E 6.2.6 Let M and N be R-complexes and let $N \xrightarrow{\simeq} I$ be a semi-injective resolution. Show that there is an isomorphisms of $\Bbbk$-modules, $\mathcal{K}(R)(M,I) \cong \mathcal{D}(R)(M,N)$.

E 6.2.7 Show that $\mathcal{K}_{\mathrm{inj}}(R)$ and $\mathcal{D}(R)$ are equivalent as triangulated categories; cf. E 6.1.16.

E 6.2.8 Let R be left hereditary. Show that there is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathcal{D}(R)$ for every R-complex M. *Hint:* E 5.2.3.

E 6.2.9 Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $(\mathcal{T},\Sigma)$. A morphism $\alpha\colon M \to N$ in $\mathcal{T}$ is called $\mathcal{S}$-*trivial* if in some, equivalently in every, distinguished triangle,

$$M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M\,,$$

the object X belongs to $\mathcal{S}$. Describe the $\mathcal{S}$-trivial morphisms in the category $\mathcal{K}(R)$ if $\mathcal{S}$ consists of all acyclic R-complexes; cf. E 6.1.11.

E 6.2.10 Let $(\mathcal{T},\Sigma)$ be a triangulated category. A commutative square in $\mathcal{T}$,

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & N \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ M & \xrightarrow[\beta]{} & V \end{array}$$

is called *homotopy cartesian* if there exists a distinguished triangle of the form

$$U \xrightarrow{\left(\begin{smallmatrix}\varphi \\ -\alpha\end{smallmatrix}\right)} \begin{matrix} M \\ \oplus \\ N \end{matrix} \xrightarrow{(\beta\ \ \psi)} V \xrightarrow{\gamma} \Sigma U\,.$$

The pair (φ,α) is called a *homotopy pullback* of (β,ψ), and (β,ψ) is called a *homotopy pushout* of (φ,α). Show that homotopy pushouts and homotopy pullbacks always exist.

E 6.2.11 Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $(\mathcal{T},\Sigma)$, and consider the homotopy cartesian square in $\mathcal{T}$ from E 6.2.10. Show that the morphism α is $\mathcal{S}$-trivial if and only if β is $\mathcal{S}$-trivial in the sense of E 6.2.9. *Hint:* See [41, lem. 1.5.8].

6.3 Derived Functors

SYNOPSIS. Induced functors; induced natural transformations; functoriality of resolutions; left and right derived functors.

FUNCTORS INDUCED FROM $\mathcal{C}$ TO $\mathcal{K}$

6.3.1 Theorem. *Let* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be a functor. If* F *maps homotopy equivalences to homotopy equivalences, then there exists a unique functor* $\tilde{\mathrm{F}}$ *that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R) & \xrightarrow{\mathrm{Q}_R} & \mathcal{K}(R) \\ {\scriptstyle \mathrm{F}}\downarrow & & \downarrow{\scriptstyle \tilde{\mathrm{F}}} \\ \mathcal{C}(S) & \xrightarrow[\mathrm{Q}_S]{} & \mathcal{K}(S)\ ; \end{array}$$

here Q *is the canonical functor from* 6.1.6. *For an* R*-complex* M *there is an equality* $\tilde{\mathrm{F}}(M) = \mathrm{F}(M)$, *and for a morphism* $[\alpha]$ *in* $\mathcal{K}(R)$ *one has* $\tilde{\mathrm{F}}([\alpha]) = [\mathrm{F}(\alpha)]$. *Furthermore, the following assertions hold.*

(a) *If* F *is* $\Bbbk$*-linear, then* $\tilde{\mathrm{F}}$ *is* $\Bbbk$*-linear.*

(b) *If* F *preserves products/coproducts, then* $\tilde{\mathrm{F}}$ *preserves products/coproducts.*

(c) *If* F *is a* Σ*-functor, then* $\tilde{\mathrm{F}}$ *is triangulated.*

PROOF. As F maps homotopy equivalences to homotopy equivalences, the composite functor $\mathrm{Q}_S\mathrm{F}$ maps homotopy equivalences in $\mathcal{C}(R)$ to isomorphisms in $\mathcal{K}(S)$; see 6.1.3. Thus the existence and uniqueness of $\tilde{\mathrm{F}}$ follow from 6.1.18.

(a): If F is $\Bbbk$-linear, then so is the functor $\mathrm{Q}_S\mathrm{F}$ by 6.1.8. It follows from 6.1.18(a) that the induced functor $\tilde{\mathrm{F}}$ is $\Bbbk$-linear.

(b): If F preserves (co)products, then so does the functor $\mathrm{Q}_S\mathrm{F}$ by 6.1.8. It follows from 6.1.18(b) that the induced functor $\tilde{\mathrm{F}}$ preserves (co)products.

(c): Assume that F is a Σ-functor. To show that $\tilde{\mathrm{F}}$ is triangulated, it suffices by 6.1.18(c) to argue that the functor $\mathrm{Q}_S\mathrm{F}$ is quasi-triangulated. By assumption there is a natural isomorphism $\varphi\colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$ such that for every morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ there exists an isomorphism $\breve{\alpha}$ in $\mathcal{C}(S)$ that makes the diagram in 4.1.6 commutative. Applying the functor Q_S to φ and using the identity $\mathrm{Q}_S\Sigma = \Sigma\mathrm{Q}_S$, see 6.1.12, one gets a natural isomorphism,

$$\mathrm{Q}_S(\varphi)\colon (\mathrm{Q}_S\mathrm{F})\Sigma \longrightarrow \Sigma(\mathrm{Q}_S\mathrm{F})\,.$$

To see that functor $\mathrm{Q}_S\mathrm{F}$ with the natural isomorphism $\mathrm{Q}_S(\varphi)$ is quasi-triangulated in the sense of 6.1.16, it must be argued that for every morphism $\alpha\colon M \to N$ of R-complexes, the following candidate triangle in $\mathcal{K}(S)$ is distinguished,

$$\mathrm{Q}_S\mathrm{F}(M) \xrightarrow{\mathrm{Q}_S\mathrm{F}(\alpha)} \mathrm{Q}_S\mathrm{F}(N) \xrightarrow{\mathrm{Q}_S\mathrm{F}\left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)} \mathrm{Q}_S\mathrm{F}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{Q}_S(\varphi^M)\circ\mathrm{Q}_S\mathrm{F}(0\ 1^{\Sigma M})} \Sigma\mathrm{Q}_S\mathrm{F}(M)\,.$$

As already noted, one has $\Sigma\mathrm{Q}_S = \mathrm{Q}_S\Sigma$, and thus the diagram above is given by application of Q_S to the top row in the following diagram in $\mathcal{C}(S)$,

$$(\star)\qquad \begin{array}{ccccccc} \mathrm{F}(M) & \xrightarrow{\mathrm{F}(\alpha)} & \mathrm{F}(N) & \xrightarrow{\mathrm{F}\left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)} & \mathrm{F}(\operatorname{Cone}\alpha) & \xrightarrow{\varphi^M\circ\mathrm{F}(0\ 1^{\Sigma M})} & \Sigma\mathrm{F}(M) \\ \Vert & & \Vert & & {\scriptstyle\cong}\downarrow{\scriptstyle\breve{\alpha}} & & \Vert \\ \mathrm{F}(M) & \xrightarrow{\mathrm{F}(\alpha)} & \mathrm{F}(N) & \xrightarrow{\left(\begin{smallmatrix} 1^{\mathrm{F}(N)} \\ 0 \end{smallmatrix}\right)} & \operatorname{Cone}\mathrm{F}(\alpha) & \xrightarrow{(0\ 1^{\Sigma\mathrm{F}(M)})} & \Sigma\mathrm{F}(M)\,. \end{array}$$

The diagram $(\star)$ is commutative by assumption on φ^M and $\breve{\alpha}$. Therefore, application of the functor Q_S to the diagram $(\star)$ yields an isomorphism of candidate triangles in $\mathcal{K}(S)$, where the lower one is strict by 6.1.14. Hence the upper candidate triangle is distinguished, as desired. □

6.3.2 Theorem. *Let* $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be a functor. If* G *maps homotopy equivalences to homotopy equivalences, then there exists a unique functor* $\tilde{\mathrm{G}}$ *that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{Q}_R^{\mathrm{op}}} & \mathcal{K}(R)^{\mathrm{op}} \\ {\scriptstyle \mathrm{G}}\downarrow & & \downarrow{\scriptstyle \tilde{\mathrm{G}}} \\ \mathcal{C}(S) & \xrightarrow[\mathrm{Q}_S]{} & \mathcal{K}(S)\,; \end{array}$$

here Q *is the canonical functor from* 6.1.6. *For an* R*-complex* M *there is an equality* $\tilde{\mathrm{G}}(M) = \mathrm{G}(M)$*, and for a morphism* $[\alpha]$ *in* $\mathcal{K}(R)^{\mathrm{op}}$ *one has* $\tilde{\mathrm{G}}([\alpha]) = [\mathrm{G}(\alpha)]$. *Furthermore, the following assertions hold.*

(a) *If* G *is* $\Bbbk$*-linear, then* $\tilde{\mathrm{G}}$ *is* $\Bbbk$*-linear.*

(b) *If* G *preserves (co)products, then* $\tilde{\mathrm{G}}$ *preserves (co)products.*

(c) *If* G *is a* Σ*-functor, then* $\tilde{\mathrm{G}}$ *is triangulated.* □

FUNCTORS INDUCED FROM $\mathcal{K}$ TO $\mathcal{D}$

6.3.3 Theorem. *Let* $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ *be a functor. If* F *maps quasi-isomorphisms to quasi-isomorphisms, then there exists a unique functor* $\tilde{\mathrm{F}}$ *that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{K}(R) & \xrightarrow{\mathrm{V}_R} & \mathcal{D}(R) \\ {\scriptstyle \mathrm{F}}\downarrow & & \downarrow{\scriptstyle \tilde{\mathrm{F}}} \\ \mathcal{K}(S) & \xrightarrow[\mathrm{V}_S]{} & \mathcal{D}(S)\,; \end{array}$$

here V *is the canonical functor from* 6.2.12. *For an* R*-complex* M *there is an equality* $\tilde{\mathrm{F}}(M) = \mathrm{F}(M)$*, and for a morphism* ${}^{\alpha}/_{\varphi}$ *in* $\mathcal{D}(R)$ *one has* $\tilde{\mathrm{F}}({}^{\alpha}/_{\varphi}) = {}^{\mathrm{F}(\alpha)}/_{\mathrm{F}(\varphi)}$. *Furthermore, the following assertions hold.*

(a) *If* F *is* $\Bbbk$*-linear, then* $\tilde{\mathrm{F}}$ *is* $\Bbbk$*-linear.*

(b) *If* F *preserves (co)products, then* $\tilde{\mathrm{F}}$ *preserves (co)products.*

(c) *If* F *is triangulated, then* $\tilde{\mathrm{F}}$ *is triangulated.*

PROOF. As F maps quasi-isomorphisms to quasi-isomorphisms, the composite functor $\mathrm{V}_S\mathrm{F}$ maps quasi-isomorphisms in $\mathcal{K}(R)$ to isomorphisms in $\mathcal{D}(S)$; see 6.2.16. Hence, the existence and uniqueness of $\tilde{\mathrm{F}}$ follow from 6.2.30. The value of $\tilde{\mathrm{F}}$ on an

R-complex M is $\tilde{\mathrm{F}}(M) = \mathrm{V}_S\mathrm{F}(M) = \mathrm{F}(M)$ since V_S is the identity on objects. By 6.2.30, 6.2.15, and 6.2.10 the value of $\tilde{\mathrm{F}}$ on a morphism ${}^{\alpha}\!/_{\varphi}$ is

$$\tilde{\mathrm{F}}({}^{\alpha}\!/_{\varphi}) = (\mathrm{V}_S\mathrm{F}(\alpha))(\mathrm{V}_S\mathrm{F}(\varphi))^{-1} = ({}^{\mathrm{F}(\alpha)}\!/_{1})({}^{\mathrm{F}(\varphi)}\!/_{1})^{-1} = ({}^{\mathrm{F}(\alpha)}\!/_{1})({}^{1}\!/_{\mathrm{F}(\varphi)}) = {}^{\mathrm{F}(\alpha)}\!/_{\mathrm{F}(\varphi)}\,.$$

By 6.2.22 and 6.2.29 the functor V_S is $\Bbbk$-linear, triangulated, and preserves (co)products. Thus, if F has any of these properties, then so does the composite functor $\mathrm{V}_S\mathrm{F}$, and the assertions in parts (a)–(c) follow from the corresponding parts in 6.2.30. □

6.3.4 Theorem. *Let* $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ *be a functor. If* G *maps quasi-isomorphisms to quasi-isomorphisms, then there exists a unique functor* $\tilde{\mathrm{G}}$ *that makes the following diagram commutative,*

$$\begin{array}{ccc}
\mathcal{K}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{V}_R^{\mathrm{op}}} & \mathcal{D}(R)^{\mathrm{op}} \\
{\scriptstyle \mathrm{G}}\downarrow & & \downarrow{\scriptstyle \tilde{\mathrm{G}}} \\
\mathcal{K}(S) & \xrightarrow[\mathrm{V}_S]{} & \mathcal{D}(S)\,;
\end{array}$$

here V *is the canonical functor from* 6.2.12*. For an* R*-complex* M *there is an equality* $\tilde{\mathrm{G}}(M) = \mathrm{G}(M)$*, and for a morphism* ${}^{\alpha}\!/_{\varphi}$ *in* $\mathcal{D}(R)$ *one has* $\tilde{\mathrm{G}}({}^{\alpha}\!/_{\varphi}) = ({}^{1}\!/_{\mathrm{G}(\varphi)})({}^{\mathrm{G}(\alpha)}\!/_{1})$. *Furthermore, the following assertions hold.*

(a) *If* G *is* $\Bbbk$*-linear, then* $\tilde{\mathrm{G}}$ *is* $\Bbbk$*-linear.*

(b) *If* G *preserves (co)products, then* $\tilde{\mathrm{G}}$ *preserves (co)products.*

(c) *If* G *is triangulated, then* $\tilde{\mathrm{G}}$ *is triangulated.* □

INDUCED NATURAL TRANSFORMATIONS

6.3.5 Proposition. *Let* $\mathrm{F},\mathrm{G}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be functors that map homotopy equivalences to homotopy equivalences, and let* $\tilde{\mathrm{F}},\tilde{\mathrm{G}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ *be the functors induced by* F *and* G*; see* 6.3.1*. If* $\eta\colon \mathrm{F} \to \mathrm{G}$ *is a natural transformation, then there is a natural transformation* $[\eta]\colon \tilde{\mathrm{F}} \to \tilde{\mathrm{G}}$ *defined by* $[\eta]^M = [\eta^M]$ *for every* R*-complex* M*.*

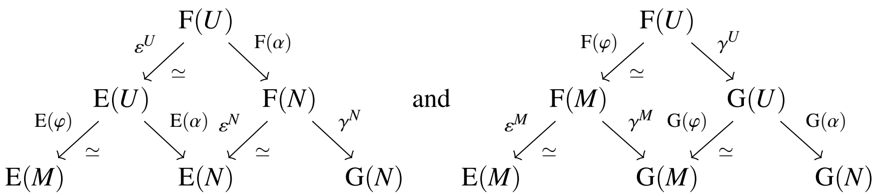
PROOF. Let $[\alpha]$ be a morphism in $\mathcal{K}(R)$, represented by some morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$. Recall from 6.3.1 that one has $\tilde{\mathrm{F}}([\alpha]) = [\mathrm{F}(\alpha)]$ and $\tilde{\mathrm{G}}([\alpha]) = [\mathrm{G}(\alpha)]$. Since η is a natural transformation, there is an equality $\eta^N\mathrm{F}(\alpha) = \mathrm{G}(\alpha)\eta^M$. It follows that $[\eta^N][\mathrm{F}(\alpha)] = [\mathrm{G}(\alpha)][\eta^M]$ holds, i.e. one has $[\eta]^N\tilde{\mathrm{F}}([\alpha]) = \tilde{\mathrm{G}}([\alpha])[\eta]^M$, which shows that $[\eta]$ is a natural transformation. □

6.3.6 Proposition. *Let* $\mathrm{E},\mathrm{F},\mathrm{G}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ *be functors that map quasi-isomorphisms to quasi-isomorphisms, and let* $\tilde{\mathrm{E}},\tilde{\mathrm{F}},\tilde{\mathrm{G}}\colon \mathcal{D}(R) \to \mathcal{D}(S)$ *be the functors induced by* E, F, *and* G*; see* 6.3.3*. If* $\mathrm{E} \xleftarrow{\varepsilon} \mathrm{F} \xrightarrow{\gamma} \mathrm{G}$ *are natural transformations and* ε^M *is a quasi-isomorphism for every* R*-complex* M*, then there is a natural transformation* ${}^{\gamma}\!/_{\varepsilon}\colon \tilde{\mathrm{E}} \to \tilde{\mathrm{G}}$ *defined by* $({}^{\gamma}\!/_{\varepsilon})^M = {}^{\gamma^M}\!/_{\varepsilon^M}$ *for every* R*-complex* M*.*

PROOF. Let ${}^{\alpha}/_{\varphi}\colon M \to N$ be a morphism in $\mathcal{D}(R)$. Recall from 6.3.3 that one has $\tilde{\mathrm{E}}({}^{\alpha}/_{\varphi}) = {}^{\mathrm{E}(\alpha)}/_{\mathrm{E}(\varphi)}$ and $\tilde{\mathrm{G}}({}^{\alpha}/_{\varphi}) = {}^{\mathrm{G}(\alpha)}/_{\mathrm{G}(\varphi)}$. Thus, to see that ${}^{\gamma}/_{\varepsilon}$ is a natural transformation, it must be shown that there is an equality,

$$(\star) \qquad \bigl({}^{\gamma^N}/_{\varepsilon^N}\bigr)\bigl({}^{\mathrm{E}(\alpha)}/_{\mathrm{E}(\varphi)}\bigr) = \bigl({}^{\mathrm{G}(\alpha)}/_{\mathrm{G}(\varphi)}\bigr)\bigl({}^{\gamma^M}/_{\varepsilon^M}\bigr)\,.$$

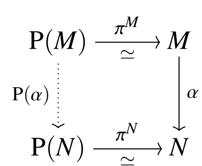
Let U be the common source of α and φ. The diagrams in $\mathcal{K}(R)$,



are commutative as ε and γ are natural transformations. Thus, the left-hand composite in $(\star)$ is ${}^{(\gamma^N\mathrm{F}(\alpha))}/_{(\mathrm{E}(\varphi)\varepsilon^U)}$ and the right-hand composite is ${}^{(\mathrm{G}(\alpha)\gamma^U)}/_{(\varepsilon^M\mathrm{F}(\varphi))}$. These two left fractions have the same numerators and denoninators, indeed, since γ and ε are natural transformations, one has $\gamma^N\mathrm{F}(\alpha) = \mathrm{G}(\alpha)\gamma^U$ and $\mathrm{E}(\varphi)\varepsilon^U = \varepsilon^M\mathrm{F}(\varphi)$. This proves the equality $(\star)$. □

FUNCTORIALITY OF RESOLUTIONS

6.3.7 Construction. For every R-complex M, choose a semi-projective resolution $\pi^M\colon \mathrm{P}(M) \xrightarrow{\simeq} M$ in $\mathcal{K}(R)$. For every morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$, denote by $\mathrm{P}(\alpha)$ the unique morphism in $\mathcal{K}(R)$ that makes the diagram



commutative; cf. 6.1.22.

6.3.8 Theorem. *The assignments* $M \mapsto \mathrm{P}(M)$ *and* $\alpha \mapsto \mathrm{P}(\alpha)$ *in* 6.3.7 *define an endofunctor on* $\mathcal{K}(R)$. *This functor* P *is* $\Bbbk$*-linear, triangulated, it preserves coproducts, and it maps quasi-isomorphisms to isomorphisms. Moreover,* π *is a natural transformation from* P *to* $\mathrm{Id}^{\mathcal{K}(R)}$.

PROOF. The fact that $\mathrm{P}(\alpha)$ is the unique morphism that makes the diagram in 6.3.7 commutative implies that P is a $\Bbbk$-linear functor. Indeed, for morphisms $\alpha\colon M \to N$ and $\beta\colon L \to M$ in $\mathcal{K}(R)$ both morphisms $\mathrm{P}(\alpha\beta)$ and $\mathrm{P}(\alpha)\mathrm{P}(\beta)$ make the following diagram commutative,

$$\begin{array}{ccc} \mathrm{P}(L) & \xrightarrow[\simeq]{\pi^L} & L \\ {\scriptstyle \mathrm{P}(\alpha\beta)}\Big\downarrow\Big\downarrow{\scriptstyle \mathrm{P}(\alpha)\mathrm{P}(\beta)} & & \Big\downarrow{\scriptstyle \alpha\beta} \\ \mathrm{P}(N) & \xrightarrow[\simeq]{\pi^N} & N\,, \end{array}$$

and hence they are identical: $\mathrm{P}(\alpha\beta) = \mathrm{P}(\alpha)\mathrm{P}(\beta)$. Similarly one finds that the equality $\mathrm{P}(1^M) = 1^{\mathrm{P}(M)}$ holds for every R-complex M and $\mathrm{P}(x\alpha+\beta) = x\mathrm{P}(\alpha)+\mathrm{P}(\beta)$ for every pair α,β of parallel morphisms in $\mathcal{K}(R)$ and every x in $\Bbbk$.

Commutativity of the diagram in 6.3.7 shows that if α is a quasi-isomorphism in $\mathcal{K}(R)$, then so is $\mathrm{P}(\alpha)$. It now follows from 6.1.24 that $\mathrm{P}(\alpha)$ is an isomorphism; thus P maps quasi-isomorphisms to isomorphisms. Commutativity of the diagram in 6.3.7 also shows that π is a natural transformation from P to $\mathrm{Id}^{\mathcal{K}(R)}$.

To see that P preserves coproducts, let $\{M^u\}_{u\in U}$ be a family of R-complexes and notice that there is a commutative diagram in $\mathcal{K}(R)$,

$$\begin{array}{ccc} \coprod_{u\in U}\mathrm{P}(M^u) & \xrightarrow{\varphi} & \mathrm{P}(\coprod_{u\in U} M^u) \\ & {\scriptstyle \coprod \pi^{M^u}}\searrow\simeq \quad \simeq\swarrow{\scriptstyle \pi^{\coprod M^u}} & \\ & \coprod_{u\in U} M^u\,, & \end{array}$$

where φ is the canonical map; cf. 3.1.10. The morphism $\pi^{\coprod M^u}$ is a quasi-isomorphism by construction and $\coprod_{u\in U}\pi^{M^u}$ is a quasi-isomorphism by 6.1.11. Therefore φ is a quasi-isomorphism, and since the complex $\coprod_{u\in U}\mathrm{P}(M^u)$ is semi-projective by 5.2.15, it follows from 6.1.24 that φ is an isomorphism.

It remains to show that the functor P is triangulated; see A.6. For each R-complex M, let ϕ^M be the unique morphism in $\mathcal{K}(R)$ that makes the diagram

$$(\star) \qquad \begin{array}{ccc} & & \mathrm{P}(\Sigma M) \\ & {\scriptstyle \phi^M}\swarrow & \simeq\Big\downarrow{\scriptstyle \pi^{\Sigma M}} \\ \Sigma\mathrm{P}(M) & \xrightarrow[\Sigma\pi^M]{\simeq} & \Sigma M \end{array}$$

commutative; see 6.1.22. It follows that ϕ^M is a quasi-isomorphism, and since its source and target are semi-projective R-complexes, 6.1.24 implies that ϕ^M is an isomorphism. In fact, $\phi\colon \mathrm{P}\Sigma \to \Sigma\mathrm{P}$ is a natural isomorphism; that is, for every morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$ one has $\phi^N\mathrm{P}(\Sigma\alpha) = (\Sigma\mathrm{P}(\alpha))\phi^M$. Indeed, as the source $\mathrm{P}(\Sigma M)$ is semi-projective and $\Sigma\pi^N$ is a quasi-isomorphism, it suffices by 6.1.25 to argue that $(\Sigma\pi^N)\phi^N\mathrm{P}(\Sigma\alpha) = (\Sigma\pi^N)(\Sigma\mathrm{P}(\alpha))\phi^M$ holds. And that is a straightforward computation using the commutativity of the diagram in 6.3.7 and of $(\star)$,

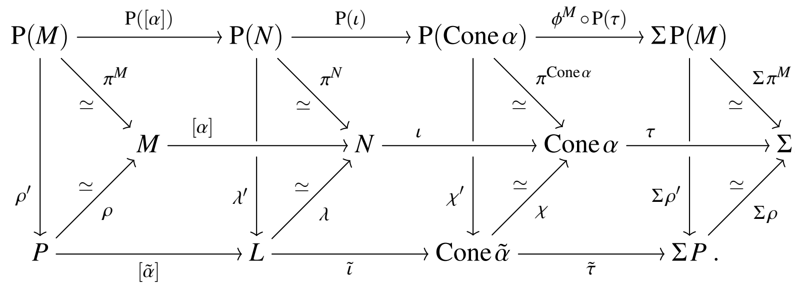
$$(\Sigma\pi^N)\phi^N\mathrm{P}(\Sigma\alpha) = \pi^{\Sigma N}\mathrm{P}(\Sigma\alpha) = (\Sigma\alpha)\pi^{\Sigma M} = (\Sigma\alpha)(\Sigma\pi^M)\phi^M = (\Sigma\pi^N)(\Sigma\mathrm{P}(\alpha))\phi^M\,.$$

We will argue that P with the natural isomorphism $\phi\colon \mathrm{P}\Sigma \to \Sigma\mathrm{P}$ is triangulated. By the definition 6.1.14 of distinguished triangles in $\mathcal{K}(R)$, it is enough to argue that P maps every strict triangle in $\mathcal{K}(R)$ to a distinguished one. Consider a morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ and the associated strict triangle in $\mathcal{K}(R)$,

$$M \xrightarrow{[\alpha]} N \xrightarrow{\iota} \operatorname{Cone}\alpha \xrightarrow{\tau} \Sigma M\ ,$$

where ι and τ are the homotopy classes of the morphisms $(1^N\ 0)^{\mathrm{T}}$ and $(0\ 1^{\Sigma M})$ in $\mathcal{C}(R)$. It must be shown that the upper candidate triangle in the following diagram is distinguished,

(‡)



Commutativity of the diagram in 6.3.7 and of $(\star)$ show that the "upper face" in (‡), i.e. the three squares that involve π, is commutative. Choose semi-projective resolutions $\rho\colon P \xrightarrow{\simeq} M$ and $\lambda\colon L \xrightarrow{\simeq} N$ in $\mathcal{K}(R)$, for example, $P = \mathrm{P}(M)$ and $L = \mathrm{P}(N)$, and let $[\tilde{\alpha}]$ be the unique morphism in $\mathcal{K}(R)$ that makes the diagram

$$\begin{array}{ccc} P & \xrightarrow[\simeq]{\rho} & M \\ {\scriptstyle [\tilde{\alpha}]}\Big\downarrow & & \Big\downarrow{\scriptstyle [\alpha]} \\ L & \xrightarrow[\simeq]{\lambda} & N \end{array}$$

commutative; cf. 6.1.22. The lower row in (‡) is the strict triangle in $\mathcal{K}(R)$ associated to some choice of morphism $\tilde{\alpha}$ in $\mathcal{C}(R)$ that represents the homotopy class $[\tilde{\alpha}]$. By axiom (TR4') for a triangulated category there exists a morphism χ that makes the "lower face" in (‡) commutative; see A.2. Since ρ and λ are quasi-isomorphisms, so is χ by 6.1.21. By 6.1.22 there exist morphisms ρ', λ', and χ' in $\mathcal{K}(R)$ that make the three leftmost "walls" in (‡) commutative; that is, one has $\rho\rho' = \pi^M$, $\lambda\lambda' = \pi^N$, and $\chi\chi' = \pi^{\operatorname{Cone}\alpha}$. It follows that also the rightmost "wall" in (‡) is commutative, and that ρ', λ', and χ' are quasi-isomorphisms. We now argue that the "back face" in (‡) is commutative, i.e. that the equalities

$$(\diamond) \qquad [\tilde{\alpha}]\rho' = \lambda'\mathrm{P}([\alpha])\ , \quad \tilde{\iota}\lambda' = \chi'\mathrm{P}(\iota)\ , \quad \text{and} \quad \tilde{\tau}\chi' = (\Sigma\rho')\phi^M\mathrm{P}(\tau)$$

hold. To prove the first equality in $(\diamond)$ it suffices by 6.1.25 to argue that one has $\lambda[\tilde{\alpha}]\rho' = \lambda\lambda'\mathrm{P}([\alpha])$. This follows from the parts of the diagram (‡) that are already known to be commutative:

$$\lambda[\tilde{\alpha}]\rho' = [\alpha]\rho\rho' = [\alpha]\pi^M = \pi^N \mathrm{P}([\alpha]) = \lambda\lambda' \mathrm{P}([\alpha])\,.$$

The second and third equalities in ($\diamond$) are proved similarly. Application of 5.2.14 to the short exact sequence $0 \to L \to \operatorname{Cone}\tilde{\alpha} \to \Sigma P \to 0$ in $\mathcal{C}(R)$ from 4.1.3 shows that the complex $\operatorname{Cone}\tilde{\alpha}$ is semi-projective. Therefore ρ', λ', and χ' are quasi-isomorphisms whose sources and targets are semi-projective complexes. It follows from 6.1.24 that ρ', λ', and χ' are isomorphisms in $\mathcal{K}(R)$, and hence the "back face" in (‡) is an isomorphism of candidate triangles in $\mathcal{K}(R)$. Since the lower candidate triangle is strict, the upper candidate triangle is distinguished, as desired. $\square$

6.3.9. Let P and $\tilde{\mathrm{P}}$ be endofunctors on $\mathcal{K}(R)$ defined by making possibly different choices of semi-projective resolutions $\pi^M\colon \mathrm{P}(M) \xrightarrow{\simeq} M$ and $\tilde{\pi}^M\colon \tilde{\mathrm{P}}(M) \xrightarrow{\simeq} M$ in $\mathcal{K}(R)$ for every R-complex M; cf. 6.3.8. By 6.1.22 there exists for every M a unique morphism φ^M in $\mathcal{K}(R)$ that makes the following diagram commutative,

$$\begin{array}{ccc} & & \mathrm{P}(M) \\ & \overset{\varphi^M}{\swarrow} & \Big\downarrow{\scriptstyle \simeq\ \pi^M} \\ \tilde{\mathrm{P}}(M) & \xrightarrow[\tilde{\pi}^M]{\simeq} & M\,. \end{array}$$

Since π^M and $\tilde{\pi}^M$ are quasi-isomorphisms, so is φ^M, and hence 6.1.24 yields that φ^M is an isomorphism in $\mathcal{K}(R)$. In fact, $\varphi\colon \mathrm{P} \to \tilde{\mathrm{P}}$ is a natural isomorphism. Indeed, for a morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$, both $\varphi^N \mathrm{P}(\alpha)$ and $\tilde{\mathrm{P}}(\alpha)\varphi^M$ make the diagram

$$\begin{array}{ccc} \mathrm{P}(M) & \xrightarrow[\simeq]{\pi^M} & M \\ {\scriptstyle \varphi^N\mathrm{P}(\alpha)}\Big\downarrow\Big\downarrow{\scriptstyle \tilde{\mathrm{P}}(\alpha)\varphi^M} & & \Big\downarrow{\scriptstyle \alpha} \\ \tilde{\mathrm{P}}(N) & \xrightarrow[\simeq]{\tilde{\pi}^N} & N \end{array}$$

commutative, whence one has $\varphi^N \mathrm{P}(\alpha) = \tilde{\mathrm{P}}(\alpha)\varphi^M$ by 6.1.25.

6.3.10 Construction. For every R-complex M, choose a semi-injective resolution $\iota^M\colon M \xrightarrow{\simeq} \mathrm{I}(M)$ in $\mathcal{K}(R)$. For every morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$, denote by $\mathrm{I}(\alpha)$ the unique morphism in $\mathcal{K}(R)$ that makes the diagram

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{\iota^M} & \mathrm{I}(M) \\ {\scriptstyle \alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{I}(\alpha)} \\ N & \xrightarrow[\simeq]{\iota^N} & \mathrm{I}(N) \end{array}$$

commutative; cf. 6.1.26.

6.3.11 Theorem. *The assignments $M \mapsto \mathrm{I}(M)$ and $\alpha \mapsto \mathrm{I}(\alpha)$ in 6.3.10 define an endofunctor on $\mathcal{K}(R)$. This functor* I *is* $\Bbbk$*-linear, triangulated, it preserves products, and it maps quasi-isomorphisms to isomorphisms. Moreover,* ι *is a natural transformation from* $\mathrm{Id}^{\mathcal{K}(R)}$ *to* I. □

6.3.12. Let I and $\tilde{\mathrm{I}}$ be endofunctors on $\mathcal{K}(R)$ defined by making possibly different choices of semi-injective resolutions in $\mathcal{K}(R)$ for every R-complex M; cf. 6.3.11. There exists then a natural isomorphism $\varphi\colon \mathrm{I} \to \tilde{\mathrm{I}}$.

LEFT AND RIGHT DERIVED FUNCTORS

6.3.13. Let $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ be a functor and let $\mathrm{P}, \mathrm{I}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ be the resolution functors from 6.3.8 and 6.3.11. Since the composite functors FP and FI map quasi-isomorphisms to isomorphisms, it follows from 6.3.3 that there are unique functors **L**F and **R**F that make the following diagrams commutative,

$$\begin{array}{ccc} \mathcal{K}(R) & \xrightarrow{\mathrm{V}_R} & \mathcal{D}(R) \\ {\scriptstyle \mathrm{FP}}\downarrow & & \downarrow{\scriptstyle \mathbf{L}\mathrm{F}} \\ \mathcal{K}(S) & \xrightarrow[\mathrm{V}_S]{} & \mathcal{D}(S) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{K}(R) & \xrightarrow{\mathrm{V}_R} & \mathcal{D}(R) \\ {\scriptstyle \mathrm{FI}}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\mathrm{F}} \\ \mathcal{K}(S) & \xrightarrow[\mathrm{V}_S]{} & \mathcal{D}(S) \end{array}.$$

6.3.14 Definition. Let $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ be a functor. The functor **L**F from 6.3.13 is called the *left derived functor* of F, and **R**F is called the *right derived functor* of F.

6.3.15. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor that maps homotopy equivalences to homotopy equivalences. By 6.3.1 it induces a functor $\tilde{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$; the derived functors $\mathbf{L}\tilde{\mathrm{F}}$ and $\mathbf{R}\tilde{\mathrm{F}}$ defined in 6.3.14 are for simplicity denoted **L**F and **R**F.

6.3.16 Definition. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor that maps homotopy equivalences to homotopy equivalences. The functor **L**F from 6.3.15 is called the *left derived functor* of F, and **R**F is called the *right derived functor* of F.

6.3.17. For an arbitrary functor $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$, and for a functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ that maps homotopy equivalences to homotopy equivalences, it follows from 6.3.1, 6.3.3, 6.3.13, and 6.3.15 that the derived functors **L**F and **R**F act as follows on objects and morphisms in the derived category.

- For an R-complex M one has $\mathbf{L}\mathrm{F}(M) = \mathrm{FP}(M)$.
- For a morphism α/φ in $\mathcal{D}(R)$ one has $\mathbf{L}\mathrm{F}(\alpha/\varphi) = \mathrm{FP}(\alpha)/\mathrm{FP}(\varphi)$.

Similarly,

- For an R-complex M one has $\mathbf{R}\mathrm{F}(M) = \mathrm{FI}(M)$.
- For a morphism α/φ in $\mathcal{D}(R)$ one has $\mathbf{R}\mathrm{F}(\alpha/\varphi) = \mathrm{FI}(\alpha)/\mathrm{FI}(\varphi)$.

6.3.18 Example. Consider the semi-projective resolution functor $\mathrm{P}\colon \mathcal{K}(R) \to \mathcal{K}(R)$. The derived functors $\mathbf{L}\mathrm{P}$ and $\mathbf{R}\mathrm{P}$ are both naturally isomorphic to $\mathrm{Id}^{\mathcal{D}(R)}$. Indeed, applying P to the natural transformations $\pi\colon \mathrm{P} \xrightarrow{\simeq} \mathrm{Id}^{\mathcal{K}(R)}$ and $\iota\colon \mathrm{Id}^{\mathcal{K}(R)} \xrightarrow{\simeq} \mathrm{I}$ from 6.3.8 and 6.3.11 yields natural isomorphisms $\mathrm{P}(\pi)$ and $\mathrm{P}(\iota)$. Thus one has two pairs of natural transformations of endofunctors on $\mathcal{K}(R)$,

$$\mathrm{PP} \xleftarrow[=]{} \mathrm{PP} \xrightarrow[\simeq]{\pi\mathrm{P}(\pi)} \mathrm{Id}^{\mathcal{K}(R)} \qquad \text{and} \qquad \mathrm{PI} \xleftarrow[\cong]{\mathrm{P}(\iota)} \mathrm{P} \xrightarrow[\simeq]{\pi} \mathrm{Id}^{\mathcal{K}(R)}\,,$$

which by 6.3.6 yield natural isomorphisms $\mathbf{L}\mathrm{P} \to \mathrm{Id}^{\mathcal{D}(R)}$ and $\mathbf{R}\mathrm{P} \to \mathrm{Id}^{\mathcal{D}(R)}$.

Similarly, the derived functors $\mathbf{L}\mathrm{I}$ and $\mathbf{R}\mathrm{I}$ of the semi-injective resolution functor $\mathrm{I}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ are both naturally isomorphic to $\mathrm{Id}^{\mathcal{D}(R)}$.

6.3.19 Example. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor that maps homotopy equivalences to homotopy equivalences and quasi-isomorphisms to quasi-isomorphisms. As the induced functor $\tilde{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ from 6.3.1 also maps quasi-isomorphisms to quasi-isomorphisms, it induces by 6.3.3 a functor $\acute{\mathrm{F}}\colon \mathcal{D}(R) \to \mathcal{D}(S)$. In a sense, $\acute{\mathrm{F}}$ is nothing but the functor F. Indeed, for an R-complex M one has $\acute{\mathrm{F}}(M) = \mathrm{F}(M)$, and for a morphism $[\alpha]/[\varphi]$ in $\mathcal{D}(R)$, where α and φ are morphisms in $\mathcal{C}(R)$ with the same source, there is an equality $\acute{\mathrm{F}}([\alpha]/[\varphi]) = [\mathrm{F}(\alpha)]/[\mathrm{F}(\varphi)]$.

The derived functors $\mathbf{L}\mathrm{F}$ and $\mathbf{R}\mathrm{F}$ are naturally isomorphic to $\acute{\mathrm{F}}$. Indeed, applying $\tilde{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ to the natural transformations $\pi\colon \mathrm{P} \xrightarrow{\simeq} \mathrm{Id}^{\mathcal{K}(R)}$ and $\iota\colon \mathrm{Id}^{\mathcal{K}(R)} \xrightarrow{\simeq} \mathrm{I}$ from 6.3.8 and 6.3.11 yields natural transformations $\tilde{\mathrm{F}}(\pi)$ and $\tilde{\mathrm{F}}(\iota)$. Thus one has two pairs of natural transformations of endofunctors on $\mathcal{K}(R)$,

$$\tilde{\mathrm{F}}\mathrm{P} \xleftarrow[=]{} \tilde{\mathrm{F}}\mathrm{P} \xrightarrow[\simeq]{\tilde{\mathrm{F}}(\pi)} \tilde{\mathrm{F}} \qquad \text{and} \qquad \tilde{\mathrm{F}}\mathrm{I} \xleftarrow[\simeq]{\tilde{\mathrm{F}}(\iota)} \tilde{\mathrm{F}} \xrightarrow[=]{} \tilde{\mathrm{F}}\,,$$

which by 6.3.6 yield natural isomorphisms $\mathbf{L}\mathrm{F} \to \acute{\mathrm{F}}$ and $\mathbf{R}\mathrm{F} \to \acute{\mathrm{F}}$.

6.3.20 Theorem. *Let $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ be a functor. The following assertions about its left derived functor hold.*

(a) *If F is $\Bbbk$-linear, then $\mathbf{L}\mathrm{F}$ is $\Bbbk$-linear.*

(b) *If F preserves coproducts, then $\mathbf{L}\mathrm{F}$ preserves coproducts.*

(c) *If F is triangulated, then $\mathbf{L}\mathrm{F}$ is triangulated.*

Furthermore, the following assertions about its right derived functor hold.

(d) *If F is $\Bbbk$-linear, then $\mathbf{R}\mathrm{F}$ is $\Bbbk$-linear.*

(e) *If F preserves products, then $\mathbf{R}\mathrm{F}$ preserves products.*

(f) *If F is triangulated, then $\mathbf{R}\mathrm{F}$ is triangulated.*

PROOF. By 6.3.8 and 6.3.11 the resolution functors P and I are $\Bbbk$-linear and triangulated. Furthermore, P preserves coproducts and I preserves products. The assertions now follow from 6.3.3. □

6.3.21 Theorem. *Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor that maps homotopy equivalences to homotopy equivalences. The next claims about its left derived functor hold.*

(a) *If F is $\Bbbk$-linear, then $\mathbf{L}\mathrm{F}$ is $\Bbbk$-linear.*

(b) *If* F *preserves coproducts, then* **L**F *preserves coproducts.*

(c) *If* F *is a* Σ*-functor, then* **L**F *is triangulated.*

Furthermore, the following claims about its right derived functor hold.

(d) *If* F *is* $\Bbbk$*-linear, then* **R**F *is* $\Bbbk$*-linear.*

(e) *If* F *preserves products, then* **R**F *preserves products.*

(f) *If* F *is a* Σ*-functor, then* **R**F *is triangulated.*

PROOF. By 6.3.15, the functors **L**F and **R**F are the left derived and right functors, in the sense of 6.3.13, of the functor $\tilde{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ induced by F. The assertions now follow from 6.3.1 and 6.3.20. □

6.3.22. Let $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ be a functor and let $\mathrm{P},\mathrm{I}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ be the resolution functors from 6.3.8 and 6.3.11. As the composite functors $\mathrm{GP}^{\mathrm{op}}$ and $\mathrm{GI}^{\mathrm{op}}$ map quasi-isomorphisms to isomorphisms, it follows from 6.3.4 that there are unique functors **L**G and **R**G that make the following diagrams commutative,

$$\begin{array}{ccc} \mathcal{K}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{V}_R^{\mathrm{op}}} & \mathcal{D}(R)^{\mathrm{op}} \\ {\scriptstyle \mathrm{GI}^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \mathbf{L}\mathrm{G}} \\ \mathcal{K}(S) & \xrightarrow[\mathrm{V}_S]{} & \mathcal{D}(S) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{K}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{V}_R^{\mathrm{op}}} & \mathcal{D}(R)^{\mathrm{op}} \\ {\scriptstyle \mathrm{GP}^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\mathrm{G}} \\ \mathcal{K}(S) & \xrightarrow[\mathrm{V}_S]{} & \mathcal{D}(S) \end{array}.$$

6.3.23 Definition. Let $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ be a functor. Then **L**G from 6.3.22 is called the *left derived functor* of G, and **R**G is called the *right derived functor* of G.

6.3.24. Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor that maps homotopy equivalences to homotopy equivalences. By 6.3.2 it induces a functor $\tilde{\mathrm{G}}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$; the derived functors $\mathbf{L}\tilde{\mathrm{G}}$ and $\mathbf{R}\tilde{\mathrm{G}}$ defined in 6.3.23 are for simplicity denoted **L**G and **R**G.

6.3.25 Definition. Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor that maps homotopy equivalences to homotopy equivalences. The functor **L**G from 6.3.24 is called the *left derived functor* of G, and **R**G is called the *right derived functor* of G.

6.3.26. For any functor $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$, and for a functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ that maps homotopy equivalences to homotopy equivalences, it follows from 6.3.2, 6.3.4, 6.3.22, and 6.3.24 that the derived functors **L**G and **R**G act as follows on objects and morphisms in the derived category.

- For an R-complex M one has $\mathbf{L}\mathrm{G}(M) = \mathrm{GI}(M)$.
- For a morphism α/φ in $\mathcal{D}(R)^{\mathrm{op}}$ one has $\mathbf{L}\mathrm{G}(\alpha/\varphi) = (1/\mathrm{GI}(\varphi))\,(\mathrm{GI}(\alpha)/1)$.

Similarly,

- For an R-complex M one has $\mathbf{R}\mathrm{G}(M) = \mathrm{GP}(M)$.
- For a morphism α/φ in $\mathcal{D}(R)^{\mathrm{op}}$ one has $\mathbf{R}\mathrm{G}(\alpha/\varphi) = (1/\mathrm{GP}(\varphi))\,(\mathrm{GP}(\alpha)/1)$.

6.3.27 Theorem. *Let* $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ *be a functor. The following assertions about its left derived functor hold.*

(a) *If* G *is* $\Bbbk$*-linear, then* **L**G *is* $\Bbbk$*-linear.*
(b) *If* G *preserves coproducts, then* **L**G *preserves coproducts.*
(c) *If* G *is triangulated, then* **L**G *is triangulated.*

Furthermore, the following assertions about its right derived functor hold.

(d) *If* G *is* $\Bbbk$*-linear, then* **R**G *is* $\Bbbk$*-linear.*
(e) *If* G *preserves products, then* **R**G *preserves products.*
(f) *If* G *is triangulated, then* **R**G *is triangulated.*

PROOF. By 6.3.11 and 6.3.8 the opposite resolution functors I^{op} and P^{op} are $\Bbbk$-linear and triangulated. Furthermore, I^{op} preserves coproducts and P^{op} preserves products. The assertions now follow from 6.3.4. □

6.3.28 Theorem. *Let* $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be a functor that maps homotopy equivalences to homotopy equivalences. The next claims about its left derived functor hold.*

(a) *If* G *is* $\Bbbk$*-linear, then* **L**G *is* $\Bbbk$*-linear.*
(b) *If* G *preserves coproducts, then* **L**G *preserves coproducts.*
(c) *If* G *is a* Σ*-functor, then* **L**G *is triangulated.*

Furthermore, the following claims about its right derived functor hold.

(d) *If* G *is* $\Bbbk$*-linear, then* **R**G *is* $\Bbbk$*-linear.*
(e) *If* G *preserves products, then* **R**G *preserves products.*
(f) *If* G *is a* Σ*-functor, then* **R**G *is triangulated.*

PROOF. By 6.3.24, the functors **L**G and **R**G are the left derived and right functors, in the sense of 6.3.22, of the functor $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ induced by G. The assertions now follow from 6.3.2 and 6.3.27. □