Chapter 6
The Derived Category

6.1 The Homotopy Category

SYNOPSIS. Homotopy category; (co)product; triangulation; universal property; unique lifting properties.

Let $\mathcal{U}$ be a category and let $\approx$ be an equivalence relation on each hom-set in $\mathcal{U}$ that is compatible with composition, i.e. a congruence relation. The quotient category $\mathcal{U}/\approx$ has the same objects as $\mathcal{U}$, and for two such objects M, N the hom-set in $\mathcal{U}/\approx$ is the set $\mathcal{U}(M,N)/\approx$ of congruence classes. Evidently, the canonical functor $\mathrm{Q}\colon \mathcal{U} \to \mathcal{U}/\approx$ has the following universal property: If $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is any functor such that $\mathrm{F}(\alpha) = \mathrm{F}(\beta)$ holds for every pair of parallel morphisms in $\mathcal{U}$, then there exists a unique functor F' that makes the following diagram commutative,

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\mathrm{Q}} & \mathcal{U}/\approx \\ {\scriptstyle \mathrm{F}}\downarrow & \swarrow {\scriptstyle \mathrm{F}'} & \\ \mathcal{V}\,. & & \end{array}$$

This section is focused on a special quotient category: the homotopy category $\mathcal{K}(R)$. It is the quotient of $\mathcal{C}(R)$ modulo homotopy equivalence. While $\mathcal{C}(R)$ is Abelian, the category $\mathcal{K}(R)$ is, in general, not. However, the mapping cone construction in $\mathcal{C}(R)$ facilitates a triangulated structure on $\mathcal{K}(R)$.

OBJECTS AND MORPHISMS

6.1.1 Definition. The *homotopy categeory* $\mathcal{K}(R)$ has the same objects as $\mathcal{C}(R)$, that is, R-complexes, and the morphisms in $\mathcal{K}(R)$ are homotopy equivalence classes of morphisms in $\mathcal{C}(R)$.

6.1.2. By 2.3.12 there is an equality $\mathcal{K}(R)(M,N) = \mathrm{H}_0(\mathrm{Hom}_R(M,N))$ of $\Bbbk$-modules for R-complexes M and N. In accordance with 2.2.12 we write $[\alpha]$ for the homotopy equivalence class of a morphism α in $\mathcal{C}(R)$. If L is also an R-complex, then the composition $\mathcal{K}(R)(M,N) \times \mathcal{K}(R)(L,M) \to \mathcal{K}(R)(L,N)$ maps $([\alpha],[\beta])$ to $[\alpha\beta]$; it follows from 2.3.4 that $[\alpha\beta]$ does not depend on the choice of representatives for $[\alpha]$ and $[\beta]$.

6.1.3. The homotopy equivalence class $[\alpha]$ of a morphism in $\mathcal{C}(R)$ is an isomorphism in $\mathcal{K}(R)$ if and only if α is a homotopy equivalence; see 2.2.25.

6.1.4. The zero complex is evidently a zero object in the homotopy category. It follows that a zero morphism in $\mathcal{K}(R)$ is the homotopy equivalence class $[0]$ of a zero morphism in $\mathcal{C}(R)$. Thus, the class $[\alpha]$ of a morphism in $\mathcal{C}(R)$ is the zero morphism in $\mathcal{K}(R)$ if and only if α is null-homotopic; see 2.2.20.

The next result shows that the zero objects in the $\mathcal{K}(R)$ are exactly the contractible complexes. Further characterizations of such complexes are given in 4.1.10.

6.1.5 Proposition. *An R-complex is isomorphic to 0 in $\mathcal{K}(R)$ if and only if it is contractible.*

PROOF. Let M be an R-complex. If M is isomorphic to 0 in $\mathcal{K}(R)$ then $\mathcal{K}(R)(M,M)$ consists of a single element. In particular, $[1^M] = [0]$ holds, so 1^M is null-homotopic, i.e. M is contractible. Conversely, if M is contractible then the morphism $M \to 0$ in $\mathcal{C}(R)$ is a homotopy equivalence, whence it represents an isomorphism in $\mathcal{K}(R)$. □

6.1.6. There is a canonical full functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$; it is the identity on objects and it maps a morphism α in $\mathcal{C}(R)$ to its homotopy equivalence class $\mathrm{Q}(\alpha) = [\alpha]$.

PRODUCTS, COPRODUCTS, AND $\Bbbk$-LINEARITY

The lemma below follows immediately from the definitions.

6.1.7 Lemma. *Let $\mathcal{U}$ and $\mathcal{V}$ be $\Bbbk$-prelinear categories that have the same objects, and let $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ be a $\Bbbk$-linear functor that is the identity on objects. If M and N are objects and if the tuple $(M \oplus N, \pi^M, \iota^M, \pi^N, \iota^N)$ is a biproduct in $\mathcal{U}$ then the tuple $(M \oplus N, \mathrm{F}(\pi^M), \mathrm{F}(\iota^M), \mathrm{F}(\pi^N), \mathrm{F}(\iota^N))$ is a biproduct in $\mathcal{V}$. In particular, if every pair of objects has a biproduct in $\mathcal{U}$ then every pair of objects has a biproduct in $\mathcal{V}$.* □

Recall that a category is said to have (co)products if all set-indexed (co)products exist in the category.

6.1.8 Theorem. *The homotopy category $\mathcal{K}(R)$ and the functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ are $\Bbbk$-linear. For every family $\{M^u\}_{u\in U}$ of R-complexes the assertions hold.*

(a) *If M with embeddings $\{\iota^u\colon M^u \rightarrowtail M\}_{u\in U}$ is the coproduct of $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)$, then M with the morphisms $\{[\iota^u]\}_{u\in U}$ is the coproduct of $\{M^u\}_{u\in U}$ in $\mathcal{K}(R)$.*

(b) *If M with projections $\{\pi^u\colon M^u \twoheadrightarrow M\}_{u\in U}$ is the product of $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)$, then M with the morphisms $\{[\pi^u]\}_{u\in U}$ is the product of $\{M^u\}_{u\in U}$ in $\mathcal{K}(R)$.*

In particular, the homotopy category $\mathcal{K}(R)$ has coproducts and products, and the canonical functor Q *preserves coproducts and products.*

PROOF. It is straightforward to verify that the category $\mathcal{K}(R)$ is $\Bbbk$-prelinear and that the canonical functor Q is $\Bbbk$-linear. The zero complex is a zero object in $\mathcal{K}(R)$, and $\mathcal{K}(R)$ has biproducts by 6.1.7. Thus $\mathcal{K}(R)$ is a $\Bbbk$-linear category.

(a): Let $\{[\alpha^u]\colon M^u \to N\}_{u\in U}$ be morphisms in $\mathcal{K}(R)$. The task is to show that there exixts a unique morphism $[\alpha]\colon M \to N$ in $\mathcal{K}(R)$ with $[\alpha\iota^u] = [\alpha^u]$ for all $u \in U$.

Existence is straightforward; indeed, by the universal property of coproducts in $\mathcal{C}(R)$, there exists a (unique) morphism $\alpha\colon M \to N$ with $\alpha\iota^u = \alpha^u$ for all $u \in U$. Applying Q to these identities one gets $[\alpha\iota^u] = [\alpha^u]$.

For uniqueness, assume that $[\alpha\iota^u] = [0]$ holds for all $u \in U$; it must be shown that $[\alpha]$ is $[0]$. Since each $\alpha\iota^u$ is null-homotopic there are degree 1 homomorphisms $\tau^u\colon M^u \to N$ such that $\alpha\iota^u = \partial^N\tau^u + \tau^u\partial^{M^u}$ holds for all $u \in U$. Now consider each homomorphism τ^u as a morphism $M^{u\natural} \to \Sigma N^\natural$ of graded R-modules. Since $M^\natural$ together with the embeddings $\{\iota^u\colon M^{u\natural} \rightarrowtail M^\natural\}_{u\in U}$ is a coproduct of $\{M^{u\natural}\}_{u\in U}$ in $\mathcal{M}_{\mathrm{gr}}(R)$, there is a morphism $\tau\colon M^\natural \to \Sigma N^\natural$ with $\tau\iota^u = \tau^u$ for all $u \in U$. Viewing τ as a degree 1 homomorphism $M \to N$, it follows that one has

$$\alpha\iota^u = \partial^N\tau^u + \tau^u\partial^{M^u} = \partial^N\tau\iota^u + \tau\iota^u\partial^{M^u} = (\partial^N\tau + \tau\partial^M)\iota^u \ ,$$

where the second equality is by definition of τ, and the third equality holds as ι^u is a morphism in $\mathcal{C}(R)$. As $\partial^N\tau + \tau\partial^M$ is a morphism of R-complexes, it follows from the universal property of coproducts in $\mathcal{C}(R)$ that one has $\alpha = \partial^N\tau + \tau\partial^M$. Thus α is null-homotopic, that is, $[\alpha]$ is $[0]$ as desired.

(b): Similar to the proof of part (a).

By construction, the canonical functor Q preserves (co)products. □

REMARK. Let $\mathcal{U}$ be a category that has products and coproducts. For a family $\{M^u\}_{u\in U}$ of objects in $\mathcal{U}$, the canonical morphisms $\prod_{u\in U} M^u \to M^u$ and $M^u \to \coprod_{u\in U} M^u$ are referred to as *projections* and *embeddings*; MacLane [35] uses the term *injections* for the latter. Embeddings need not be monomorphisms and projections need not be epimorphisms. Our use of the term embedding for inclusions of subobjects is thus a slight abuse of terminology.

6.1.9. Let $\{[\alpha^u]\colon M^u \to N^u\}_{u\in U}$ and $\{[\beta^u]\colon N^u \to M^u\}_{u\in U}$ be families of morphisms in $\mathcal{K}(R)$. By the universal properties of (co)products there are unique morphisms $[\alpha]$ and $[\beta]$ that make the following diagrams commutative for every $u \in U$,

$$\begin{array}{ccc} M^u & \longrightarrow & \coprod_{u\in U} M^u \\ \downarrow{\scriptstyle [\alpha^u]} & & \downarrow{\scriptstyle [\alpha]} \\ N^u & \longrightarrow & \coprod_{u\in U} N^u \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \prod_{u\in U} N^u & \xrightarrow{[\beta]} & \prod_{u\in U} M^u \\ \downarrow & & \downarrow \\ N^u & \xrightarrow{[\beta^u]} & M^u \ . \end{array}$$

The horizontal morphisms in the left-hand diagram are embeddings and the vertical morphisms in the right-hand diagram are projections. The morphism $[\alpha]$ is called the *coproduct* in $\mathcal{K}(R)$ of $\{[\alpha^u]\}_{u\in U}$ and denoted $\coprod_{u\in U}[\alpha^u]$. Similarly, $[\beta]$ is called the *product* in $\mathcal{K}(R)$ of $\{[\beta^u]\}_{u\in U}$ and denoted $\prod_{u\in U}[\beta^u]$. From the construction of (co)products in $\mathcal{K}(R)$ given in 6.1.8, it follows that one has

$$\coprod_{u\in U}[\alpha^u] = \Big[\coprod_{u\in U}\alpha^u\Big] \qquad \text{and} \qquad \prod_{u\in U}[\beta^u] = \Big[\prod_{u\in U}\beta^u\Big]\,,$$

where $\coprod_{u\in U}\alpha^u$ and $\prod_{u\in U}\beta^u$ are the coproduct and the product of $\{\alpha^u\}_{u\in U}$ and $\{\beta^u\}_{u\in U}$ in $\mathcal{C}(R)$; see 3.1.5 and 3.1.18.

6.1.10 Definition. A morphism $[\alpha]$ in $\mathcal{K}(R)$ is called a *quasi-isomorphism* if some, equivalently every, morphism in $\mathcal{C}(R)$ that represents the homotopy equivalence class $[\alpha]$ is a quasi-isomorphism.

We apply the terminology from Definitions 5.1.6, 5.2.12, and 5.3.12 to quasi-isomorphisms in the homotopy category. That is, a quasi-isomorphism $X \xrightarrow{\simeq} M$ in $\mathcal{K}(R)$, where X is a semi-free/-projective complex is called a semi-free/-projective resolution of M; similarly a quasi-isomorphism $X \xrightarrow{\simeq} Y$, where Y is semi-injective, is called a semi-injective resolution of M.

The next result is immediate from 4.2.7 and 6.1.9.

6.1.11 Proposition. *Let $\{[\alpha^u]\colon M^u \to N^u\}_{u\in U}$ be a family of morphisms in $\mathcal{K}(R)$. If $[\alpha^u]$ is a quasi-isomorphism for every $u\in U$, then the coproduct $\coprod_{u\in U}[\alpha^u]$ and the product $\prod_{u\in U}[\alpha^u]$ are quasi-isomorphisms.* □

6.1.12 Proposition. *There is a unique endofunctor on $\mathcal{K}(R)$ that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R) & \xrightarrow{\ \mathrm{Q}\ } & \mathcal{K}(R) \\ {\scriptstyle\Sigma}\downarrow & & \vdots \\ \mathcal{C}(R) & \xrightarrow[\ \mathrm{Q}\ ]{} & \mathcal{K}(R)\,. \end{array}$$

This functor is denoted $\Sigma_{\mathcal{K}}$; it is $\Bbbk$-linear and an isomorphism. For a morphism $[\alpha]$ in $\mathcal{K}(R)$ one has $\Sigma_{\mathcal{K}}([\alpha]) = [\Sigma\alpha]$.

PROOF. It is elementary to verify that the endofunctor on $\mathcal{K}(R)$ that maps an object M to ΣM and a morphism $[\alpha]$ to $[\Sigma\alpha]$ has the asserted properties. □

When there is no risk of ambiguity, we write Σ for the functor $\Sigma_{\mathcal{K}}$.

TRIANGULATION

Consider the $\Bbbk$-linear category $\mathcal{K}(R)$, see 6.1.8, equipped with the $\Bbbk$-linear autofunctor $\Sigma = \Sigma_{\mathcal{K}}$ from 6.1.12. One may now speak of candidate triangles in $\mathcal{K}(R)$ in the sense of A.1.

6.1.13 Lemma. *Let* $\alpha\colon M \to N$ *be a morphism in* $\mathcal{C}(R)$. *The image under the canonical functor* $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ *of the diagram*

$$M \xrightarrow{\ \alpha\ } N \xrightarrow{\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M$$

is a candidate triangle in $\mathcal{K}(R)$.

PROOF. We must prove that the three composites in $\mathcal{C}(R)$,

$$\varphi = \begin{pmatrix}1^N\\ 0\end{pmatrix}\alpha = \begin{pmatrix}\alpha\\ 0\end{pmatrix}, \ \ \psi = \begin{pmatrix}0 & 1^{\Sigma M}\end{pmatrix}\begin{pmatrix}1^N\\ 0\end{pmatrix} = 0, \ \text{ and } \ \chi = (\Sigma\alpha)\begin{pmatrix}0 & 1^{\Sigma M}\end{pmatrix} = \begin{pmatrix}0 & \Sigma\alpha\end{pmatrix}$$

are null-homotopic. Since ψ is even zero in $\mathcal{C}(R)$, we are left to consider φ and χ. Define degree 1 homomorphisms $\varrho\colon M \to \operatorname{Cone}\alpha$ and $\tau\colon \operatorname{Cone}\alpha \to \Sigma N$ by

$$\varrho = \begin{pmatrix}0\\ \varsigma_1^M\end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix}\varsigma_1^N & 0\end{pmatrix},$$

where $\varsigma_s^M\colon M \to \Sigma^s M$ is the map introduced in 2.2.3. From the fact that ς_s^M is a degree s chain map and from commutativity of the diagram (2.2.3.1), it follows that there are equalities $\partial^{\operatorname{Cone}\alpha}\varrho + \varrho\partial^M = \varphi$ and $\partial^{\Sigma N}\tau + \tau\partial^{\operatorname{Cone}\alpha} = \chi$. Indeed, one has

$$\begin{pmatrix}\partial^N & \alpha\varsigma_{-1}^{\Sigma M}\\ 0 & \partial^{\Sigma M}\end{pmatrix}\begin{pmatrix}0\\ \varsigma_1^M\end{pmatrix} + \begin{pmatrix}0\\ \varsigma_1^M\end{pmatrix}\partial^M = \begin{pmatrix}\alpha\\ 0\end{pmatrix}$$

and

$$\partial^{\Sigma N}\begin{pmatrix}\varsigma_1^N & 0\end{pmatrix} + \begin{pmatrix}\varsigma_1^N & 0\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma_{-1}^{\Sigma M}\\ 0 & \partial^{\Sigma M}\end{pmatrix} = \begin{pmatrix}0 & \Sigma\alpha\end{pmatrix}. \qquad \square$$

6.1.14 Definition. A candidate triangle in $\mathcal{K}(R)$ of the form considered in 6.1.13 is called a *strict triangle*. A candidate triangle in $\mathcal{K}(R)$ that is isomorphic, in the sense of A.1, to a strict triangle is called a *distinguished triangle*.

Triangulated categories are defined in A.3.

6.1.15 Theorem. *The homotopy category* $\mathcal{K}(R)$, *equipped with the autofunctor* Σ *and the collection of distinguished triangles defined in* 6.1.14, *is triangulated.*

PROOF. We verify the axioms in A.3.

(TR0): Evidently, the collection of distinguished triangles is closed under isomorphisms. Furthermore, it follows from 4.1.10 and 6.1.5 that application of the canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ to the following diagram in $\mathcal{C}(R)$,

$$M \xrightarrow{\ 1^M\ } M \xrightarrow{\left(\begin{smallmatrix}1^M\\ 0\end{smallmatrix}\right)} \operatorname{Cone}(1^M) \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M,$$

yields, up to isomorphism in $\mathcal{K}(R)$, the candidate triangle $M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma M$ which, therefore, is distinguished.

(TR1): By the definition of morphisms in $\mathcal{K}(R)$, every morphism in this category fits into a distinguished (even a strict) triangle; see. 6.1.13.

(TR2'): By A.4 it it sufficient to verify that (TR2) holds. Thus, let

$$\varDelta = M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} \Sigma M'$$

be a distinguished triangle in $\mathcal{K}(R)$. We must argue that the candidate triangles

$$\varDelta' = N' \xrightarrow{\beta'} X \xrightarrow{\gamma'} \Sigma M' \xrightarrow{-\Sigma\alpha'} \Sigma N' \quad \text{and} \quad \varDelta'' = \Sigma^{-1} X' \xrightarrow{-\Sigma^{-1}\gamma'} M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X'$$

are distinguished. Up to isomorphism, $\varDelta$ is given by application of the canonical functor Q to a diagram in $\mathcal{C}(R)$ of the form,

$$M \xrightarrow{\quad\alpha\quad} N \xrightarrow{\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M\ .$$

Thus, the candidate triangles $\varDelta'$ and $\varDelta''$ are, up to isomorphism, given by application of Q to the following diagrams in $\mathcal{C}(R)$,

$$N \xrightarrow{\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M \xrightarrow{\quad -\Sigma\alpha\quad} \Sigma N\ ,$$

and

$$\Sigma^{-1}\operatorname{Cone}\alpha \xrightarrow{(0\ -1^M)} M \xrightarrow{\quad\alpha\quad} N \xrightarrow{\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)} \operatorname{Cone}\alpha\ .$$

These two diagrams in $\mathcal{C}(R)$ are the top rows in $(\star)$ and $(\ddagger)$ below. By definition, the bottom rows in $(\star)$ and $(\ddagger)$ give strict triangles in $\mathcal{K}(R)$ when the functor Q is applied; see 6.1.14. Thus, to show that $\varDelta'$ and $\varDelta''$ are distinguished triangles in $\mathcal{K}(R)$, it suffices to argue that $(\star)$ and $(\ddagger)$ are commutative up to homotopy, and that the vertical morphisms in both diagrams are homotopy equivalences.

$$(\star)\qquad \begin{array}{ccccccc}
N & \xrightarrow{\iota=\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)} & \operatorname{Cone}\alpha & \xrightarrow{(0\ 1^{\Sigma M})} & \Sigma M & \xrightarrow{\quad -\Sigma\alpha\quad} & \Sigma N \\
\| & & \| & \vartheta=(0\ 1^{\Sigma M}\ 0) & \Big\uparrow\Big\downarrow & \varphi=\left(\begin{smallmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{smallmatrix}\right) & \| \\
N & \xrightarrow[\iota=\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)]{} & \operatorname{Cone}\alpha & \xrightarrow[\left(\begin{smallmatrix}1^N & 0\\ 0 & 1^{\Sigma M}\\ 0 & 0\end{smallmatrix}\right)]{} & \operatorname{Cone}\iota & \xrightarrow[(0\ 0\ 1^{\Sigma N})]{} & \Sigma N
\end{array}$$

$$
(\ddagger)\qquad
\begin{array}{ccccccc}
\Sigma^{-1}\operatorname{Cone}\alpha & \xrightarrow{\pi=(0\ -1^M)} & M & \xrightarrow{\ \alpha\ } & N & \xrightarrow{\begin{pmatrix}1^N\\0\end{pmatrix}} & \operatorname{Cone}\alpha \\
\Vert & & \Vert & \xi=(\alpha\ 1^N\ 0)\ \Big\uparrow\ \ \Big\downarrow\ \psi=\begin{pmatrix}0\\1^N\\0\end{pmatrix} & & & \Vert \\
\Sigma^{-1}\operatorname{Cone}\alpha & \xrightarrow[\pi=(0\ -1^M)]{} & M & \xrightarrow[\begin{pmatrix}1^M\\0\\0\end{pmatrix}]{} & \operatorname{Cone}\pi & \xrightarrow[\begin{pmatrix}0&1^N&0\\0&0&1^{\Sigma M}\end{pmatrix}]{} & \operatorname{Cone}\alpha
\end{array}
$$

First consider the diagram $(\star)$. Note that φ and ϑ are morphisms, as one has

$$
\partial^{\operatorname{Cone}\iota}\varphi = \begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix}\begin{pmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{pmatrix} = \begin{pmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{pmatrix}\partial^{\Sigma M} = \varphi\partial^{\Sigma M}
$$

and

$$
\partial^{\Sigma M}\vartheta = \begin{pmatrix}0 & \partial^{\Sigma M} & 0\end{pmatrix} = \begin{pmatrix}0 & 1^{\Sigma M} & 0\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix} = \vartheta\partial^{\operatorname{Cone}\iota}\,.
$$

Next we argue that φ is a homotopy equivalence with homotopy inverse ϑ. Evidently one has $\vartheta\varphi = 1^{\Sigma M}$, so it remains to show that the morphism

$$
1^{\operatorname{Cone}\iota} - \varphi\vartheta = \begin{pmatrix}1^N & 0 & 0\\ 0 & 1^{\Sigma M} & 0\\ 0 & 0 & 1^{\Sigma N}\end{pmatrix} - \begin{pmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{pmatrix}\begin{pmatrix}0 & 1^{\Sigma M} & 0\end{pmatrix} = \begin{pmatrix}1^N & 0 & 0\\ 0 & 0 & 0\\ 0 & \Sigma\alpha & 1^{\Sigma N}\end{pmatrix}
$$

is null-homotopic. The degree 1 homomorphism $\sigma\colon \operatorname{Cone}\iota \to \operatorname{Cone}\iota$ given by

$$
\sigma = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \varsigma^N_1 & 0 & 0\end{pmatrix}
$$

is the desired homotopy. Indeed, one has

$$
\begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \varsigma^N_1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \varsigma^N_1 & 0 & 0\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma N}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix} = \begin{pmatrix}1^N & 0 & 0\\ 0 & 0 & 0\\ 0 & \Sigma\alpha & 1^{\Sigma N}\end{pmatrix};
$$

that is, $\partial^{\operatorname{Cone}\iota}\sigma + \sigma\partial^{\operatorname{Cone}\iota} = 1^{\operatorname{Cone}\iota} - \varphi\vartheta$ holds. Thus ϑ is a homotopy inverse of φ.

Now we turn to the issue of commutativity of $(\star)$. The left- and right-hand squares in $(\star)$ are even commutative in $\mathcal{C}(R)$. For the commutativity, up to homotopy, of the middle square, it must be proved that the morphism $\beta\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\iota$, given by

$$
\beta = \begin{pmatrix}1^N & 0\\ 0 & 1^{\Sigma M}\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{pmatrix}\begin{pmatrix}0 & 1^{\Sigma M}\end{pmatrix} = \begin{pmatrix}1^N & 0\\ 0 & 0\\ 0 & \Sigma\alpha\end{pmatrix},
$$

is null-homotopic. Let $\tau\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\iota$ be the degree one homomorphism

$$\tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varsigma_1^N & 0 \end{pmatrix}.$$

It is straightforward to verify the equality

$$\begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} & \varsigma_{-1}^{\Sigma N} \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & 0 & \partial^{\Sigma N} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varsigma_1^N & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varsigma_1^N & 0 \end{pmatrix} \begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \begin{pmatrix} 1^N & 0 \\ 0 & 0 \\ 0 & \Sigma\alpha \end{pmatrix};$$

that is, $\partial^{\operatorname{Cone}\iota}\tau + \tau\partial^{\operatorname{Cone}\alpha} = \beta$ holds, and hence β is null-homotopic.

Similar arguemnts show that the second diagram (‡) is commutative up to homotopy, in particular, that ψ is a homotopy equivalence with homotopy inverse ξ.

(TR4'): Consider the following diagram in $\mathcal{C}(R)$, where the rows and the morphisms φ and ψ are given, and the left-hand square is commutative up to homotopy,

$$(\diamond) \qquad \begin{array}{ccccccc} M & \xrightarrow{\alpha} & N & \xrightarrow{\left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)} & \operatorname{Cone}\alpha & \xrightarrow{(0\ 1^{\Sigma M})} & \Sigma M \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \vdots\,{\scriptstyle \chi = \left(\begin{smallmatrix} \chi^{11} & \chi^{12} \\ \chi^{21} & \chi^{22} \end{smallmatrix}\right)} & & \downarrow{\scriptstyle\Sigma\varphi} \\ M' & \xrightarrow[\alpha']{} & N' & \xrightarrow[\left(\begin{smallmatrix} 1^{N'} \\ 0 \end{smallmatrix}\right)]{} & \operatorname{Cone}\alpha' & \xrightarrow[(0\ 1^{\Sigma M'})]{} & \Sigma M'. \end{array}$$

To verify that (TR4') holds we are, in view of the definition of distinguished triangles in $\mathcal{K}(R)$, required to prove that there exists a morphism $\chi\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\alpha'$ with the following two properties: First of all, χ must make $(\diamond)$ commutative up to homotopy. In this case, $(\mathrm{Q}(\varphi), \mathrm{Q}(\psi), \mathrm{Q}(\chi))$ is a morphism of distinguished triangles in $\mathcal{K}(R)$, and its mapping cone candidate triangle is given by application of the functor Q to the following diagram in $\mathcal{C}(R)$,

$$(\S) \qquad \begin{array}{c} M' \\ \oplus \\ N \end{array} \xrightarrow{\begin{pmatrix} \alpha' & \psi \\ 0 & -1^N \\ 0 & 0 \end{pmatrix}} \begin{array}{c} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{array} \xrightarrow{\begin{pmatrix} 1^{N'} & \chi^{11} & \chi^{12} \\ 0 & \chi^{21} & \chi^{22} \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}} \begin{array}{c} \operatorname{Cone}\alpha' \\ \oplus \\ \Sigma M \end{array} \xrightarrow{\begin{pmatrix} 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix}} \begin{array}{c} \Sigma M' \\ \oplus \\ \Sigma N \end{array}.$$

Secondly, Q applied to (§) must yield a distinguished triangle in $\mathcal{K}(R)$.

We start by constructing a morphism χ that makes $(\diamond)$ commutative up to homotopy. By assumption, there is a degree 1 homomorphism $\sigma\colon M \to N'$ such that the equality $\psi\alpha - \alpha'\varphi = \partial^{N'}\sigma + \sigma\partial^M$ holds. Define $\chi\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\alpha'$ as follows:

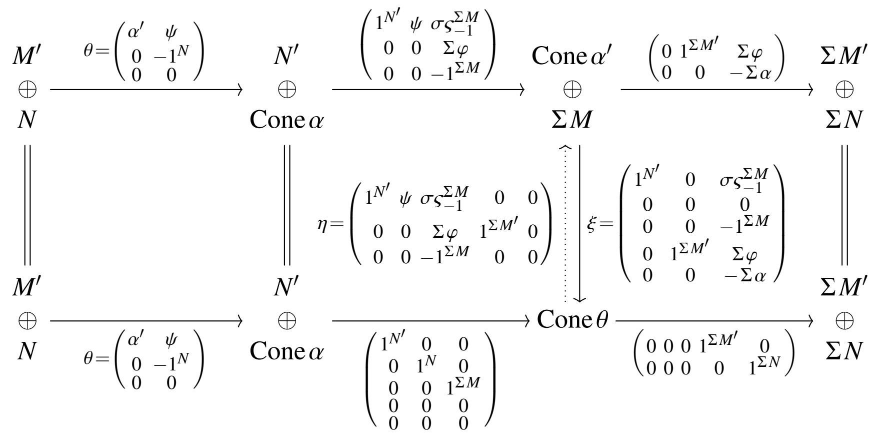
$$\chi = \begin{pmatrix} \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & \Sigma\varphi \end{pmatrix}.$$

It is straightforward verify that it is a morphism; that is, one has

$$\partial^{\operatorname{Cone}\alpha'}\chi = \begin{pmatrix} \partial^{N'} & \alpha'\varsigma_{-1}^{\Sigma M'} \\ 0 & \partial^{\Sigma M'} \end{pmatrix}\begin{pmatrix} \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & \Sigma\varphi \end{pmatrix} = \begin{pmatrix} \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & \Sigma\varphi \end{pmatrix}\begin{pmatrix} \partial^{N} & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \chi\partial^{\operatorname{Cone}\alpha}\,.$$

Notice that χ makes the middle and right-hand squares in $(\diamond)$ commutative in $\mathcal{C}(R)$.

Finally, to see that application of the functor Q to (§) yields a distinguished triangle in $\mathcal{K}(R)$, note that (§) is the top row in following diagram, and that the bottom row yields a strict triangle in $\mathcal{K}(R)$ when Q is applied. Thus, it suffices to argue that the diagram below is commutative up to homotopy, and that the vertical morphisms are homotopy equivalences.



The differentials on the complexes $\operatorname{Cone}\alpha' \oplus \Sigma M$ and $\operatorname{Cone}\theta$ are given by

$$\begin{pmatrix} \partial^{N'} & \alpha'\varsigma_{-1}^{\Sigma M'} & 0 \\ 0 & \partial^{\Sigma M'} & 0 \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \partial^{N'} & 0 & 0 & \alpha'\varsigma_{-1}^{\Sigma M'} & \psi\varsigma_{-1}^{\Sigma N} \\ 0 & \partial^{N} & \alpha\varsigma_{-1}^{\Sigma M} & 0 & -\varsigma_{-1}^{\Sigma N} \\ 0 & 0 & \partial^{\Sigma M} & 0 & 0 \\ 0 & 0 & 0 & \partial^{\Sigma M'} & 0 \\ 0 & 0 & 0 & 0 & \partial^{\Sigma N} \end{pmatrix}.$$

It is straightforward to verify that ξ and η are morphisms. Evidently there is an equality $\eta\xi = 1^{\operatorname{Cone}\alpha'\oplus\Sigma M}$. Furthermore, the morphism $\xi\eta - 1^{\operatorname{Cone}\theta}$ is null-homotopic, as the degree 1 homomorphism $\tau\colon \operatorname{Cone}\theta \to \operatorname{Cone}\theta$ given by

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varsigma_1^N & 0 & 0 & 0 \end{pmatrix}$$

satisfies the identity $\partial^{\operatorname{Cone}\theta}\tau + \tau\partial^{\operatorname{Cone}\theta} = \xi\eta - 1^{\operatorname{Cone}\theta}$. Hence ξ is a homotopy equivalence with homotopy inverse η.

The left-hand and right-hand squares in the diagram are commutative in $\mathcal{C}(R)$. The diagram's middle square is commutative up to homotopy; indeed, the difference morphism $\gamma\colon N' \oplus \operatorname{Cone}\alpha \to \operatorname{Cone}\theta$, given by

$$\gamma = \begin{pmatrix} 1^{N'} & 0 & \sigma\varsigma^{\Sigma M}_{-1} \\ 0 & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \\ 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix} \begin{pmatrix} 1^{N'} & \psi & \sigma\varsigma^{\Sigma M}_{-1} \\ 0 & 0 & \Sigma\varphi \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix} - \begin{pmatrix} 1^{N'} & 0 & 0 \\ 0 & 1^{N} & 0 \\ 0 & 0 & 1^{\Sigma M} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \psi & 0 \\ 0 & -1^{N} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma\alpha \end{pmatrix},$$

is null-homotopic. This follows as $\varrho\colon N' \oplus \operatorname{Cone}\alpha \to \operatorname{Cone}\theta$, given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varsigma^{N}_{1} & 0 \end{pmatrix},$$

is a degree 1 homomorphism with $\partial^{\operatorname{Cone}\theta}\varrho + \varrho\partial^{N' \oplus \operatorname{Cone}\alpha} = \gamma$. □

THE UNIVERSAL PROPERTY

6.1.16 Definition. Let $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be a triangulated category. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ is called *quasi-triangulated* if there is a natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$ such that

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}\left(\begin{smallmatrix} 1^{N} \\ 0 \end{smallmatrix}\right)} \mathrm{F}(\operatorname{Cone}\alpha) \xrightarrow{\phi^{M} \circ \mathrm{F}(0\ \ 1^{\Sigma M})} \Sigma_{\mathcal{U}}\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{U}$ for every morphism of R-complexes $\alpha\colon M \to N$.

6.1.17 Example. The canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ is quasi-triangulated.

The quasi-triangulated functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ has a universal property described in the next theorem.

6.1.18 Theorem. *Let $\mathcal{U}$ be a category and let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ be a functor. If F maps homotopy equivalences to isomorphisms, then there exists a unique functor F' that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R) & \xrightarrow{\mathrm{Q}} & \mathcal{K}(R) \\ {\scriptstyle \mathrm{F}}\big\downarrow & \swarrow{\scriptstyle \mathrm{F}'} & \\ \mathcal{U}\,; & & \end{array}$$

here Q *is the canonical functor from* 6.1.6*. For an* R*-complex* M *there is an equality* $\mathrm{F}'(M) = \mathrm{F}(M)$*, and for a morphism* $[\alpha]$ *in* $\mathcal{K}(R)$ *one has* $\mathrm{F}'([\alpha]) = \mathrm{F}(\alpha)$*. Furthermore, the following assertions hold.*

(a) *Assume that* $\mathcal{U}$ *is* $\Bbbk$*-prelinear; then* F' *is* $\Bbbk$*-linear if and only if* F *is* $\Bbbk$*-linear.*

(b) *Assume that* $\mathcal{U}$ *has (co)products; then* F' *preserves (co)products if and only if* F *preserves (co)products.*

(c) *If* $\mathcal{U}$ *is triangulated and* F *is quasi-triangulated, then* F' *is triangulated.*

PROOF. Uniqueness of F' follows as Q is the identity on objects and full.

For existence of F', set $\mathrm{F}'(M) = \mathrm{F}(M)$ for every R-complex M and $\mathrm{F}'([\alpha]) = \mathrm{F}(\alpha)$ for every morphism α of R-complexes. To see that this makes sense—in which case the the identity $\mathrm{F}'\mathrm{Q} = \mathrm{F}$ evidently holds—let $\alpha,\beta\colon M \to N$ be homotopic morphisms of R-complexes. It must be shown that one has $\mathrm{F}(\alpha) = \mathrm{F}(\beta)$. To this end, define an R-complex C as follows,

$$C^\natural = \begin{matrix} M^\natural \\ \oplus \\ M^\natural \\ \oplus \\ \Sigma M^\natural \end{matrix} \qquad \text{and} \qquad \partial^C = \begin{pmatrix} \partial^M & 0 & -\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^M & \varsigma_{-1}^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix}.$$

The following three maps are morphisms in $\mathcal{C}(R)$,

$$M \underset{\iota = \begin{pmatrix} 0 \\ 1^M \\ 0 \end{pmatrix}}{\overset{\varepsilon = \begin{pmatrix} 1^M \\ 0 \\ 0 \end{pmatrix}}{\rightrightarrows}} C \xrightarrow{\pi = (1^M \;\; 1^M \;\; 0)} M.$$

Moreover, ι is a homotopy equivalence with homotopy inverse π, indeed, the equality $\pi\iota = 1^M$ evidently holds. To prove that the morphism

$$\iota\pi - 1^C = \begin{pmatrix} 0 \\ 1^M \\ 0 \end{pmatrix} \begin{pmatrix} 1^M & 1^M & 0 \end{pmatrix} - \begin{pmatrix} 1^M & 0 & 0 \\ 0 & 1^M & 0 \\ 0 & 0 & 1^{\Sigma M} \end{pmatrix} = \begin{pmatrix} -1^M & 0 & 0 \\ 1^M & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}$$

is null-homotopic, consider the degree 1 homomorphism $\sigma\colon C \to C$ given by

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varsigma_1^M & 0 & 0 \end{pmatrix}.$$

One readily verifies the equality $\partial^C\sigma + \sigma\partial^C = \iota\pi - 1^C$, that is,

$$\begin{pmatrix} \partial^M & 0 & -\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^M & \varsigma_{-1}^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varsigma_1^M & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varsigma_1^M & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial^M & 0 & -\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^M & \varsigma_{-1}^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} = \begin{pmatrix} -1^M & 0 & 0 \\ 1^M & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}.$$

These arguments show that ι is a homotopy equivalence with homotopy inverse π. It follows that $\mathrm{F}(\iota)$ is an isomorphism in $\mathcal{U}$ with inverse $\mathrm{F}(\pi)$. As $\pi\varepsilon = 1^M$, and hence $\mathrm{F}(\pi)\mathrm{F}(\varepsilon) = 1^{\mathrm{F}(M)}$, holds it follows that one has $\mathrm{F}(\varepsilon) = \mathrm{F}(\iota)$. Since the morphisms $\alpha,\beta\colon M \to N$ are homotopic, there exists a degree 1 homomorphism $\varrho\colon M \to N$ with $\beta - \alpha = \partial^N\varrho + \varrho\partial^M$. The degree 0 homomorphism $\gamma = \begin{pmatrix}\alpha & \beta & \varrho\varsigma^{\Sigma M}_{-1}\end{pmatrix} : C \to N$ is a morphism, as one has

$$\partial^N\gamma = \partial^N\begin{pmatrix}\alpha & \beta & \varrho\varsigma^{\Sigma M}_{-1}\end{pmatrix} = \begin{pmatrix}\alpha & \beta & \varrho\varsigma^{\Sigma M}_{-1}\end{pmatrix}\begin{pmatrix}\partial^M & 0 & -\varsigma^{\Sigma M}_{-1}\\ 0 & \partial^M & \varsigma^{\Sigma M}_{-1}\\ 0 & 0 & \partial^{\Sigma M}\end{pmatrix} = \gamma\partial^C\,.$$

From the equalities $\alpha = \gamma\varepsilon$ and $\gamma\iota = \beta$ one gets $\mathrm{F}(\alpha) = \mathrm{F}(\gamma)\mathrm{F}(\varepsilon) = \mathrm{F}(\gamma)\mathrm{F}(\iota) = \mathrm{F}(\beta)$.

It remains to prove the assertions (a), (b), and (c).

(a): If F' is $\Bbbk$-linear then so is $\mathrm{F} = \mathrm{F}'\mathrm{Q}$, as a composite of two $\Bbbk$-linear functors. Conversely, if F is $\Bbbk$-linear then so is F', since the equalities

$$\mathrm{F}'(x[\alpha]+[\beta]) = \mathrm{F}'([x\alpha+\beta]) = \mathrm{F}(x\alpha+\beta) = x\mathrm{F}(\alpha)+\mathrm{F}(\beta) = x\mathrm{F}'([\alpha])+\mathrm{F}'([\beta])$$

hold for every pair α,β of parallel morphisms in $\mathcal{C}(R)$ and every element x in $\Bbbk$.

(b): Let $\{M^u\}_{u\in U}$ be a family of R-complexes. Since the functor Q preserves coproducts; see 6.1.8, the canonical morphism

$$\coprod_{u\in U} M^u = \coprod_{u\in U}\mathrm{Q}(M^u) \xrightarrow{\;\psi\;} \mathrm{Q}(\coprod_{u\in U} M^u)$$

in $\mathcal{K}(R)$ is an isomorphism; cf. 3.1.9. Application of F' yields an isomorphism

$$\mathrm{F}'(\coprod_{u\in U} M^u) \xrightarrow{\mathrm{F}'(\psi)} \mathrm{F}'\mathrm{Q}(\coprod_{u\in U} M^u) = \mathrm{F}(\coprod_{u\in U} M^u)$$

in $\mathcal{U}$ such that there is a commutative diagram

$$\begin{array}{ccc}\mathrm{F}'(\coprod_{u\in U} M^u) & \xrightarrow[\cong]{\mathrm{F}'(\psi)} & \mathrm{F}(\coprod_{u\in U} M^u)\\ \Big\downarrow{\scriptstyle\varphi'} & & \Big\downarrow{\scriptstyle\varphi}\\ \coprod_{u\in U}\mathrm{F}'(M^u) & = & \coprod_{u\in U}\mathrm{F}(M^u)\,,\end{array}$$

where φ and φ' are the canonical morphisms. It follows that φ is and isomorphism if and only φ' is an isomorphism; and hence the functor F preserves coproducts if and only if F' preserves coproducts.

The assertion about products is proved similarly.

(c): Let $\phi\colon \mathrm{F}\Sigma_{\mathcal{C}} \to \Sigma_{\mathcal{U}}\mathrm{F}$ be a natural isomorphism as in 6.1.16. By 6.1.12 there are equalities $\mathrm{F}\Sigma_{\mathcal{C}} = \mathrm{F}'\mathrm{Q}\Sigma_{\mathcal{C}} = \mathrm{F}'\Sigma_{\mathcal{K}}\mathrm{Q}$, and one has $\Sigma_{\mathcal{U}}\mathrm{F} = \Sigma_{\mathcal{U}}\mathrm{F}'\mathrm{Q}$. Since Q is the identity on objects, ϕ can be viewed as a natural isomorphism $\phi'\colon \mathrm{F}'\Sigma_{\mathcal{K}} \to \Sigma_{\mathcal{U}}\mathrm{F}'$. We verify that the functor $\mathrm{F}'\colon \mathcal{K}(R) \to \mathcal{U}$ with the isomorphism ϕ', is triangulated. By definition, every distinguished triangle in $\mathcal{K}(R)$ is isomorphic to a strict triangle, that is, to a diagram of the form

$$M \xrightarrow{\mathrm{Q}(\alpha)} N \xrightarrow{\mathrm{Q}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{\mathrm{Q}(0\ 1^{\Sigma M})} \Sigma_{\mathcal{K}} M\ ,$$

where $\alpha\colon M \to N$ is a morphism in $\mathcal{C}(R)$. Thus, it must be shown that the following candidate triangle in $\mathcal{U}$ is distinguished,

$$\mathrm{F}'(M) \xrightarrow{\mathrm{F}'\mathrm{Q}(\alpha)} \mathrm{F}'(N) \xrightarrow{\mathrm{F}'\mathrm{Q}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \mathrm{F}'(\operatorname{Cone}\alpha) \xrightarrow{\phi'^M \circ \mathrm{F}'\mathrm{Q}(0\ 1^{\Sigma M})} \Sigma_{\mathcal{U}}\mathrm{F}'(M)\ .$$

However, this diagram is identical to the one in 6.1.16, which is distinguished triangle in $\mathcal{U}$ since F is assumed to be quasi-triangulated. □

REMARK. The crux of 6.1.18 is that the category $\mathcal{K}(R)$ is the localization of $\mathcal{C}(R)$ with respect to the collection of homotopy equivalences. In the next section, we treat the further localization of $\mathcal{K}(R)$ with respect to the collection of quasi-isomorphisms; this leads to the derived category.

6.1.19 Definition. Let $(\mathcal{V}, \Sigma_{\mathcal{V}})$ be a triangulated category. A functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ is called *quasi-triangulated* if the functor G^{op} from $\mathcal{C}(R)$ to the triangulated category $(\mathcal{V}^{\mathrm{op}}, \Sigma_{\mathcal{V}}^{-1})$, see A.5, is quasi-triangulated in the sense of 6.1.16. Explicitly, this means that there is a natural isomorphism $\psi\colon \Sigma_{\mathcal{V}}^{-1}\mathrm{G} \to \mathrm{G}\Sigma$ of functors $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$, such that the candidate triangle

$$\Sigma_{\mathcal{V}}^{-1}\mathrm{G}(M) \xrightarrow{\mathrm{G}(0\ 1^{\Sigma M})\circ\psi^M} \mathrm{G}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{G}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \mathrm{G}(N) \xrightarrow{\mathrm{G}(\alpha)} \mathrm{G}(M)$$

is a distinguished triangle in $\mathcal{V}$ for every morphism of R-complexes $\alpha\colon M \to N$.

A morphism in $\mathcal{C}(R)^{\mathrm{op}}$ is called a homotopy equivalence if the corresponding morphism in $\mathcal{C}(R)$ is a homotopy equivalence in the sense of 2.2.25. To parse and prove the next result, recall further that if $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is a functor between categories with products (coproducts), then $\mathrm{F}^{\mathrm{op}}\colon \mathcal{U}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ is a functor between categories with coproducts (products), and F preserves products (coproducts) if and only if F^{op} preserves coproducts (products).

6.1.20 Theorem. *Let $\mathcal{V}$ be a category and let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ be a functor. If G maps homotopy equivalences to isomorphisms, then there exists a unique functor G' that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{Q}^{\mathrm{op}}} & \mathcal{K}(R)^{\mathrm{op}} \\ {\scriptstyle \mathrm{G}}\downarrow & \swarrow{\scriptstyle \mathrm{G}'} & \\ \mathcal{V}\ ; & & \end{array}$$

here Q is the canonical functor from 6.1.6. For an R-complex M there is an equality $\mathrm{G}'(M) = \mathrm{G}(M)$, and for a morphism $[\alpha]$ in $\mathcal{K}(R)^{\mathrm{op}}$ one has $\mathrm{G}'([\alpha]) = \mathrm{G}(\alpha)$. Furthermore, the following assertions hold.

(a) *Assume that $\mathcal{V}$ is $\Bbbk$-prelinear; then G' is $\Bbbk$-linear if and only if G is $\Bbbk$-linear.*

(b) *Assume that $\mathcal{V}$ has (co)products; then G' preserves (co)products if and only if G preserves (co)products.*

(c) *If $\mathcal{V}$ is triangulated and G is quasi-triangulated, then G' is triangulated.*

PROOF. Apply 6.1.18 to the functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{C}(R) \to \mathcal{V}^{\mathrm{op}}$. □

UNIQUE LIFTING PROPERTIES

In the balance of this chapter, we use Greek letters for morphisms in $\mathcal{K}(R)$; that is, $\alpha, \beta, \gamma, \ldots$ denote homotopy equivalence classes of morphisms in $\mathcal{C}(R)$.

Now we rephrase 5.2.16 and 5.3.21 in the language of the homotopy category.

6.1.21. Let P be a semi-projective R-complex. If $\alpha\colon P \to N$ is a morphism and $\beta\colon M \to N$ is a quasi-isomorphism in $\mathcal{K}(R)$, then there exists a unique morphism γ that makes the following diagram in $\mathcal{K}(R)$ commutative,

$$\begin{array}{ccc} & & P \\ & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle\alpha} \\ M & \xrightarrow[\beta]{\simeq} & N\,. \end{array}$$

6.1.22 Lemma. *Let P be a semi-projective R-complex. For morphisms in $\mathcal{K}(R)$,*

$$P \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M \xrightarrow[\simeq]{\varphi} N\,,$$

where φ is a quasi-isomorphism, one has $\alpha = \beta$ if $\varphi\alpha = \varphi\beta$ holds. □

6.1.23. Let I be a semi-injective R-complex. If $\alpha\colon M \to I$ is a morphism and $\beta\colon M \to N$ is a quasi-isomorphism in $\mathcal{K}(R)$, then there exists a unique morphism γ that makes the following diagram in $\mathcal{K}(R)$ commutative,

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{\beta} & N \\ {\scriptstyle\alpha}\downarrow & \underset{\gamma}{\swarrow} & \\ I\,. & & \end{array}$$

6.1.24 Lemma. *Let I be a semi-injective R-complex. For morphisms in $\mathcal{K}(R)$,*

$$M \xrightarrow[\simeq]{\varphi} N \underset{\beta}{\overset{\alpha}{\rightrightarrows}} I\,,$$

where φ is a quasi-isomorphism, one has $\alpha = \beta$ if $\alpha\varphi = \beta\varphi$ holds. □

We also recast 5.2.17 and 5.3.22 in the language of the homotopy category.

6.1.25. If $\beta\colon M \to P$ is a quasi-isomorphism in $\mathcal{K}(R)$ and P is semi-projective, then β has a right inverse in $\mathcal{K}(R)$ which is also a quasi-isomorphism.

6.1.26. If $\beta\colon I \to M$ is a quasi-isomorphism in $\mathcal{K}(R)$ and I is semi-injective, then β has a left inverse in $\mathcal{K}(R)$ which is also a quasi-isomorphism.

Exercises

E 6.1.1 Show that a morphism in a triangulated category is a monomorphism (epimorphism) if and only if it has a left (right) inverse. Conclude that every object in a triangulated category is both injective and projective.

E 6.1.2 Show that the homotopy category $\mathcal{K}(\mathbb{Z})$ is not Abelian.

E 6.1.3 Show that the homotopy category $\mathcal{K}(R)$ is Abelian if R is semi-simple.

E 6.1.4 Two homomorphisms of R-modules $\alpha,\beta\colon M \to N$ are called stably equivalent if $\alpha-\beta$ factors through a projective R-module. The *stable module category* $\underline{\mathcal{M}}(R)$ has as objects all R-modules. The hom-set $\underline{\mathcal{M}}(R)(M,N)$, often written as $\underline{\operatorname{Hom}}_R(M,N)$, is the set of classes of stably equivalent homomorphisms $M \to N$. (a) Show that $\underline{\mathcal{M}}(R)$ is a $\Bbbk$-linear category with coproducts. (b) For an R-module M, let $\Omega(M)$ be the kernel of any projective precover $P \twoheadrightarrow M$. Show that Ω is a well-defined $\Bbbk$-linear endofunctor on $\underline{\mathcal{M}}(R)$. (c) Show that the category $\underline{\mathcal{M}}(R)$ is triangulated if R is quasi-Frobenius.

E 6.1.5 Show that the stable module category $\underline{\mathcal{M}}(\mathbb{Q}[x]/(x^2))$ is not Abelian.

E 6.1.6 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{K}(R)$. Show that the complex X in a distinguished triangle $M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M$ in $\mathcal{K}(R)$ is unique up to isomorphism.

E 6.1.7 (Cf. A.5) Let $(\mathcal{T},\Sigma)$ be a triangulated category. Show that $(\mathcal{T}^{\mathrm{op}},\Sigma^{-1})$ is triangulated in the canonical way: A candidate triangle $M \to N \to X \to \Sigma^{-1}M$ in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the candidate triangle $\Sigma^{-1}M \to X \to N \to M$ is distinguished in $\mathcal{T}$.

E 6.1.8 Let $(\mathcal{T},\Sigma)$ be a triangulated category and let $\mathcal{S}$ be a subcategory of $\mathcal{T}$ that is closed under isomorphisms. Show that $\mathcal{S}$ is a triangulated subcategory if and only if $(\mathcal{S},\Sigma)$ is a triangulated category and the embedding functor $\mathcal{S} \to \mathcal{T}$ is full and triangulated.

E 6.1.9 Let $\mathcal{S}$ be a triangulated subcategory of $(\mathcal{T},\Sigma)$ and let $M \to N \to X \to \Sigma M$ be a distinguished triangle in $\mathcal{T}$. Show that if two of the objects M, N, and X are in $\mathcal{S}$, then the third object is also in $\mathcal{S}$.

E 6.1.10 Show that the full subcategory of $\mathcal{K}(R)$ whose objects are all acyclic R-complexes is triangulated.

E 6.1.11 Show that the full subcategories of $\mathcal{K}(R)$ defined by specifying their objects as follows:

$$\mathcal{K}_{\sqsubset}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded above complex } M' \text{ with } M \cong M' \text{ in } \mathcal{K}(R)\}\,,$$
$$\mathcal{K}_{\square}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded complex } M' \text{ with } M \cong M' \text{ in } \mathcal{K}(R)\}\,, \text{ and}$$
$$\mathcal{K}_{\sqsupset}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded below complex } M' \text{ with } M \cong M' \text{ in } \mathcal{K}(R)\}$$

are triangulated subcategories of $\mathcal{K}(R)$.

E 6.1.12 Show that the full subcategory of $\mathcal{K}(R)$ defined by specifying its objects,

$$\mathcal{K}(\operatorname{Prj} R) = \{P \in \mathcal{K}(R) \mid P \text{ is a complex of projective modules}\}\,,$$

is a triangulated category but, in general, not a triangulated subcategory of $\mathcal{K}(R)$. Show that the inclusion functor $\mathcal{K}(\operatorname{Prj} R) \to \mathcal{K}(R)$ is triangulated.

E 6.1.13 Show that the full subcategory of $\mathcal{K}_{\mathrm{prj}}(R)$ defined by specifying its objects,

$$\mathcal{K}_{\mathrm{prj}}(R) = \{P \in \mathcal{K}(\mathrm{Prj}\, R) \mid P \text{ is semi-projective}\}\ ,$$

is a triangulated subcategory of $\mathcal{K}(\mathrm{Prj}\, R)$.

E 6.1.14 Show that the full subcategory of $\mathcal{K}(R)$ defined by specifying its objects,

$$\mathcal{K}(\mathrm{Inj}\, R) = \{I \in \mathcal{K}(R) \mid I \text{ is a complex of injective modules}\}\ ,$$

is a triangulated category but, in general, not a triangulated subcategory of $\mathcal{K}(R)$. Show that the inclusion functor $\mathcal{K}(\mathrm{Inj}\, R) \to \mathcal{K}(R)$ is triangulated.

E 6.1.15 Show that the full subcategory of $\mathcal{K}(R)$ defined by specifying its objects,

$$\mathcal{K}_{\mathrm{inj}}(R) = \{I \in \mathcal{K}(\mathrm{Inj}\, R) \mid \text{ is semi-injective}\}\ ,$$

is a triangulated subcategory of $\mathcal{K}(\mathrm{Inj}\, R)$.

6.2 The Derived Category

SYNOPSIS. Derived category; (co)product; triangulation; universal property.

Let $\mathcal{U}$ be a category and let $\mathcal{S}$ be a collection of morphisms in $\mathcal{U}$. One may seek a category $\mathcal{S}^{-1}\mathcal{U}$—called the *localization* of $\mathcal{U}$ with respect to $\mathcal{S}$—together with a functor $\mathrm{Q}\colon \mathcal{U} \to \mathcal{S}^{-1}\mathcal{U}$ that has the following universal property: The functor Q maps $\mathcal{S}$ to isomorphisms in $\mathcal{S}^{-1}\mathcal{U}$, and for any functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ that maps $\mathcal{S}$ to isomorphisms in $\mathcal{V}$ there is a unique functor F' that makes the next diagram commutative,

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\mathrm{Q}} & \mathcal{S}^{-1}\mathcal{U} \\ {\scriptstyle \mathrm{F}}\downarrow & \swarrow{\scriptstyle \mathrm{F}'} & \\ \mathcal{V}\ . & & \end{array}$$

There is a formal way to construct $\mathcal{S}^{-1}\mathcal{U}$. However, this constuction may result in a "category" in which the hom-sets are not sets but proper classes. Thus, the localization of $\mathcal{U}$ with respect to $\mathcal{S}$ might not exist. An early motivation for the theory of model categories was to avoid such such set theoretic problems.

As asserted in 6.1.18, the homotopy category $\mathcal{K}(R)$ is the localization of $\mathcal{C}(R)$ with respect to the collection of homotopy equivalences. Now we proceed with the further localization of $\mathcal{K}(R)$ with respect to the collection of quasi-isomorphisms. The result is a category $\mathcal{D}(R)$—the existence of semi-projective resolutions can be harnessed to show that the hom-sets in $\mathcal{D}(R)$ are actual sets—called the derived category over R; it inherits a triangulated structure fom $\mathcal{K}(R)$.

OBJECTS AND MORPHISMS

Recall that we use Greek letters for morphisms in $\mathcal{K}(R)$, that is, $\alpha,\beta,\gamma,\ldots$ denote homotopy equivalence classes of morphisms in $\mathcal{C}(R)$.

6.2.1. If $\beta\colon M \to V$ is a morphism and $\psi\colon N \to V$ is a quasi-isomorphism in $\mathcal{K}(R)$, then there exist a morphism α and a quasi-isomorphism φ such that the following diagram in $\mathcal{K}(R)$ is commutative,

(6.2.1.1)
$$\begin{array}{ccc} U & \xrightarrow{\alpha} & N \\ {\scriptstyle \varphi}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \psi} \\ M & \xrightarrow[\beta]{} & V \end{array} .$$

For example, choose by 5.2.13 a semi-projective resolution $\varphi\colon U \xrightarrow{\simeq} M$ and apply 6.1.21 to get the morphism α.

Similarly, from the existence of semi-injective resolutions 5.3.18, it follows that given a morphism $\alpha\colon U \to N$ and a quasi-isomorphism $\varphi\colon U \to M$ in $\mathcal{K}(R)$ there exist a morphism β and a quasi-isomorphism ψ such that (6.2.1.1) is commutative.

REMARK. One does not need semi-projective and semi-injective resolutions to prove the assertions in 6.2.1; in fact, they may be proved using only that the homotopy category is triangulated; see E 6.2.8–E 6.2.10.

6.2.2 Definition. Let M and N be objects in $\mathcal{K}(R)$. A *left prefraction* from M to N is a pair (α,φ) of morphisms in $\mathcal{K}(R)$ such that α and φ have the same source, the target of φ is M, the target of α is N, and φ is a quasi-isomorphism:

$$M \xleftarrow[\simeq]{\varphi} U \xrightarrow{\alpha} N .$$

Two left prefractions (α^1,φ^1) and (α^2,φ^2) from M to N are equivalent, in symbols $(\alpha^1,\varphi^1) \equiv (\alpha^2,\varphi^2)$, if there exist a third left prefraction (α,φ) from M to N and morphisms μ^1 and μ^2 that make the following diagram in $\mathcal{K}(R)$ commutative,

(6.2.2.1)
$$\begin{array}{ccccc} & & U^1 & & \\ & {\scriptstyle \varphi^1}\swarrow & \uparrow{\scriptstyle \mu^1} & \searrow{\scriptstyle \alpha^1} & \\ M & \xleftarrow[\varphi]{} & U & \xrightarrow{\alpha} & N \\ & {\scriptstyle \varphi^2}\nwarrow & \downarrow{\scriptstyle \mu^2} & \nearrow{\scriptstyle \alpha^2} & \\ & & U^2 & & \end{array} .$$

Note that the morphisms μ^1 and μ^2 in (6.2.2.1) are quasi-isomorphisms.

6.2.3 Lemma. *Let M and N be objects in $\mathcal{K}(R)$. The relation $\equiv$ introduced in* 6.2.2 *is an equivalence relation on the class of left prefractions from M to N.*

PROOF. It is evident that $\equiv$ is reflexive and symmetric. To prove transitivity, assume that there are relations $(\alpha_1,\varphi_1) \equiv (\alpha_2,\varphi_2)$ and $(\alpha_2,\varphi_2) \equiv (\alpha_3,\varphi_3)$; that is, there exist commutative diagrams in $\mathcal{K}(R)$,

$$\begin{array}{ccccc} & & U_1 & & \\ & \swarrow^{\varphi_1} & \uparrow{\scriptstyle\mu_1} & \searrow^{\alpha_1} & \\ M & \xleftarrow[\varphi]{} & U & \xrightarrow{\alpha} & N \\ & \nwarrow_{\varphi_2} & \downarrow{\scriptstyle\mu_2} & \nearrow_{\alpha_2} & \\ & & U_2 & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & U_2 & & \\ & \swarrow^{\varphi_2} & \uparrow{\scriptstyle\mu_2'} & \searrow^{\alpha_2} & \\ M & \xleftarrow[\varphi']{} & U' & \xrightarrow{\alpha'} & N \\ & \nwarrow_{\varphi_3} & \downarrow{\scriptstyle\mu_3'} & \nearrow_{\alpha_3} & \\ & & U_3 & & \end{array},$$

where (α,φ) and (α',φ') are left prefractions. By 6.2.1 there exist quasi-isomorphisms θ and θ' that make the following diagram in $\mathcal{K}(R)$ commutative,

$$\begin{array}{ccc} W & \xrightarrow[\simeq]{\theta'} & U \\ {\scriptstyle\theta}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\mu_2} \\ U' & \xrightarrow[\mu_2']{\simeq} & U_2 \end{array}.$$

Note that $\varphi\theta' = \varphi'\theta$ holds since one has $\varphi\theta' = \varphi_2\mu_2\theta' = \varphi_2\mu_2'\theta = \varphi'\theta$; similarly, the equality $\alpha\theta' = \alpha'\theta$ holds. Thus, there is a commutative diagram,

$$\begin{array}{ccccc} & & U_1 & & \\ & \swarrow^{\varphi_1} & \uparrow{\scriptstyle\mu_1\theta'} & \searrow^{\alpha_1} & \\ M & \xleftarrow[\varphi\theta'=\varphi'\theta]{\simeq} & W & \xrightarrow{\alpha\theta'=\alpha'\theta} & N \\ & \nwarrow_{\varphi_3} & \downarrow{\scriptstyle\mu_3'\theta} & \nearrow_{\alpha_3} & \\ & & U_3 & & \end{array},$$

which shows that there is a relation $(\alpha_1,\varphi_1) \equiv (\alpha_3,\varphi_3)$. □

6.2.4 Definition. Let M and N be objects in $\mathcal{K}(R)$. For a left prefraction (α,φ) from M to N, denote by α/φ the equivalence class containing (α,φ). The class α/φ is called a *left fraction* from M to N, and the collection of all such is denoted $\mathcal{D}(R)(M,N)$.

The notation introduced in 6.2.4 is suggestive and, indeed, we shall shortly prove that there is a category $\mathcal{D}(R)$ whose objects are all R-complexes and in which the hom-set $\mathcal{D}(R)(M,N)$ is the collection of all left fractions from M to N.

6.2.5. Consider a diagram in $\mathcal{K}(R)$ of the form

$$\begin{array}{ccccc} M & \xleftarrow[\simeq]{\varphi} & U & \xrightarrow{\alpha} & N \\ & & {\scriptstyle\psi}\uparrow{\scriptstyle\simeq} & & \\ & & V & & \end{array},$$

where φ and ψ are quasi-isomorphisms. The commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & \overset{\varphi}{\swarrow} & \big\uparrow{\scriptstyle \psi} & \overset{\alpha}{\searrow} & \\ M & \xleftarrow[\varphi\psi]{} & V & \xrightarrow{\alpha\psi} & N \\ & \underset{\varphi\psi}{\nwarrow} & \big\| & \underset{\alpha\psi}{\nearrow} & \\ & & V & & \end{array}$$

shows that the left prefractions (α,φ) and $(\alpha\psi,\varphi\psi)$ are equivalent; whence there is an equality ${}^{\alpha}/_{\varphi} = {}^{(\alpha\psi)}/_{(\varphi\psi)}$.

6.2.6 Lemma. *Let M and N be objects in $\mathcal{K}(R)$ and let ${}^{\alpha_1}/_{\varphi_1}$ and ${}^{\alpha_2}/_{\varphi_2}$ be left fractions from M to N. There exist morphisms α_1',α_2' and a quasi-isomorphism φ in $\mathcal{K}(R)$ such that the equalities ${}^{\alpha_1}/_{\varphi_1} = {}^{\alpha_1'}/_{\varphi}$ and ${}^{\alpha_2}/_{\varphi_2} = {}^{\alpha_2'}/_{\varphi}$ hold.*

PROOF. Since the quasi-isomorphisms $\varphi_1 : U_1 \to M$ and $\varphi_2 : U_2 \to M$ have the same target, 6.2.1 yields quasi-isomorphisms ψ_1 and ψ_2, illustrated below,

$$\begin{array}{ccccc} M & \xleftarrow[\simeq]{\varphi_1} & U_1 & \xrightarrow{\alpha_1} & N \\ & & {\scriptstyle\psi_1}\big\uparrow{\scriptstyle\simeq} & & \\ & & V & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} M & \xleftarrow[\simeq]{\varphi_2} & U_2 & \xrightarrow{\alpha_2} & N \\ & & {\scriptstyle\psi_2}\big\uparrow{\scriptstyle\simeq} & & \\ & & V\,, & & \end{array}$$

such that $\varphi_1\psi_1 = \varphi_2\psi_2$ holds. Note that this composite is a quasi-isomorphism, and denote it by φ. Now 6.2.5 yields ${}^{\alpha_i}/_{\varphi_i} = {}^{(\alpha_i\psi_i)}/_{(\varphi_i\psi_i)} = {}^{(\alpha_i\psi_i)}/_{\varphi}$ for $i = 1,2$. □

The collection of all left prefractions from M to N is a proper class (as opposed to a set); however, the collection of equivalence classes of these left prefractions turns out to be a set.

6.2.7 Lemma. *For R-complexes M and N, the collection $\mathcal{D}(R)(M,N)$ of left fractions from M to N is a set. Moreover, $\mathcal{D}(R)(M,N)$ is $\Bbbk$-module with addition and $\Bbbk$-multiplication defined as follows.*

- *For ${}^{\alpha_1}/_{\varphi_1}$ and ${}^{\alpha_2}/_{\varphi_2}$ in $\mathcal{D}(R)(M,N)$ set*

$$ {}^{\alpha_1}/_{\varphi_1} + {}^{\alpha_2}/_{\varphi_2} = {}^{(\alpha_1' + \alpha_2')}/_{\varphi} $$

  for any choice of left prefractions (α_i',φ) with ${}^{\alpha_i}/_{\varphi_i} = {}^{\alpha_i'}/_{\varphi}$ for $i = 1,2$; cf. 6.2.6.
- *For x in $\Bbbk$ and ${}^{\alpha}/_{\varphi}$ in $\mathcal{D}(R)(M,N)$ set*

$$ x\,({}^{\alpha}/_{\varphi}) = {}^{(x\alpha)}/_{\varphi}\,. $$

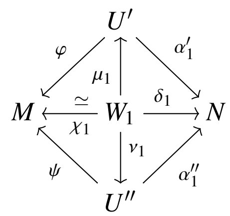
The equivalence class ${}^{0}/_{1^M}$ containing the left prefraction $M \xleftarrow{1^M} M \xrightarrow{0} N$ is the zero element in the $\Bbbk$-module $\mathcal{D}(R)(M,N)$.

PROOF. To show that $\mathcal{D}(R)(M,N)$ is a set, let $\pi\colon P \xrightarrow{\simeq} M$ be a semi-projective resolution of M and consider the map

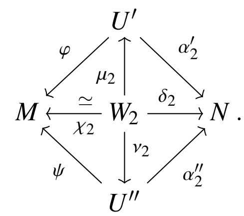
$$(\star) \qquad \mathcal{K}(R)(P,N) \longrightarrow \mathcal{D}(R)(M,N) \quad \text{given by} \quad \beta \longmapsto \beta/\pi\,.$$

Let (α,φ) be a left prefraction from M to N, and denote by U the common source of α and φ. By 6.1.21 there exists a morphism $\gamma\colon P \to U$ in $\mathcal{K}(R)$ with $\varphi\gamma = \pi$. Note that γ is a quasi-isomorphism. It follows from 6.2.1 that $\alpha/\varphi = (\alpha\gamma)/(\varphi\gamma) = (\alpha\gamma)/\pi$ holds, and hence $(\star)$ is surjective. Since $\mathcal{K}(R)(P,N)$ is a set, so is $\mathcal{D}(R)(M,N)$.

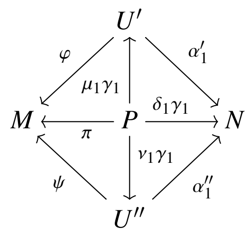
To prove that addition in $\mathcal{D}(R)(M,N)$ is well-defined, assume that $\alpha_1'/\varphi = \alpha_1''/\psi$ and $\alpha_2'/\varphi = \alpha_2''/\psi$ hold. It must be argued that one has $(\alpha_1'+\alpha_2')/\varphi = (\alpha_1''+\alpha_2'')/\psi$. By assumption, there exist commutative diagrams,



and



Let $\pi\colon P \xrightarrow{\simeq} M$ be a semi-projective resolution of M. By 6.1.21 there are morphisms $\gamma_1 : P \to W_1$ and $\gamma_2 : P \to W_2$ such that $\chi_1\gamma_1 = \pi = \chi_2\gamma_2$ holds. Hence there are commutative diagrams,



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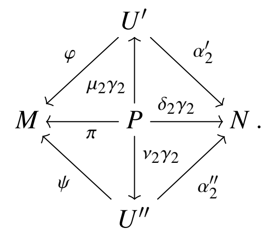
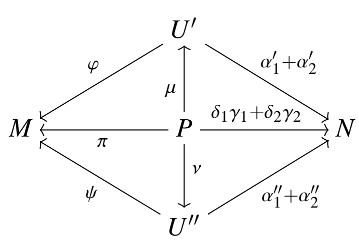


By the uniqueness part in 6.1.21, it follows that $\mu_1\gamma_1 = \mu_2\gamma_2$ and $\nu_1\gamma_1 = \nu_2\gamma_2$ hold. Set $\mu = \mu_i\gamma_i$ and $\nu = \nu_i\gamma_i$; the commutative diagram



shows that $(\alpha_1'+\alpha_2')/\varphi = (\alpha_1''+\alpha_2'')/\psi$ holds, as desired.

It is straightforward to see that $\Bbbk$-multiplication is well-defined.

Finally, to see that $\mathcal{D}(R)(M,N)$ is a $\Bbbk$-module with zero element ${}^{0}/_{1^M}$ notice that the surjective map $(\star)$ preserves addition and $\Bbbk$-multiplication. $\square$

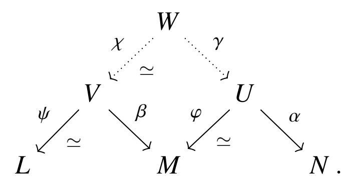
6.2.8 Lemma. *Let L, M, and N be R-complexes. There is a map*

$$\mathcal{D}(R)(M,N) \times \mathcal{D}(R)(L,M) \longrightarrow \mathcal{D}(R)(L,N)\ ,$$

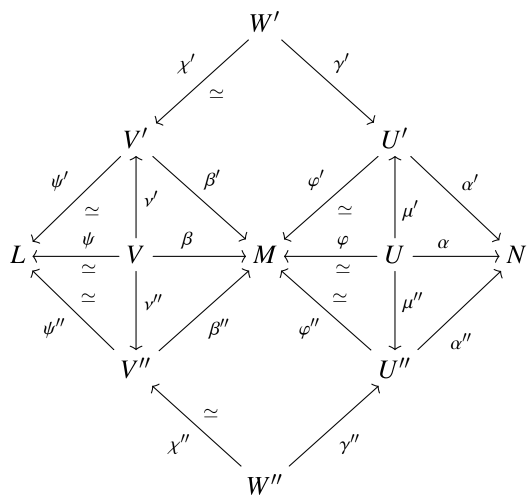
given by

$$({}^{\alpha}/_{\varphi}, {}^{\beta}/_{\psi}) \longmapsto ({}^{\alpha}/_{\varphi})({}^{\beta}/_{\psi}) = {}^{(\alpha\gamma)}/_{(\psi\chi)}\ ,$$

where ${}^{\gamma}/_{\chi}$ is any left fraction that makes the following diagram in $\mathcal{K}(R)$ commutative, cf. 6.2.1,

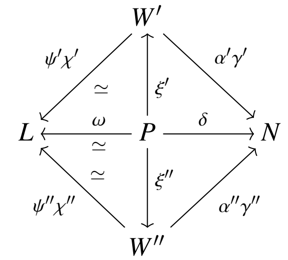


PROOF. To prove that the map is well-defined, it must be verified that given a commutative diagram

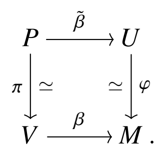


in $\mathcal{K}(R)$ one can construct a commutative diagram of the form

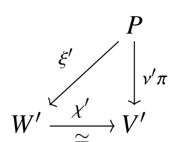
(⋆)



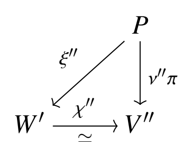
Choose a semi-projective resolution $\pi\colon P \xrightarrow{\simeq} V$ and let $\tilde{\beta}$ be the morphism, see 6.1.21, that makes the following diagram commutative,



Set $\omega = \psi\pi$ and $\delta = \alpha\tilde{\beta}$. Let ξ' and ξ'' be the morphisms that make the diagrams



and



commutative. The two left-hand triangles in (⋆) are commutative, as one has $\psi'\chi'\xi' = \psi'\nu'\pi = \psi\pi = \omega$ and $\psi''\chi''\xi'' = \psi''\nu''\pi = \psi\pi = \omega$. Next we argue that the upper right-hand triangle in (⋆) is commutative, i.e. that $\alpha'\gamma'\xi' = \delta$ holds. As one has $\delta = \alpha\tilde{\beta} = \alpha'\mu'\tilde{\beta}$, it suffices to verify the identity $\gamma'\xi' = \mu'\tilde{\beta}$. By 6.1.22 it is sufficient to show that one has $\varphi'\gamma'\xi' = \varphi'\mu'\tilde{\beta}$, and that is a straightforward computation: $\varphi'\gamma'\xi' = \beta'\chi'\xi' = \beta'\nu'\pi = \beta\pi = \varphi\tilde{\beta} = \varphi'\mu'\tilde{\beta}$. A similar argument shows that the lower right-hand triangle in (⋆) is commutative. □

PRODUCTS, COPRODUCTS, AND $\Bbbk$-LINEARITY

The following definition is justified by the subsequent lemma.

6.2.9 Definition. The *derived category* $\mathcal{D}(R)$ has the same objects as $\mathcal{C}(R)$ and $\mathcal{K}(R)$, that is, R-complexes. For R-complexes M and N, the hom-set $\mathcal{D}(R)(M,N)$ is the set of all left fractions from M to N; cf. 6.2.4. Composition in $\mathcal{D}(R)$ is given by the map in 6.2.8. Isomorphisms in $\mathcal{D}(R)$ are marked by the symbol '$\simeq$'.

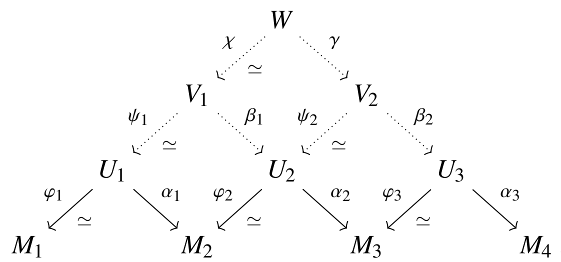
6.2.10. Notice the following special cases of composition in $\mathcal{D}(R)$.

(a) For composable morphisms $^{\alpha}/_{1}, {}^{\beta}/_{\psi}$ in $\mathcal{D}(R)$ one has $(^{\alpha}/_{1})(^{\beta}/_{\psi}) = {}^{(\alpha\beta)}/_{\psi}$.

(b) For composable morphisms $1/\varphi, \beta/\psi$ in $\mathcal{D}(R)$ one has $(\beta/\psi)(1/\varphi) = \beta/(\varphi\psi)$.

6.2.11 Lemma. *The derived category $\mathcal{D}(R)$ is a category. The identity morphism in $\mathcal{D}(R)$ for an R-complex M is $1^M/1^M$.*

PROOF. It follows from 6.2.10 that $1^M/1^M$ is an identity for M. Let α_1/φ_1, α_2/φ_2, and α_3/φ_3 be composable morphisms in $\mathcal{D}(R)$. Apply 6.2.1 to get morphisms β_1/ψ_1, β_2/ψ_2, and γ/χ in $\mathcal{D}(R)$ that make the following diagram in $\mathcal{K}(R)$ commutative,



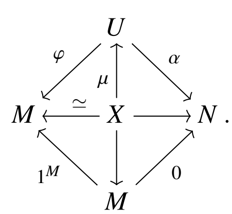
One has $((\alpha_3/\varphi_3)(\alpha_2/\varphi_2))(\alpha_1/\varphi_1) = (\alpha_3\beta_2\gamma)/(\varphi_1\psi_1\chi) = (\alpha_3/\varphi_3)((\alpha_2/\varphi_2)(\alpha_1/\varphi_1))$ by 6.2.8, and hence composition in $\mathcal{D}(R)$ is associative. $\square$

6.2.12. There is a canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$; it is the identity on objects and it maps a morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$ to the left fraction $\alpha/1^M$. Indeed, for composable morphisms $\alpha\colon M \to N$ and $\beta\colon L \to M$ in $\mathcal{K}(R)$, it follows from 6.2.10 that $\mathrm{V}(\alpha\beta) = (\alpha\beta)/1^L = (\alpha/1^M)(\beta/1^L) = \mathrm{V}(\alpha)\mathrm{V}(\beta)$ holds. Furtheremore, for every R-complex M, the morphism $\mathrm{V}(1^M) = 1^M/1^M$ in $\mathcal{D}(R)$ is the identity for M.

6.2.13 Lemma. *A left fraction α/φ in $\mathcal{D}(R)(M,N)$ equals $0/1^M$ if and only if there exists a quasi-isomorphism μ in $\mathcal{K}(R)$ such that $\alpha\mu = 0$ in $\mathcal{K}(R)$.*

PROOF. "If": Assume that there exists a quasi-isomorphism μ with $\alpha\mu = 0$. Then 6.2.5 yields $\alpha/\varphi = (\alpha\mu)/(\varphi\mu) = 0/(\varphi\mu) = (0\varphi\mu)/(1^M\varphi\mu) = 0/1^M$.

"Only if": Assume that there is an equality $\alpha/\varphi = 0/1^M$, that is, there exists a commutative diagram in $\mathcal{K}(R)$ of the following form,



It follows that μ is a quasi-isomorphism with $\alpha\mu = 0$. $\square$

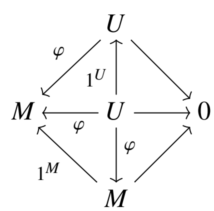
6.2.14 Theorem. *The derived category $\mathcal{D}(R)$ and the functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ are $\Bbbk$-linear. For every family $\{M^u\}_{u\in U}$ of R-complexes the following assertions hold.*

(a) *If* M *with embeddings* $\{\iota^u : M^u \to M\}_{u\in U}$ *is the coproduct of* $\{M^u\}_{u\in U}$ *in* $\mathcal{K}(R)$*, then* M *with the morphisms* $\{\iota^u/1^{M^u}\}_{u\in U}$ *is the coproduct of* $\{M^u\}_{u\in U}$ *in* $\mathcal{D}(R)$*.*

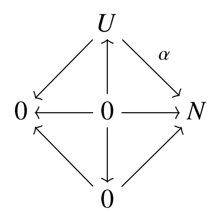
(b) *If* M *with projections* $\{\pi^u : M^u \to M\}_{u\in U}$ *is the product of* $\{M^u\}_{u\in U}$ *in* $\mathcal{K}(R)$*, then* M *with the morphisms* $\{\pi^u/1^{M^u}\}_{u\in U}$ *is the product of* $\{M^u\}_{u\in U}$ *in* $\mathcal{D}(R)$*.*

In particular, the derived category $\mathcal{D}(R)$ *has coproducts and products, and the canonical functor* V *preserves coproducts and products.*

PROOF. By 6.2.7 the hom-sets in $\mathcal{D}(R)$ are $\Bbbk$-modules, and it is straightforward to verify that composition of morphisms in $\mathcal{D}(R)$ is $\Bbbk$-bilinear. Thus the category $\mathcal{D}(R)$ is $\Bbbk$-prelinear. The functor V is $\Bbbk$-linear; indeed, for x in $\Bbbk$ and a morphism α in $\mathcal{K}(R)$ one has $\mathrm{V}(x\alpha) = {}^{(x\alpha)}/_1 = x({}^{\alpha}/_1) = x\mathrm{V}(\alpha)$. To show that $\mathcal{D}(R)$ is $\Bbbk$-linear, it must be argued that it has biproducts and a zero object. Since $\mathcal{K}(R)$ has biproducts, see 6.1.8, it follows from 6.1.7 applied to the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ that $\mathcal{D}(R)$ has biproducts as well. The zero complex 0 is a zero object in $\mathcal{D}(R)$, that is, 0 is both an initial and a terminal object in $\mathcal{D}(R)$. Indeed, the commutative diagrams



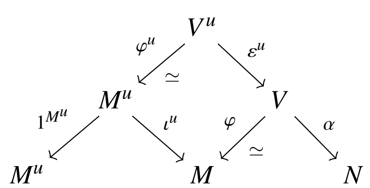
and



in $\mathcal{K}(R)$ show that one has $\mathcal{D}(R)(M,0) = \{{}^0/_{1^M}\}$ and $\mathcal{D}(R)(0,N) = \{{}^0/_{1^0}\}$.

(a): Let $\{{}^{\alpha^u}/_{\varphi^u} : M^u \to N\}_{u\in U}$ be morphisms in $\mathcal{D}(R)$. It must be shown that there is a unique morphism ${}^{\alpha}/_{\varphi}\colon M \to N$ in $\mathcal{D}(R)$ with $({}^{\alpha}/_{\varphi})({}^{\iota^u}/_{1^{M^u}}) = {}^{\alpha^u}/_{\varphi^u}$ for all $u \in U$.

For existence, let V^u be the common source of α^u and φ^u, and denote by V the coproduct of $\{V^u\}_{u\in U}$ in $\mathcal{K}(R)$ with embeddings $\varepsilon^u : V^u \to V$. Let $\alpha\colon V \to N$ be the unique morphism in $\mathcal{K}(R)$ with $\alpha\varepsilon^u = \alpha^u$ for all $u \in U$, and let $\varphi\colon V \to M$ be the coproduct in $\mathcal{K}(R)$ of the family of quasi-isomorphisms $\{\varphi^u : V^u \to M^u\}_{u\in U}$. Then φ is a quasi-isomorphism by 6.1.11, and there is a commutative diagram



from which it follows that $({}^{\alpha}/_{\varphi})({}^{\iota^u}/_{1^{M^u}}) = {}^{\alpha^u}/_{\varphi^u}$ holds.

For uniqueness, assume that $({}^{\alpha}/_{\varphi})({}^{\iota^u}/_{1^{M^u}})$ is zero in $\mathcal{D}(R)$ for all $u \in U$; it must be shown that ${}^{\alpha}/_{\varphi}$ is zero in $\mathcal{D}(R)$. By definition, each composite $({}^{\alpha}/_{\varphi})({}^{\iota^u}/_{1^{M^u}})$ is equal to ${}^{(\alpha\beta^u)}/_{\psi^u}$ for any choice of morphism β^u and quasi-isomorphism ψ^u that make the following diagram in $\mathcal{K}(R)$ commutative,

$$
\begin{array}{ccccccccc}
 & & & & W^u & & & & \\
 & & & {\scriptstyle \psi^u}\swarrow{\scriptstyle \simeq} & & \searrow{\scriptstyle \beta^u} & & & \\
 & & M^u & & & & V & & \\
 & {\scriptstyle 1^{M^u}}\swarrow & & \searrow{\scriptstyle \iota^u} & & {\scriptstyle \varphi}\swarrow{\scriptstyle \simeq} & & \searrow{\scriptstyle \alpha} & \\
M^u & & & & M & & & & N\,.
\end{array}
$$

Let W denote the coproduct of $\{W^u\}_{u\in U}$ in $\mathcal{K}(R)$ with embeddings $\omega^u\colon W^u \to W$. Let $\beta\colon W \to V$ be the unique morphism in $\mathcal{K}(R)$ with $\beta\omega^u = \beta^u$, and let $\psi\colon W \to M$ be the coproduct in $\mathcal{K}(R)$ of the family $\{\psi^u\colon W^u \xrightarrow{\simeq} M^u\}_{u\in U}$. Then ψ is a quasi-isomorphism by 6.1.11, and it is the unique morphism $W \to M$ with $\psi\omega^u = \iota^u\psi^u$ for all $u \in U$; see 6.1.9. Since one also has $(\varphi\beta)\omega^u = \varphi\beta^u = \iota^u\psi^u$, we conclude $\varphi\beta = \psi$. Since φ and ψ are quasi-isomorphisms, so is β. Hence 6.2.5 yields ${}^{\alpha}/_{\varphi} = {}^{(\alpha\beta)}/_{(\varphi\beta)}$. In view of this, 6.2.13 implies that ${}^{\alpha}/_{\varphi}$ is zero in $\mathcal{D}(R)$ if and only if $\alpha\beta\mu$ is zero in $\mathcal{K}(R)$ for some quasi-isomorphism μ. By assumption, ${}^{(\alpha\beta^u)}/_{\psi^u}$ is zero, and thus for every $u \in U$ there is a quasi-isomorphism $\mu^u\colon X^u \to W^u$ with $\alpha\beta^u\mu^u = 0$. The coproduct $\mu\colon X \to W$ in $\mathcal{K}(R)$ of the quasi-isomorphisms $\{\mu^u\}_{u\in U}$ is a quasi-isomorphism; since both $\alpha\beta\mu$ and 0 make the diagrams

$$
\begin{array}{ccc}
X^u & \longrightarrow & X \\
{\scriptstyle \alpha\beta^u\mu^u=0}\big\downarrow & \swarrow{\scriptstyle \cdots} & \\
N & &
\end{array}
$$

commutative, the universal property of coproducts implies that $\alpha\beta\mu = 0$ holds.

(b): Let $\{{}^{\alpha^u}/_{\varphi^u}\colon N \to M^u\}_{u\in U}$ be morphisms in $\mathcal{D}(R)$. It must be shown that there is a unique morphism ${}^{\alpha}/_{\varphi}\colon N \to M$ in $\mathcal{D}(R)$ with $({}^{\pi^u}/_{1^M})({}^{\alpha}/_{\varphi}) = {}^{\alpha^u}/_{\varphi^u}$ for all $u \in U$.

For every $u \in U$ let X^u be the common source of α^u and φ^u. Let $\lambda^u\colon M^u \xrightarrow{\simeq} I^u$ be a semi-injective resolution, and let $\beta^u\colon N \to I^u$ be the unique morphism, see 6.1.23, that makes the following diagram in $\mathcal{K}(R)$ commutative,

$$
\begin{array}{ccc}
X^u & \xrightarrow{\ \alpha^u\ } & M^u \\
{\scriptstyle \varphi^u}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \lambda^u} \\
N & \xrightarrow[\ \beta^u\ ]{} & I^u\,.
\end{array}
$$

Let I be the product of $\{I^u\}_{u\in U}$ in $\mathcal{K}(R)$ with projections $\rho^u\colon I \to I^u$. Denote by $\lambda\colon M \to I$ the product in $\mathcal{K}(R)$ of the quasi-isomorphisms $\{\lambda^u\colon M^u \to I^u\}_{u\in U}$. Then λ is a quasi-isomorphism by 6.1.11, and it is the unique morphism in $\mathcal{K}(R)$ with $\rho^u\lambda = \lambda^u\pi^u$ for all $u \in U$; cf. 6.1.9. Furtermore, let $\beta\colon N \to I$ be the unique morphism in $\mathcal{K}(R)$ that satisfies $\rho^u\beta = \beta^u$ for all $u \in U$. Choose a semi-projective resolution $\varphi\colon P \xrightarrow{\simeq} N$ and let $\alpha\colon P \to M$ be the unique morphism that makes the following diagram in $\mathcal{K}(R)$ commutative, see 6.1.21,

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha\ } & M \\ {\scriptstyle\varphi}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\lambda} \\ N & \xrightarrow[\ \beta\ ]{} & I\,. \end{array}$$

We claim that $(\pi^u/1^M)(\alpha/\varphi) = \alpha^u/\varphi^u$ holds for all $u \in U$. Let $\tau^u \colon P \to X^u$ be the unique morphism in $\mathcal{K}(R)$ that makes the next diagram commutative, see 6.1.21,

$$\begin{array}{ccc} & & P \\ & {\scriptstyle\tau^u}\swarrow & \big\downarrow{\scriptstyle\varphi} \\ X^u & \xrightarrow[\varphi^u]{\simeq} & N\,. \end{array}$$

Note that τ^u is a quasi-isomorphism, as φ and φ^u are quasi-isomorphisms. The morphism τ^u satisfies $\alpha^u\tau^u = \pi^u\alpha$; indeed, by 6.1.22 the equality $\alpha^u\tau^u = \pi^u\alpha$ holds if one has $\lambda^u\alpha^u\tau^u = \lambda^u\pi^u\alpha$; and that is a simple computation: $\lambda^u\alpha^u\tau^u = \beta^u\varphi^u\tau^u = \beta^u\varphi = \rho^u\beta\varphi = \rho^u\lambda\alpha = \lambda^u\pi^u\alpha$. Thus, it follows from 6.2.10 and 6.2.5 that one has

$$(\pi^u/1^M)\,(\alpha/\varphi) \;=\; (\pi^u\alpha)/\varphi \;=\; (\alpha^u\tau^u)/(\varphi^u\tau^u) \;=\; \alpha^u/\varphi^u\,.$$

This proves the existence of the desired morphism α/φ.

For uniqueness, assume that $(\pi^u/1^M)(\alpha/\varphi) = (\pi^u\alpha)/\varphi$ is zero in $\mathcal{D}(R)$ for all $u \in U$; it must be shown that α/φ is zero in $\mathcal{D}(R)$. Denote by X the common source of α and φ. As $(\pi^u\alpha)/\varphi$ is zero there is by 6.2.13 a quasi-isomorphism $\mu^u \colon Y^u \to X$ with $\pi^u\alpha\mu^u = 0$. Let $\rho \colon Q \xrightarrow{\simeq} X$ be a semi-projective resolution, and let $\nu^u \colon Q \to Y^u$ be the unique morphism in $\mathcal{K}(R)$ that makes the following diagram commutative,

$$\begin{array}{ccc} & & Q \\ & {\scriptstyle\nu^u}\swarrow & \big\downarrow{\scriptstyle\rho} \\ Y^u & \xrightarrow[\mu^u]{\simeq} & X\,. \end{array}$$

Note that $\pi^u\alpha\rho = \pi^u\alpha\mu^u\nu^u = 0\nu^u = 0$ holds for all $u \in U$, and hence the universal property of products in $\mathcal{K}(R)$ implies that one has $\alpha\rho = 0$. Since ρ is a quasi-isomorphism, another application of 6.2.13 gives that α/φ is zero in $\mathcal{D}(R)$.

By construction, the canonical functor V preserves (co)products. □

REMARK. Let M and N be R-complexes. A *right prefraction* from M to N is a diagram in $\mathcal{K}(R)$,

$$(*) \qquad M \xrightarrow{\ \beta\ } V \xleftarrow[\simeq]{\ \psi\ } N\,,$$

where ψ is a quasi-isomorphism. Dually to 6.2.2 one can define an equivalence relation on the collection of right prefractions from M to N; the equivalence class containing the right prefraction $(*)$ is denoted $\psi\backslash\beta$ and called a *right fraction*. Like $\mathcal{D}(R)(M,N)$, the collection $\mathcal{D}'(R)(M,N)$ of all right fractions from M to N is a set. The collection of all such sets provide the hom-sets for a category $\mathcal{D}'(R)$ whose objects are all R-complexes. There is a functor $\mathcal{D}'(R) \to \mathcal{D}(R)$; it is the identity on objects and it maps a right fraction $\psi\backslash\beta$ to the left fraction ${}^{\alpha}/_{\varphi}$ for any choice of morphism α and quasi-isomorphism φ such that the diagram (6.2.1.1) in $\mathcal{K}(R)$ is commutative. The functor $\mathcal{D}'(R) \to \mathcal{D}(R)$ is an equivalence and, consequently, the derived category may just as well be constructed using right fractions. We shall soon prove that $\mathcal{D}(R)$ is a triangulated category; similarly so is $\mathcal{D}'(R)$. The equivalence $\mathcal{D}'(R) \to \mathcal{D}(R)$ is actually a triangulated functor, and hence $\mathcal{D}'(R)$ and $\mathcal{D}(R)$ are equivalent even as triangulated categories.

We shall not pursue the right fraction point of view beyond this remark, even though it does have certain advantages. For example, as the proof of 6.2.14 reveals, the argument for existence of coproducts in $\mathcal{D}(R)$ is more straightforward than the one proving existence of products. This is because left fractions mesh better with coproducts than with products. Dually, it is straightforward to show existence of products in $\mathcal{D}'(R)$, but slightly more involved to establish the existence of coproducts. Had we proved the equivalence between $\mathcal{D}'(R)$ and $\mathcal{D}(R)$, existence of products in $\mathcal{D}(R)$ would follow immediately from the existence of products in $\mathcal{D}'(R)$.

6.2.15 Lemma. *Let α, β, and γ be morphisms in $\mathcal{K}(R)$. If $\alpha\beta$ and $\beta\gamma$ are quasi-isomorphisms then α, β, and γ are quasi-isomorphisms.*

PROOF. By 6.1.10, we may assume that α, β, and γ are morphisms in $\mathcal{C}(R)$. Since $\mathrm{H}(\alpha)\,\mathrm{H}(\beta) = \mathrm{H}(\alpha\beta)$ is an isomorphism, $\mathrm{H}(\beta)$ has a left-inverse; and as $\mathrm{H}(\beta)\,\mathrm{H}(\gamma) = \mathrm{H}(\beta\gamma)$ is an isomorphism, $\mathrm{H}(\beta)$ has a right-inverse. Consequently, $\mathrm{H}(\beta)$ is an isomorphism. It follows that $\mathrm{H}(\alpha)$ and $\mathrm{H}(\gamma)$ are isomorphisms as well. □

6.2.16. It is straightforward to verify that if (α,φ) and (α',φ') are left prefractions such that ${}^{\alpha}/_{\varphi} = {}^{\alpha'}/_{\varphi'}$ holds in $\mathcal{D}(R)$, then α is a quasi-isomorphism if and only if α' is a quasi-isomorphism; cf. (6.2.2.1).

The next result describes the isomorphisms in the derived category. It also explains why isomorphisms in $\mathcal{D}(R)$ are marked by the same symbol '$\simeq$' as quasi-isomorphisms in $\mathcal{C}(R)$ and $\mathcal{K}(R)$ and not by the usual '$\cong$', which is used for isomorphisms in abstract categories—triangulated categories included; see Appn. A.

6.2.17 Proposition. *A morphism ${}^{\alpha}/_{\varphi}$ in $\mathcal{D}(R)$ is an isomorphism if and only if α is a quasi-isomorphism, in which case one has $({}^{\alpha}/_{\varphi})^{-1} = {}^{\varphi}/_{\alpha}$.*

PROOF. Let ${}^{\alpha}/_{\varphi}$ be a morphism from M to N.

If α is a quasi-isomorphism, then ${}^{\varphi}/_{\alpha}$ is a morphism in $\mathcal{D}(R)$ from N to M. One has $({}^{\varphi}/_{\alpha})({}^{\alpha}/_{\varphi}) = {}^{\varphi}/_{\varphi} = (1^M\varphi)/(1^M\varphi) = 1^M/1^M$, where the last equality follows from 6.2.5. Similarly, one has $({}^{\alpha}/_{\varphi})({}^{\varphi}/_{\alpha}) = {}^{\alpha}/_{\alpha} = (1^N\alpha)/(1^N\alpha) = 1^N/1^N$, whence ${}^{\alpha}/_{\varphi}$ is an isomorphism with inverse ${}^{\varphi}/_{\alpha}$.

Conversely, assume that ${}^{\alpha}/_{\varphi}$ is an isomorphism. Denote by U the common source of α and φ. By the arguments above, ${}^{1^U}/_{\varphi}$ is an isomorphism, and since one has ${}^{\alpha}/_{\varphi} = ({}^{\alpha}/1^U)({}^{1^U}/_{\varphi})$ by 6.2.10, it follows that ${}^{\alpha}/1^U$ is an isomorphism; denote by β/ψ its inverse. From 6.2.16 and the equalities $1^N/1^N = ({}^{\alpha}/1^U)(\beta/\psi) = (\alpha\beta)/\psi$ it follows that $\alpha\beta$ is a quasi-isomorphism. Furthermore, one has $1^U/1^U = (\beta/\psi)({}^{\alpha}/1^U) = (\beta\gamma)/\chi$ for some morphism γ and quasi-isomorphism χ. Another application of 6.2.16 gives that $\beta\gamma$ is a quasi-isomorphism, and hence α is a quasi-isomorphism by 6.2.15. □

Quasi-isomorphisms of complexes yield isomorphisms in the derived category.

6.2.18 Corollary. *For every quasi-isomorphism $M \xrightarrow{\alpha} N$ of R-complexes the fraction $\mathrm{V}(\alpha) = {}^{\alpha}/1^{M}$ is an isomorphism in $\mathcal{D}(R)$ with inverse ${}_{1}{}^{M}/\alpha$.* □

For complexes with certain lifting properties there is a conceptual converse to the corollary. That is, isomorphisms in $\mathcal{D}(R)$ yield quasi-isomorphisms of complexes.

6.2.19 Proposition. *Let P and M be R-complexes. If P is semi-projective and M and P are isomorphic in $\mathcal{D}(R)$, then there is a quasi-isomorphism $P \xrightarrow{\simeq} M$ in $\mathcal{C}(R)$.*

PROOF. If P and M are isomorphic in $\mathcal{D}(R)$, then by 6.2.17 there are quasi-isomorphisms $P \xleftarrow{\simeq} U \xrightarrow{\simeq} M$. By 6.1.25 there exists a quasi-isomorphism $P \xrightarrow{\simeq} U$ which composes with $U \xrightarrow{\simeq} M$ to give the desired quasi-isomorphism. □

6.2.20 Proposition. *Let I and M be R-complexes. If I is semi-injective and M and I are isomorphic in $\mathcal{D}(R)$, then there is a quasi-isomorphism $M \xrightarrow{\simeq} I$ in $\mathcal{C}(R)$.*

PROOF. If M and I are isomorphic in $\mathcal{D}(R)$, then there exist quasi-isomorphisms $M \xleftarrow{\simeq} U \xrightarrow{\simeq} I$ by 6.2.17. Now 6.2.1 yields quasi-isomorphisms $M \xrightarrow{\simeq} V \xleftarrow{\simeq} I$. It follows from 6.1.26 that there is a quasi-isomorphism $V \xrightarrow{\simeq} I$ which composes with $M \xrightarrow{\simeq} V$ to give the desired quasi-isomorphism. □

REMARK. Existence of an isomorphism $M \xrightarrow{\simeq} N$ in $\mathcal{D}(R)$ does not imply that there is even a morphism $M \to N$ in $\mathcal{C}(R)$; see E 6.2.2.

It follows from 6.2.17 that complexes that are isomorphic in the derived category have isomorphic homology. As the next example shows, the converse is not true.

6.2.21 Example. Over the ring $\mathbb{Z}/4\mathbb{Z}$ consider the complexes

$$P = 0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0 \quad \text{and}$$

$$M = 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

concentrated in degrees 0 and 1. Evidently, one has $\mathrm{H}(P) \cong M \cong \mathrm{H}(M)$. The complex P is semi-projective by 5.2.7, so if P and M were isomorphic in $\mathcal{D}(\mathbb{Z}/4\mathbb{Z})$, then by 6.2.19 there would exist a quasi-isomorphism $P \to M$. However, it is straightforward to verify that every morphism $\alpha\colon P \to M$ in $\mathcal{C}(\mathbb{Z}/4\mathbb{Z})$ has $\mathrm{H}_1(\alpha) = 0$.

The complexes that are isomorphic to 0 in the homotopy category are precisely the contractible complexes. The next result explains what is means for a complex to be isomorphic to 0 in the derived category.

6.2.22 Proposition. *An R-complex is isomorphic to 0 in $\mathcal{D}(R)$ if and only if it is acyclic.*

PROOF. If M is acyclic, then the morphism $0 \to M$ in $\mathcal{C}(R)$, and hence also $0 \to M$ in $\mathcal{K}(R)$, is a quasi-isomorphism. By 6.2.17 the left prefraction $0 \xleftarrow{=} 0 \xrightarrow{\simeq} M$ in $\mathcal{K}(R)$ gives an isomorphism from 0 to M in $\mathcal{D}(R)$.

If M is isomorphic to 0 in $\mathcal{D}(R)$, then $\mathcal{D}(R)(M,M)$ consists of a single element. In particular, ${}^{1^M}\!/\!{}_{1^M} = {}^{0}\!/\!{}_{1^M}$ holds, and 6.2.16 implies that the zero morphism $M \to M$ in $\mathcal{K}(R)$, and hence in $\mathcal{C}(R)$, is a quasi-isomorphism. Thus M is acyclic. □

6.2.23 Lemma. *Assume that R is a left principal ideal domain. For every R-complex L of free modules there is a quasi-isomorphism $L \xrightarrow{\simeq} \mathrm{H}(L)$.*

PROOF. For every $v \in \mathbb{Z}$ the exact sequence $0 \to \mathrm{Z}_v(L) \to L_v \to \mathrm{B}_{v-1}(L) \to 0$ splits. Indeed, $\mathrm{B}_{v-1}(L)$ is a submodule of the free module L_{v-1}, so by 1.3.10 it is itself free and, in particular, projective. Thus, by 1.3.17 there are isomorphisms

$$(\star) \qquad L_v \longrightarrow \mathrm{Z}_v(L) \oplus \mathrm{B}_{v-1}(L)\ .$$

For every v let K^v be the complex $0 \to \mathrm{B}_v(L) \to \mathrm{Z}_v(L) \to 0$ concentrated in degrees $v+1$ and v; notice that one has $\mathrm{H}(K^v) = \mathrm{H}_v(L)$. The isomorphisms $(\star)$ yield an isomorphism of complexes $L \to \coprod_{v\in\mathbb{Z}} K^v = K$, and there is an obvious quasi-isomorphism $K \xrightarrow{\simeq} \mathrm{H}(K) \cong \mathrm{H}(L)$. □

6.2.24 Proposition. *Assume that R is a left principal ideal domain. For every R-complex M there is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathcal{D}(R)$.*

PROOF. Pick by 5.1.7 a semi-free resolution $\pi\colon L \xrightarrow{\simeq} M$. It follows from 6.2.23 that there there is a quasi-isomorphism $\alpha\colon L \to \mathrm{H}(L)$; now ${}^{\mathrm{H}(\pi)\alpha}\!/\!{}_{\pi}$ is an isomorphism in $\mathcal{D}(R)$ from M to $\mathrm{H}(M)$. □

6.2.25 Proposition. *There is a unique endofunctor on $\mathcal{D}(R)$ that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{K}(R) & \xrightarrow{\ \mathrm{V}\ } & \mathcal{D}(R) \\ {\scriptstyle\Sigma}\big\downarrow & & \vdots \\ \mathcal{K}(R) & \xrightarrow[\ \mathrm{V}\ ]{} & \mathcal{D}(R)\ . \end{array}$$

This functor is denoted $\Sigma_{\mathcal{D}}$; it is $\Bbbk$-linear and an isomorphism. For a morphism ${}^{\alpha}\!/\!{}_{\varphi}$ in $\mathcal{D}(R)$ one has $\Sigma_{\mathcal{D}}({}^{\alpha}\!/\!{}_{\varphi}) = {}^{(\Sigma\alpha)}\!/\!{}_{(\Sigma\varphi)}$.

PROOF. It is elementary to verify that the endofunctor on $\mathcal{D}(R)$ that maps an object M to ΣM and a morphism ${}^{\alpha}\!/\!{}_{\varphi}$ in $\mathcal{D}(R)$ to ${}^{(\Sigma\alpha)}\!/\!{}_{(\Sigma\varphi)}$ has the asserted properties. □

When there is no risk of ambiguity, we write Σ for the functor $\Sigma_{\mathcal{D}}$.

TRIANGULATION

Consider the $\Bbbk$-linear category $\mathcal{D}(R)$, see 6.2.14, equipped with the $\Bbbk$-linear autofunctor $\Sigma = \Sigma_{\mathcal{D}}$ from 6.2.25. One may now speak of candidate triangles in $\mathcal{D}(R)$ in the sense of A.1.

6.2.26 Definition. A candidate triangle in $\mathcal{D}(R)$ is called a *distinguished triangle* if it is isomorphic to the image of a distinguished triangle in $\mathcal{K}(R)$ under the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$.

The canoncal functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ maps a morphism $\alpha\colon M \to N$ to ${}^{\alpha}/1^{M}$. It is often convenient to omit the superscript M and use the abridged notation ${}^{\alpha}/1$.

6.2.27 Lemma. *Let $\pi\colon P \xrightarrow{\simeq} M$ and $\rho\colon F \xrightarrow{\simeq} N$ be semi-projective resolutions. For every morphism ${}^{\alpha}/\varphi\colon M \to N$ in $\mathcal{D}(R)$ there exists a unique morphism $\tilde{\alpha}\colon P \to F$ in $\mathcal{K}(R)$ such that the following diagram in $\mathcal{D}(R)$ is commutative,*

$$\begin{array}{ccc} P & \xrightarrow{\pi/1} & M \\ {\scriptstyle \tilde{\alpha}/1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha/\varphi} \\ F & \xrightarrow{\rho/1} & N\,. \end{array}$$

PROOF. For uniqueness, it must be shown that the only morphism $\beta\colon P \to F$ in $\mathcal{K}(R)$ with $({}^{\rho}/1)({}^{\beta}/1) = {}^{0}/1$ is $\beta = 0$. The composite $({}^{\rho}/1)({}^{\beta}/1)$ is equal to $(\rho\beta)/1$, and thus 6.2.13 yields a quasi-isomorphism μ with $\rho\beta\mu = 0$. Since μ is a quasi-isomorphism with the semi-projective complex P as target, μ has a right inverse by by 6.1.25. Thus the equality $\rho\beta\mu = 0$ implies $\rho\beta = 0$. As β has the semi-projective complex P as source, it follows from $\rho\beta = 0$ and 6.1.22 that one has $\beta = 0$.

For existence, denote by U the domain of α and φ, and apply 6.1.21 to get morphisms γ and $\tilde{\alpha}$ that make the following diagrams in $\mathcal{K}(R)$ commutative,

$$\begin{array}{ccc} & & P \\ & {\scriptstyle\gamma}\swarrow & \big\downarrow{\scriptstyle\pi} \\ U & \xrightarrow[\varphi]{\simeq} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & P \\ & {\scriptstyle\tilde{\alpha}}\swarrow & \big\downarrow{\scriptstyle\alpha\gamma} \\ F & \xrightarrow[\rho]{\simeq} & N\,. \end{array}$$

Now one has $({}^{\alpha}/\varphi)({}^{\pi}/1) = (\alpha\gamma)/1$, since there is a commutative diagram,

$$\begin{array}{ccccccc} & & & P & & & \\ & & {\scriptstyle 1}\swarrow & \simeq & \searrow{\scriptstyle\gamma} & & \\ & P & & & & U & \\ {\scriptstyle 1}\swarrow\ \simeq & & \searrow{\scriptstyle\pi} & & {\scriptstyle\varphi}\swarrow\ \simeq & & \searrow{\scriptstyle\alpha} \\ P & & & M & & & N\,. \end{array}$$

Thus there are equalities $({}^{\rho}/1)({}^{\tilde{\alpha}}/1) = (\rho\tilde{\alpha})/1 = (\alpha\gamma)/1 = ({}^{\alpha}/\varphi)({}^{\pi}/1)$. $\square$

6.2.28 Lemma. *Consider a distinguished triangle $M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X \xrightarrow{\gamma'} \Sigma M'$ in $\mathcal{K}(R)$, and a commutative diagram in $\mathcal{D}(R)$,*

(6.2.28.1)
$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha'/1} & N' & \xrightarrow{\beta'/1} & X' & \xrightarrow{\gamma'/1} & \Sigma_{\mathcal{D}}M' \\ \downarrow\simeq & & \downarrow\simeq & & \vdots & & \downarrow\simeq \\ M & \xrightarrow[\alpha/\varphi]{} & N & \xrightarrow[\beta/\psi]{} & X & \xrightarrow[\gamma/\chi]{} & \Sigma_{\mathcal{D}}M \end{array},$$

where the bottom row is a distinguished triangle, $M' \to M$ and $N' \to N$ are isomorphisms, and the isomorphism $\Sigma_{\mathcal{D}}M' \to \Sigma_{\mathcal{D}}M$ is induced by $M' \to M$. There exists an isomorphism $X' \to X$ in $\mathcal{D}(R)$ that makes (6.2.28.1) *commutative.*

PROOF. By the definition of distinguished triangles in $\mathcal{D}(R)$, see 6.2.26, there exist a distinguished triangle in $\mathcal{K}(R)$, say,

$$M'' \xrightarrow{\alpha''} N'' \xrightarrow{\beta''} X'' \xrightarrow{\gamma''} \Sigma M'' ,$$

and an isomorphism of distinguished triangles in $\mathcal{D}(R)$,

$(\star)$
$$\begin{array}{ccccccc} M & \xrightarrow{\alpha/\varphi} & N & \xrightarrow{\beta/\psi} & X & \xrightarrow{\gamma/\chi} & \Sigma_{\mathcal{D}}M \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ M'' & \xrightarrow[\alpha''/1]{} & N'' & \xrightarrow[\beta''/1]{} & X'' & \xrightarrow[\gamma''/1]{} & \Sigma_{\mathcal{D}}M'' \end{array}.$$

Denote the composite isomorphisms $M' \to M \to M''$ and $N' \to N \to N''$ by ${}^{\mu}\!/\zeta$ and ${}^{\nu}\!/\eta$, respectively. Then there is a commutative diagram in $\mathcal{D}(R)$,

$(\ddagger)$
$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha'/1} & N' & \xrightarrow{\beta'/1} & X' & \xrightarrow{\gamma'/1} & \Sigma_{\mathcal{D}}M' \\ \simeq\downarrow \mu/\zeta & & \simeq\downarrow \nu/\eta & & \vdots & & \simeq\downarrow \Sigma_{\mathcal{D}}(\mu/\zeta) \\ M'' & \xrightarrow[\alpha''/1]{} & N'' & \xrightarrow[\beta''/1]{} & X'' & \xrightarrow[\gamma''/1]{} & \Sigma_{\mathcal{D}}M'' \end{array}.$$

If one can construct an isomorphism $X' \to X''$ in $\mathcal{D}(R)$ that makes $(\ddagger)$ commutative, then the composite of this isomorphism with the inverse of the isomorphism $X \to X''$ from $(\star)$ evidently gives an isomorphism $X' \to X$ that makes (6.2.28.1) commutative, as desired. To find $X' \to X''$, apply [40, Lemma 2.1.38]. □

6.2.29 Theorem. *The derived category $\mathcal{D}(R)$, equipped with the autofunctor Σ and the collection of distinguished triangles defined in* 6.2.26, *is triangulated. Furthermore, the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ is triangulated.*

PROOF. By 6.2.25 one has $\mathrm{V}\Sigma = \Sigma_{\mathcal{D}}\mathrm{V}$. Thus, once it has been established that the category $\mathcal{D}(R)$ is triangulated, the functor V is evidently triangulated by 6.2.26.

(TR0): Follows immediately from the definition of distinguished triangles in $\mathcal{D}(R)$ and the fact that $(\mathcal{K}(R),\Sigma)$ satisfies (TR0).

(TR1): Let $\alpha/\varphi\colon M \to N$ be a morphism in $\mathcal{D}(R)$ and denote by U the common source of α and φ. As $(\mathcal{K}(R),\Sigma)$ satisfies (TR1), the morphism α fits in a distinguished triangle in $\mathcal{K}(R)$, say,

$$U \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma U\ .$$

Applying V to this distinguished triangle in $\mathcal{K}(R)$, one gets by 6.2.26 a distinguished triangle in $\mathcal{D}(R)$, namely the top row in the commutative diagram

$$\begin{array}{ccccccc}
U & \xrightarrow{\mathrm{V}(\alpha)} & N & \xrightarrow{\mathrm{V}(\beta)} & X & \xrightarrow{\mathrm{V}(\gamma)} & \Sigma_{\mathcal{D}}U \\
{\scriptstyle \mathrm{V}(\varphi)}\downarrow & & \| & & \| & & \downarrow{\scriptstyle \Sigma_{\mathcal{D}}\mathrm{V}(\varphi)} \\
M & \xrightarrow[\alpha/\varphi]{} & N & \xrightarrow[\mathrm{V}(\beta)]{} & X & \xrightarrow[\Sigma_{\mathcal{D}}\mathrm{V}(\varphi)\circ\mathrm{V}(\gamma)]{} & \Sigma_{\mathcal{D}}M\ .
\end{array}$$

Since $\mathrm{V}(\varphi) = {}^{\varphi}/1^{U}$ is an isomorphism by 6.2.17, the bottom candidate triangle is also distinguished.

(TR2'): Follows immediately since $(\mathcal{K}(R),\Sigma)$ satisfies (TR2').

(TR4'): Consider a commutative diagram in $\mathcal{D}(R)$,

$$(\star)\qquad \begin{array}{ccccccc}
M^1 & \xrightarrow{\alpha^1/\varphi^1} & N^1 & \xrightarrow{\beta^1/\psi^1} & X^1 & \xrightarrow{\gamma^1/\chi^1} & \Sigma_{\mathcal{D}}M^1 \\
\downarrow{\scriptstyle \mu/\zeta} & & \downarrow{\scriptstyle \nu/\eta} & & & & \downarrow{\scriptstyle \Sigma_{\mathcal{D}}(\mu/\zeta)} \\
M^2 & \xrightarrow[\alpha^2/\varphi^2]{} & N^2 & \xrightarrow[\beta^2/\psi^2]{} & X^2 & \xrightarrow[\gamma^2/\chi^2]{} & \Sigma_{\mathcal{D}}M^2\ ,
\end{array}$$

where the rows are distinguished triangles. A morphism $\lambda/\theta\colon X^1 \to X^2$ must be constructed that makes $(\star)$ commutative, and such that the mapping cone candidate triangle of $(\mu/\zeta, \nu/\eta, \lambda/\theta)$ in $\mathcal{D}(R)$ is distinguished. Choose in $\mathcal{K}(R)$ semi-projective resolutions $\pi^i\colon P^i \xrightarrow{\simeq} M^i$ and $\rho^i\colon F^i \xrightarrow{\simeq} N^i$ for $i = 1,2$, and let

$$\tilde{\alpha}^i : P^i \longrightarrow F^i\ , \quad \tilde{\mu} : P^1 \longrightarrow P^2\ , \quad \text{and} \quad \tilde{\nu} : F^1 \longrightarrow F^2$$

be the unique morphisms in $\mathcal{K}(R)$ that make the following diagrams commutative,

$$\begin{array}{ccc}
P^i & \xrightarrow{\pi^i/1} & M^i \\
{\scriptstyle \tilde{\alpha}^i/1}\downarrow & & \downarrow{\scriptstyle \alpha^i/\varphi^i} \\
F^i & \xrightarrow{\rho^i/1} & N^i
\end{array}\ , \qquad
\begin{array}{ccc}
P^1 & \xrightarrow{\pi^1/1} & M^1 \\
{\scriptstyle \tilde{\mu}/1}\downarrow & & \downarrow{\scriptstyle \mu/\zeta} \\
P^2 & \xrightarrow{\pi^2/1} & M^2
\end{array}\ , \quad \text{and} \quad
\begin{array}{ccc}
F^1 & \xrightarrow{\rho^1/1} & N^1 \\
{\scriptstyle \tilde{\nu}/1}\downarrow & & \downarrow{\scriptstyle \nu/\eta} \\
F^2 & \xrightarrow{\rho^2/1} & N^2
\end{array}\ ;$$

see 6.2.27. Since the homotopy category is triangulated, the morphisms $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ fit in distinguished triangles in $\mathcal{K}(R)$, which are the rows in the following diagram,

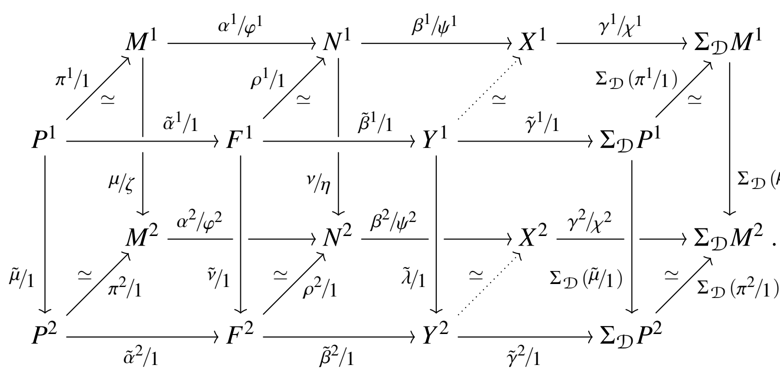
$$(\ddagger)\qquad \begin{array}{ccccccc} P^1 & \xrightarrow{\tilde{\alpha}^1} & F^1 & \xrightarrow{\tilde{\beta}^1} & Y^1 & \xrightarrow{\tilde{\gamma}^1} & \Sigma P^1 \\ {\scriptstyle \tilde{\mu}}\downarrow & & \downarrow{\scriptstyle \tilde{\nu}} & & \vdots\,{\scriptstyle \tilde{\lambda}} & & {\scriptstyle \Sigma\tilde{\mu}}\downarrow \\ P^2 & \xrightarrow{\tilde{\alpha}^2} & F^2 & \xrightarrow{\tilde{\beta}^2} & Y^2 & \xrightarrow{\tilde{\gamma}^2} & \Sigma P^2 \end{array}.$$

The left-hand square in (‡) is commutative. Indeed, as $(\alpha^2/\varphi^2)(\mu/\zeta) = (\nu/\eta)(\alpha^1/\varphi^1)$ holds, both morphisms $\tilde{\alpha}^2\tilde{\mu}$ and $\tilde{\nu}\tilde{\alpha}^1$ make the diagram

$$\begin{array}{ccc} P^1 & \cdots\cdots\cdots\longrightarrow & F^2 \\ {\scriptstyle \pi^1/1}\downarrow & & \downarrow{\scriptstyle \rho^2/1} \\ M^1 & \xrightarrow{(\alpha^2/\varphi^2)(\mu/\zeta) = (\nu/\eta)(\alpha^1/\varphi^1)} & N^2 \end{array}$$

in $\mathcal{D}(R)$ commutative, and it follows from 6.2.27 that one has $\tilde{\alpha}^2\tilde{\mu} = \tilde{\nu}\tilde{\alpha}^1$. As $\mathcal{K}(R)$ is triangulated, there exists a morphism $\tilde{\lambda}\colon Y^1 \to Y^2$ in $\mathcal{K}(R)$ that makes (‡) commutative, and such that the mapping cone candidate triangle $\tilde{\Delta}$ in $\mathcal{K}(R)$ of the morphism $(\tilde{\mu},\tilde{\nu},\tilde{\lambda})$ is distinguished.

In the diagram (⋄) below, the "front" is the image of (‡) under the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$, and the "back" is the diagram (⋆),

(⋄) 

The mapping cone candidate triangle Δ' in $\mathcal{D}(R)$ of the morphism $(\tilde{\mu}/1, \tilde{\nu}/1, \tilde{\lambda}/1)$ is isomorphic to the image of the mapping cone distinguished triangle $\tilde{\Delta}$ of $(\tilde{\mu},\tilde{\nu},\tilde{\lambda})$ under the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$. Hence, Δ' is distinguished.

The morphisms $\pi^i/1 : P^i \to M^i$ and $\rho^i/1 : F^i \to N^i$ for $i = 1,2$ are isomorphisms in $\mathcal{D}(R)$ by 6.2.17, and thus 6.2.28 yields (dotted) isomorphisms $Y_1 \to X_1$ and $Y_2 \to X_2$ that make the "top" and "bottom" in (⋄) commutative. Now, let λ/θ be the unique morphism in $\mathcal{D}(R)$ that makes the "interior wall"

$$\begin{array}{ccc} Y^1 & \xrightarrow{\simeq} & X^1 \\ {\scriptstyle \tilde{\lambda}/1}\downarrow & & \vdots\,{\scriptstyle \lambda/\theta} \\ Y^2 & \xrightarrow{\simeq} & X^2 \end{array}$$

in (⋄) commutative. A straightforward diagram chase reveals that this morphism λ/θ makes the "back" in (⋄) commutative. It follows from A.8 that the mapping cone candidate triangle $\varDelta$ of $({}^{\mu}\!/\zeta, {}^{\nu}\!/\eta, \lambda/\theta)$ is isomorphic to the mapping cone candidate triangle $\varDelta'$ of $({}^{\tilde{\mu}}\!/1, {}^{\tilde{\nu}}\!/1, \tilde{\lambda}/1)$, and since $\varDelta'$ is distinguished, so is $\varDelta$. □

REMARK. Let $\mathcal{K}_{\mathrm{prj}}(R)$ denote the full subcategory of $\mathcal{K}(R)$ whose objects are the semi-projective R-complexes. The proof of 6.1.15 shows that $(\mathcal{K}_{\mathrm{prj}}(R), \Sigma)$ is a triangulated category, albeit not a triangulated subcategory of $\mathcal{K}(R)$; see E 6.1.13. The composite functor $\mathcal{K}_{\mathrm{prj}}(R) \to \mathcal{K}(R) \to \mathcal{D}(R)$ is a triangulated equivalence; see E 6.2.4.

THE UNIVERSAL PROPERTY

The triangulated functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ has a universal property described in the next theorem.

6.2.30 Theorem. *Let $\mathcal{U}$ be a category and let $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{U}$ be a functor. If* F *maps quasi-isomorphisms to isomorphisms, then there exists a unique functor* F' *that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{K}(R) & \xrightarrow{\ \mathrm{V}\ } & \mathcal{D}(R) \\ {\scriptstyle \mathrm{F}}\big\downarrow & \swarrow{\scriptstyle \mathrm{F}'} & \\ \mathcal{U}\,; & & \end{array}$$

here V *is the canonical functor from* 6.2.12*. For an* R*-complex* M *there is an equality* $\mathrm{F}'(M) = \mathrm{F}(M)$*, and for a morphism* ${}^{\alpha}\!/\varphi$ *in* $\mathcal{D}(R)$ *one has* $\mathrm{F}'({}^{\alpha}\!/\varphi) = \mathrm{F}(\alpha)\mathrm{F}(\varphi)^{-1}$*. Furthermore, the following assertions hold.*

(a) *Assume that* $\mathcal{U}$ *is* $\Bbbk$*-prelinear; then* F' *is* $\Bbbk$*-linear if and only if* F *is* $\Bbbk$*-linear.*

(b) *Assume that* $\mathcal{U}$ *has (co)products; then* F' *preserves (co)products if and only if* F *preserves (co)products.*

(c) *Assume that* $\mathcal{U}$ *is triangulated; then* F' *is triangulated if and only if* F *is triangulated.*

PROOF. For uniqueness of the functor F', assume that $\mathrm{F}'\mathrm{V} = \mathrm{F}$ holds. Since V is the identity on objects, one has $\mathrm{F}'(M) = \mathrm{F}(M)$ for every R-complex M. A morphism ${}^{\alpha}\!/\varphi$ in $\mathcal{D}(R)$ can be written ${}^{\alpha}\!/\varphi = ({}^{\alpha}\!/1)({}^{1}\!/\varphi) = ({}^{\alpha}\!/1)({}^{\varphi}\!/1)^{-1} = \mathrm{V}(\alpha)\mathrm{V}(\varphi)^{-1}$ by 6.2.10 and 6.2.17, and thus there are equalities

$$\mathrm{F}'({}^{\alpha}\!/\varphi) = \mathrm{F}'(\mathrm{V}(\alpha)\mathrm{V}(\varphi)^{-1}) = (\mathrm{F}'\mathrm{V}(\alpha))(\mathrm{F}'\mathrm{V}(\varphi))^{-1} = \mathrm{F}(\alpha)\mathrm{F}(\varphi)^{-1}\,.$$

Consequently, the functor F' is uniquely determined by F.

For existence, notice that if ${}^{\alpha_1}\!/\varphi_1 = {}^{\alpha_2}\!/\varphi_2$ holds in $\mathcal{D}(R)$, then there is an equality $\mathrm{F}(\alpha_1)\mathrm{F}(\varphi_1)^{-1} = \mathrm{F}(\alpha_2)\mathrm{F}(\varphi_2)^{-1}$ in $\mathcal{U}$. Indeed, if one has ${}^{\alpha_1}\!/\varphi_1 = {}^{\alpha_2}\!/\varphi_2$, then by 6.2.2 there exist a morphism α and quasi-isomorphisms φ, μ_1, and μ_2 in $\mathcal{K}(R)$ such that

$\alpha_1\mu_1 = \alpha = \alpha_2\mu_2$ and $\varphi_1\mu_1 = \varphi = \varphi_2\mu_2$ hold. It follows that $\mathrm{F}(\alpha_1)\mathrm{F}(\mu_1) = \mathrm{F}(\alpha)$ and $\mathrm{F}(\varphi_1)\mathrm{F}(\mu_1) = \mathrm{F}(\varphi)$ hold and, consequently, there are equalities

$$\mathrm{F}(\alpha_1)\mathrm{F}(\varphi_1)^{-1} = \mathrm{F}(\alpha)\mathrm{F}(\mu_1)^{-1}\mathrm{F}(\mu_1)\mathrm{F}(\varphi)^{-1} = \mathrm{F}(\alpha)\mathrm{F}(\varphi)^{-1} .$$

Similarly one finds $\mathrm{F}(\alpha_2)\mathrm{F}(\varphi_2)^{-1} = \mathrm{F}(\alpha)\mathrm{F}(\varphi)^{-1}$. Thus, one can set $\mathrm{F}'(M) = \mathrm{F}(M)$ for R-complexes M and $\mathrm{F}'({}^{\alpha}/\!_{\varphi}) = \mathrm{F}(\alpha)\mathrm{F}(\varphi)^{-1}$ for morphisms ${}^{\alpha}/\!_{\varphi}$ in $\mathcal{D}(R)$. With this definition, one evidently has $\mathrm{F}'\mathrm{V} = \mathrm{F}$.

In order for F' to be a functor, it must preserve identity morphisms and respect composition. By definition, $\mathrm{F}'({}^{1^M}/\!_{1^M}) = \mathrm{F}(1^M)\mathrm{F}(1^M)^{-1} = 1^{\mathrm{F}(M)} = 1^{\mathrm{F}'(M)}$ holds for every R-complex M. Let ${}^{\alpha}/\!_{\varphi}$ and ${}^{\beta}/\!_{\psi}$ be composable morphisms in $\mathcal{D}(R)$. By 6.2.8 the composition $({}^{\alpha}/\!_{\varphi})({}^{\beta}/\!_{\psi})$ is ${}^{(\alpha\gamma)}/\!_{(\psi\chi)}$ for any choice of morphism γ and quasi-isomorphism χ in $\mathcal{K}(R)$ with $\beta\chi = \varphi\gamma$. Thus there are equalities,

$$\begin{aligned}
\mathrm{F}'(({}^{\alpha}/\!_{\varphi})({}^{\beta}/\!_{\psi})) &= \mathrm{F}'({}^{(\alpha\gamma)}/\!_{(\psi\chi)}) \\
&= \mathrm{F}(\alpha\gamma)\mathrm{F}(\psi\chi)^{-1} \\
&= \mathrm{F}(\alpha)\mathrm{F}(\gamma)\mathrm{F}(\chi)^{-1}\mathrm{F}(\psi)^{-1} \\
&= \mathrm{F}(\alpha)\mathrm{F}(\varphi)^{-1}\mathrm{F}(\beta)\mathrm{F}(\psi)^{-1} \\
&= \mathrm{F}'({}^{\alpha}/\!_{\varphi})\mathrm{F}'({}^{\beta}/\!_{\psi}) .
\end{aligned}$$

(a): If F' is $\Bbbk$-linear, then so is $\mathrm{F} = \mathrm{F}'\mathrm{V}$ as a composite of two $\Bbbk$-linear functors. Conversely, assume that F is $\Bbbk$-linear, and let ${}^{\alpha_1}/\!_{\varphi_1}$ and ${}^{\alpha_2}/\!_{\varphi_2}$ be parallel morphisms and let x be an element in $\Bbbk$. Write ${}^{\alpha_1}/\!_{\varphi_1} = {}^{\alpha_1'}/\!_{\varphi}$ and ${}^{\alpha_2}/\!_{\varphi_2} = {}^{\alpha_2'}/\!_{\varphi}$ for morphisms α_1', α_2' and a quasi-isomorphism φ, see 6.2.6. Now 6.2.7 yields

$$\begin{aligned}
\mathrm{F}'(x({}^{\alpha_1}/\!_{\varphi_1}) + {}^{\alpha_2}/\!_{\varphi_2}) &= \mathrm{F}'({}^{(x\alpha_1' + \alpha_2')}/\!_{\varphi}) \\
&= \mathrm{F}(x\alpha_1' + \alpha_2')\mathrm{F}(\varphi)^{-1} \\
&= (x\mathrm{F}(\alpha_1') + \mathrm{F}(\alpha_2'))\mathrm{F}(\varphi)^{-1} \\
&= x\mathrm{F}(\alpha_1')\mathrm{F}(\varphi)^{-1} + \mathrm{F}(\alpha_2')\mathrm{F}(\varphi)^{-1} \\
&= x\mathrm{F}'({}^{\alpha_1'}/\!_{\varphi}) + \mathrm{F}'({}^{\alpha_2'}/\!_{\varphi}) \\
&= x\mathrm{F}'({}^{\alpha_1}/\!_{\varphi_1}) + \mathrm{F}'({}^{\alpha_2}/\!_{\varphi_2}) ,
\end{aligned}$$

and hence F' is $\Bbbk$-linear.

(b): The proof of 6.1.18(b) applies to prove part (b) in this theorem; only one has to replace the functor Q by V and the reference to 6.1.8 by one to 6.2.14.

(c): If F' is triangulated, then so is $\mathrm{F} = \mathrm{F}'\mathrm{V}$ as a composite of two triangulated functors; cf. 6.2.29. Conversely, assume that F is triangulated. By definition, there exists a natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$ such that

$$(\star) \qquad \mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X) \xrightarrow{\phi^M \circ \mathrm{F}(\gamma)} \Sigma_{\mathcal{U}}\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{U}$ for every distinguished triangle

$$(\ddagger) \qquad M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$$

in $\mathcal{K}(R)$. As one has $\mathrm{F}\Sigma = \mathrm{F}'\mathrm{V}\Sigma = \mathrm{F}'\Sigma_{\mathcal{D}}\mathrm{V}$ and $\Sigma_{\mathcal{U}}\mathrm{F} = \Sigma_{\mathcal{U}}\mathrm{F}'\mathrm{V}$, and since V is the identity on objects, ϕ can be viewed as a natural isomorphism $\phi' \colon \mathrm{F}'\Sigma_{\mathcal{D}} \to \Sigma_{\mathcal{U}}\mathrm{F}'$. We verify that the functor F' with the natural isomorphism ϕ', is triangulated. By definition, a distinguished triangle in $\mathcal{D}(R)$ has, up to isomorphism, the form

$$M \xrightarrow{\mathrm{V}(\alpha)} N \xrightarrow{\mathrm{V}(\beta)} X \xrightarrow{\mathrm{V}(\gamma)} \Sigma_{\mathcal{D}}M$$

for some distinguished triangle ($\ddagger$) in $\mathcal{K}(R)$. Hence, it must be argued that

$$(\diamond) \qquad \mathrm{F}'(M) \xrightarrow{\mathrm{F}'\mathrm{V}(\alpha)} \mathrm{F}'(N) \xrightarrow{\mathrm{F}'\mathrm{V}(\beta)} \mathrm{F}'(X) \xrightarrow{\phi'^{M} \circ \mathrm{F}'\mathrm{V}(\gamma)} \Sigma_{\mathcal{U}}\mathrm{F}'(M)$$

is a distinguished triangle; however, ($\diamond$) is exactly the distinguished triangle ($\star$). □

A morphism in $\mathcal{K}(R)^{\mathrm{op}}$ is called a quasi-isomorphism if the corresponding morphism in $\mathcal{K}(R)$ is a quasi-isomorphism in the sense of 6.1.10.

6.2.31 Theorem. *Let $\mathcal{V}$ be a category and let $\mathrm{G} \colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$ be a functor. If* G *maps quasi-isomorphisms to isomorphisms, then there exists a unique functor* G' *that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{K}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{V}^{\mathrm{op}}} & \mathcal{D}(R)^{\mathrm{op}} \\ {\scriptstyle \mathrm{G}}\downarrow & \swarrow{\scriptstyle \mathrm{G}'} & \\ \mathcal{V}\,; & & \end{array}$$

here V *is the canonical functor from* 6.2.12. *For an R-complex M there is an equality* $\mathrm{G}'(M) = \mathrm{G}(M)$, *and for a morphism* $^{\alpha}/_{\varphi}$ *in* $\mathcal{D}(R)^{\mathrm{op}}$ *one has* $\mathrm{G}'(^{\alpha}/_{\varphi}) = \mathrm{G}(\varphi)^{-1}\mathrm{G}(\alpha)$. *Furthermore, the following assertions hold.*

(a) *Assume that $\mathcal{V}$ is $\Bbbk$-prelinear; then G' is $\Bbbk$-linear if and only if* G *is $\Bbbk$-linear.*

(b) *Assume that $\mathcal{V}$ has (co)products; then G' preserves (co)products if and only if* G *preserves (co)products.*

(c) *Assume that $\mathcal{V}$ is triangulated; then G' is triangulated if and only if* G *is triangulated.*

PROOF. Apply 6.2.30 to the functor $\mathrm{G}^{\mathrm{op}} \colon \mathcal{K}(R) \to \mathcal{V}^{\mathrm{op}}$. □

EXERCISES

E 6.2.1 Let α be a morphism in $\mathcal{K}(R)$. Show that there exists a quasi-isomorphism μ with $\alpha\mu = 0$ if and only if there exists a quasi-isomorphism ν with $\nu\alpha = 0$.

E 6.2.2 Show that the complexes in 4.2.4 are isomorphic in $\mathcal{D}(R)$. *Hint:* E 5.1.8.

E 6.2.3 Let M and N be R-complexes and let $P \xrightarrow{\simeq} M$ be a semi-projective resolution. Show that there is an isomorphisms of $\Bbbk$-modules, $\mathcal{K}(R)(P,N) \cong \mathcal{D}(R)(M,N)$.

E 6.2.4 Show that $\mathcal{K}_{\mathrm{prj}}(R)$ and $\mathcal{D}(R)$ are equivalent as triangulated categories; cf. E 6.1.13.

E 6.2.5 Let M and N be R-complexes and let $N \xrightarrow{\simeq} I$ be a semi-injective resolution. Show that there is an isomorphisms of $\Bbbk$-modules, $\mathcal{K}(R)(M,I) \cong \mathcal{D}(R)(M,N)$.

E 6.2.6 Show that $\mathcal{K}_{\mathrm{inj}}(R)$ and $\mathcal{D}(R)$ are equivalent as triangulated categories; cf. E 6.1.15.

E 6.2.7 Let R be left hereditary. Show that there is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathcal{D}(R)$ for every R-complex M. *Hint:* E 5.2.3.

E 6.2.8 Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $(\mathcal{T},\Sigma)$. A morphism $\alpha\colon M \to N$ in $\mathcal{T}$ is called $\mathcal{S}$-*trivial* if in some, equivalently in every, distinguished triangle,

$$M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M\,,$$

the object X belongs to $\mathcal{S}$. Describe the $\mathcal{S}$-trivial morphisms in the category $\mathcal{K}(R)$ if $\mathcal{S}$ consists of all acyclic R-complexes; cf. E 6.1.10.

E 6.2.9 Let $(\mathcal{T},\Sigma)$ be a triangulated category. A commutative square in $\mathcal{T}$,

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & N \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ M & \xrightarrow[\beta]{} & V \end{array}$$

is called *homotopy cartesian* if there exists a distinguished triangle of the form

$$U \xrightarrow{\left(\begin{smallmatrix}\varphi\\ -\alpha\end{smallmatrix}\right)} \begin{matrix} M \\ \oplus \\ N \end{matrix} \xrightarrow{(\beta\ \ \psi)} V \xrightarrow{\ \gamma\ } \Sigma U\,.$$

The pair (φ,α) is called a *homotopy pullback* of (β,ψ), and (β,ψ) is called a *homotopy pushout* of (φ,α). Show that homotopy pushouts and homotopy pullbacks always exist.

E 6.2.10 Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $(\mathcal{T},\Sigma)$, and consider the homotopy cartesian square in $\mathcal{T}$ from E 6.2.9. Show that the morphism α is $\mathcal{S}$-trivial if and only if β is $\mathcal{S}$-trivial in the sense of E 6.2.8. *Hint:* See [40, lem. 1.5.8].