinguished. In the corresponding diagram in $\mathcal{T}$,

$$\begin{array}{ccccccc} \Sigma^{-1} M & \xrightarrow{\gamma} & X & \xrightarrow{\beta} & N & \xrightarrow{\alpha} & M \\ \cong\big\downarrow \Sigma^{-1}\varphi & & \cong\big\downarrow \chi & & \cong\big\downarrow \psi & & \cong\big\downarrow \varphi \\ \Sigma^{-1} M' & \xrightarrow{\gamma'} & X' & \xrightarrow{\beta'} & N' & \xrightarrow{\alpha'} & M' \end{array},$$

the top row belongs to $\triangle$, and by (TR0) so does the bottom row. Hence, the top row in (§) is a distinguished triangle in $\mathcal{T}^{\mathrm{op}}$. $\square$

A.6 Definition. Let $(\mathcal{T}, \Sigma_{\mathcal{T}})$ and $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be triangulated categories. A *triangulated functor* $\mathrm{F}\colon \mathcal{T} \to \mathcal{U}$ is an additive functor with a natural isomorphism $\phi\colon \mathrm{F}\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}}\mathrm{F}$ such that for every distinguished triangle in $\mathcal{T}$,

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma_{\mathcal{T}} M\ ,$$

the induced candidate triangle in $\mathcal{U}$,

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X) \xrightarrow{\phi_M \circ \mathrm{F}(\gamma)} \Sigma_{\mathcal{U}}\mathrm{F}(M)\ ,$$

is distinguished.

A.7 Definition. Let $(\mathcal{T}, \Sigma)$ be a triangulated category. A *triangulated subcategory* of $\mathcal{T}$ is a full additive subcategory $\mathcal{S}$ that satisfies the following conditions.

(1) If N and N' are isomorphic objects in $\mathcal{T}$, then N is in $\mathcal{S}$ if and only if N' is in $\mathcal{S}$.
(2) An object N is in $\mathcal{S}$ if an only if ΣN is in $\mathcal{S}$.
(3) For every distinguished triangle $M \to N \to X \to \Sigma M$ in $\mathcal{T}$, such that the objects M and N are in $\mathcal{S}$, also X is in $\mathcal{S}$.

Note that if $\mathcal{S}$ is a triangulated subcategory of $(\mathcal{T}, \Sigma)$, then $(\mathcal{S}, \Sigma)$ is a triangulated category.