Chapter 6
The Derived Category

6.1 The Homotopy Category

SYNOPSIS. Homotopy category; (co)product; triangulation; universal property; unique lifting properties.

OBJECTS AND MORPHISMS

6.1.1 Definition. The *homotopy categeory* $\mathcal{K}(R)$ has the same objects as $\mathcal{C}(R)$, that is, R-complexes, and the morphisms in $\mathcal{K}(R)$ are homotopy equivalence classes of morphisms in $\mathcal{C}(R)$.

6.1.2. By 2.3.11 there is an equality $\mathcal{K}(R)(M,N) = \mathrm{H}_0(\mathrm{Hom}_R(M,N))$ of $\Bbbk$-modules for R-complexes M and N. In accordance with 2.2.12 we write $[\alpha]$ for the homotopy equivalence class of a morphism α in $\mathcal{C}(R)$. If K is yet another R-complex then the composition $\mathcal{K}(R)(M,N) \times \mathcal{K}(R)(K,M) \to \mathcal{K}(R)(K,N)$ maps $([\alpha],[\beta])$ to $[\alpha\beta]$.

6.1.3. Let α be a morphism in $\mathcal{C}(R)$. By definition, $[\alpha]$ is the zero morphism in $\mathcal{K}(R)$ if and only if α is null-homotopic; see 2.2.20. Furthermore, $[\alpha]$ is an isomorphism in $\mathcal{K}(R)$ if and only if α is a homotopy equivalence; see 2.2.25.

6.1.4. There is a canonical full functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$; it is the identity on objects and it maps a morphism α in $\mathcal{C}(R)$ to its homotopy equivalence class $\mathrm{Q}(\alpha) = [\alpha]$.

PRODUCTS, COPRODUCTS, AND $\Bbbk$-LINEARITY

The lemma below follows immediately from the definitions.

6.1.5 Lemma. *Let $\mathcal{U}$ and $\mathcal{V}$ be $\Bbbk$-prelinear categories that have the same objects, and let $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ be a $\Bbbk$-linear functor that is the identity on objects. If M and N*

are objects and if the tuple $(M \oplus N, \pi^M, \iota^M, \pi^N, \iota^N)$ *is a biproduct in* $\mathcal{U}$ *then the tuple* $(M \oplus N, \mathrm{F}(\pi^M), \mathrm{F}(\iota^M), \mathrm{F}(\pi^N), \mathrm{F}(\iota^N))$ *is a biproduct in* $\mathcal{V}$. *In particular, if every pair of objects has a biproduct in* $\mathcal{U}$ *then every pair of objects has a biproduct in* $\mathcal{V}$. □

When we say that a category "has (co)products", we mean that all set-indexed (co)products exist in the category; such objects are unique up to isomorphism.

6.1.6 Theorem. *The homotopy category* $\mathcal{K}(R)$ *and the functor* $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ *are* $\Bbbk$*-linear. For every family* $\{M^u\}_{u\in U}$ *of* R*-complexes the next assertions hold.*

(a) *If* M *with embeddings* $\{\iota^u\colon M^u \rightarrowtail M\}_{u\in U}$ *is the coproduct of* $\{M^u\}_{u\in U}$ *in* $\mathcal{C}(R)$, *then* M *with the morphisms* $\{[\iota^u]\}_{u\in U}$ *is the coproduct of* $\{M^u\}_{u\in U}$ *in* $\mathcal{K}(R)$.

(b) *If* M *with projections* $\{\pi^u\colon M^u \twoheadrightarrow M\}_{u\in U}$ *is the product of* $\{M^u\}_{u\in U}$ *in* $\mathcal{C}(R)$, *then* M *with the morphisms* $\{[\pi^u]\}_{u\in U}$ *is the product of* $\{M^u\}_{u\in U}$ *in* $\mathcal{K}(R)$.

In particular, the homotopy category $\mathcal{K}(R)$ *has coproducts and products, and the canonical functor* Q *preserves coproducts and products.*

PROOF. It is straightforward to verify that the category $\mathcal{K}(R)$ is $\Bbbk$-prelinear and that the canonical functor Q is $\Bbbk$-linear. Evidently, the zero complex is a zero object in $\mathcal{K}(R)$, and $\mathcal{K}(R)$ has biproducts by 6.1.5. Thus $\mathcal{K}(R)$ is a $\Bbbk$-linear category.

(a): Let $\{[\alpha^u]\colon M^u \to N\}_{u\in U}$ be morphisms in $\mathcal{K}(R)$. The task is to show that there exixts a unique morphism $[\alpha]\colon M \to N$ in $\mathcal{K}(R)$ with $[\alpha\iota^u] = [\alpha^u]$ for all $u \in U$.

Existence is straightforward; indeed, by the universal property of coproducts in $\mathcal{C}(R)$, there exists a (unique) morphism $\alpha\colon M \to N$ with $\alpha\iota^u = \alpha^u$ for all $u \in U$. Applying Q to these identities one gets $[\alpha\iota^u] = [\alpha^u]$.

For uniqueness, assume that $[\alpha\iota^u] = [0]$ holds for all $u \in U$; it must be shown that $[\alpha]$ is $[0]$. Since each $\alpha\iota^u$ is null-homotopic there are degree 1 homomorphisms $\tau^u\colon M^u \to N$ such that $\alpha\iota^u = \partial^N\tau^u + \tau^u\partial^{M^u}$ holds for all $u \in U$. Now consider each homomorphism τ^u as a morphism $M^{u\natural} \to \Sigma N^\natural$ of graded R-modules. Since $M^\natural$ together with the embeddings $\{\iota^u\colon M^{u\natural} \rightarrowtail M^\natural\}_{u\in U}$ is a coproduct of $\{M^{u\natural}\}_{u\in U}$ in $\mathcal{M}_{\mathrm{gr}}(R)$, there is a morphism $\tau\colon M^\natural \to \Sigma N^\natural$ with $\tau\iota^u = \tau^u$ for all $u \in U$. Viewing τ as a degree 1 homomorphism $M \to N$, it follows that one has

$$\alpha\iota^u = \partial^N\tau^u + \tau^u\partial^{M^u} = \partial^N\tau\iota^u + \tau\iota^u\partial^{M^u} = (\partial^N\tau + \tau\partial^M)\iota^u\ ,$$

where the second equality is by definition of τ, and the third equality holds as ι^u is a morphism in $\mathcal{C}(R)$. As $\partial^N\tau + \tau\partial^M$ is a morphism of R-complexes, it follows from the universal property of coproducts in $\mathcal{C}(R)$ that one has $\alpha = \partial^N\tau + \tau\partial^M$. Thus α is null-homotopic, that is, $[\alpha]$ is $[0]$ as desired.

(b): Similar to the proof of part (a).

By construction, the canonical functor Q preserves (co)products. □

The following result shows that the zero objects in the homotopy category are exactly the contractible complexes. Further characterizations of such complexes are given in 4.1.23.

6.1.7 Proposition. *An* R*-complex is isomorphic to* 0 *in* $\mathcal{K}(R)$ *if and only if it is contractible.*

PROOF. Let M be an R-complex. If M is isomorphic to 0 in $\mathcal{K}(R)$ then $\mathcal{K}(R)(M,M)$ consists of a single element. In particular, $[1^M] = [0]$ holds, so 1^M is null-homotopic, i.e. M is contractible. Conversely, if M is contractible then the morphism $M \to 0$ in $\mathcal{C}(R)$ is a homotopy equivalence, whence it represents an isomorphism in $\mathcal{K}(R)$. □

6.1.8 Proposition. *There is a unique endofunctor on $\mathcal{K}(R)$ that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R) & \xrightarrow{\mathrm{Q}} & \mathcal{K}(R) \\ {\scriptstyle\Sigma}\downarrow & & \vdots \\ \mathcal{C}(R) & \xrightarrow[\mathrm{Q}]{} & \mathcal{K}(R)\,. \end{array}$$

This functor is denoted $\Sigma_{\mathcal{K}}$; it is $\Bbbk$-linear and invertible.

PROOF. It is elementary to verify that the endofunctor on $\mathcal{K}(R)$ that maps an object M to ΣM and a morphism $[\alpha]$ to $[\Sigma\alpha]$ has the asserted properties. □

When there is no risk of ambiguity, we write Σ for the functor $\Sigma_{\mathcal{K}}$.

TRIANGULATION

Consider the $\Bbbk$-linear category $\mathcal{K}(R)$, see 6.1.6, equipped with the $\Bbbk$-linear and invertible endofunctor $\Sigma = \Sigma_{\mathcal{K}}$ from 6.1.8. One may now speak of candidate triangles in $\mathcal{K}(R)$ in the sense of A.1.

6.1.9 Lemma. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{C}(R)$. The image under the canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ of the diagram*

$$M \xrightarrow{\;\alpha\;} N \xrightarrow{\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\;1^{\Sigma M})} \Sigma M$$

is a candidate triangle in $\mathcal{K}(R)$.

PROOF. We must prove that the three composites in $\mathcal{C}(R)$,

$$\varphi = \begin{pmatrix}1^N\\0\end{pmatrix}\alpha = \begin{pmatrix}\alpha\\0\end{pmatrix}, \;\; \psi = \begin{pmatrix}0 & 1^{\Sigma M}\end{pmatrix}\begin{pmatrix}1^N\\0\end{pmatrix} = 0, \;\text{ and }\; \chi = (\Sigma\alpha)\begin{pmatrix}0 & 1^{\Sigma M}\end{pmatrix} = \begin{pmatrix}0 & \Sigma\alpha\end{pmatrix}$$

are null-homotopic. Since ψ is even zero in $\mathcal{C}(R)$, we are left to consider φ and χ. Define degree 1 homomorphisms $\varrho\colon M \to \operatorname{Cone}\alpha$ and $\tau\colon \operatorname{Cone}\alpha \to \Sigma N$ by

$$\varrho = \begin{pmatrix}0\\ \sigma_1^M\end{pmatrix} \quad\text{and}\quad \tau = \begin{pmatrix}\sigma_1^N & 0\end{pmatrix},$$

where $\sigma_s^M \colon M \to \Sigma^s M$ is the map introduced in 2.2.3. From the fact that σ_s^M is a degree s chain map and from commutativity of the diagram (2.2.3.1), it follows that there are equalities $\partial^{\operatorname{Cone}\alpha}\varrho + \varrho\partial^M = \varphi$ and $\partial^{\Sigma N}\tau + \tau\partial^{\operatorname{Cone}\alpha} = \chi$. Indeed, one has

$$\begin{pmatrix} \partial^N & \sigma_{-1}^{\Sigma N}\Sigma\alpha \\ 0 & \partial^{\Sigma M} \end{pmatrix}\begin{pmatrix} 0 \\ \sigma_1^M \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma_1^M \end{pmatrix}\partial^M = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

and

$$\partial^{\Sigma N}\begin{pmatrix} \sigma_1^N & 0 \end{pmatrix} + \begin{pmatrix} \sigma_1^N & 0 \end{pmatrix}\begin{pmatrix} \partial^N & \sigma_{-1}^{\Sigma N}\Sigma\alpha \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \begin{pmatrix} 0 & \Sigma\alpha \end{pmatrix}\ . \qquad \square$$

6.1.10 Definition. A candidate triangle in $\mathcal{K}(R)$ of the form considered in 6.1.9 is called a *strict triangle*. A candidate triangle in $\mathcal{K}(R)$ that is isomorphic, in the sense of A.1, to a strict triangle is called a *distinguished triangle*.

6.1.11 Theorem. *The category $\mathcal{K}(R)$, equipped with the invertible endofunctor Σ and the notion of distinguished triangles from* 6.1.10*, is triangulated.*

PROOF. We verify the axioms in A.3.

(TR0): Evidently, the collection of distinguished triangles is closed under isomorphisms. Furthermore, it follows from 4.1.23 and 6.1.7 that application of the canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ to the following diagram in $\mathcal{C}(R)$,

$$M \xrightarrow{\;1^M\;} M \xrightarrow{\left(\begin{smallmatrix} 1^M \\ 0 \end{smallmatrix}\right)} \operatorname{Cone}(1^M) \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M\ ,$$

yields, up to isomorphism in $\mathcal{K}(R)$, the candidate triangle $M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma M$ which, therefore, is distinguished.

(TR1): By the definition of morphisms in $\mathcal{K}(R)$, every morphism in this category fits into a distinguished (even a strict) triangle; see. 6.1.9.

(TR2'): By A.4 it it sufficient to verify that (TR2) holds. Thus, let

$$\Delta = M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} \Sigma M'$$

be a distinguished triangle in $\mathcal{K}(R)$. We must argue that the candidate triangles

$$\Delta' = N' \xrightarrow{\beta'} X \xrightarrow{\gamma'} \Sigma M' \xrightarrow{-\Sigma\alpha'} \Sigma N' \quad \text{and} \quad \Delta'' = \Sigma^{-1} X' \xrightarrow{-\Sigma^{-1}\gamma'} M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X'$$

are distinguished. Up to isomorphism, Δ is given by application of the canonical functor Q to a diagram in $\mathcal{C}(R)$ of the form,

$$M \xrightarrow{\;\alpha\;} N \xrightarrow{\left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M\ .$$

Thus, the candidate triangles Δ' and Δ'' are, up to isomorphism, given by application of Q to the following diagrams in $\mathcal{C}(R)$,

$$N \xrightarrow{\binom{1^N}{0}} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N\,,$$

$$\Sigma^{-1}\operatorname{Cone}\alpha \xrightarrow{(0\ -1^M)} M \xrightarrow{\ \alpha\ } N \xrightarrow{\binom{1^N}{0}} \operatorname{Cone}\alpha\,.$$

These two diagrams in $\mathcal{C}(R)$ are the top rows in $(\star)$ and $(\ddagger)$ below. By definition, the bottom rows in $(\star)$ and $(\ddagger)$ give strict triangles in $\mathcal{K}(R)$ when the functor Q is applied; see 6.1.10. Thus, to show that Δ' and Δ'' are distinguished triangles in $\mathcal{K}(R)$, it suffices to argue that $(\star)$ and $(\ddagger)$ are commutative up to homotopy, and that all vertical morphisms are homotopy equivalences.

$$(\star)\qquad \begin{array}{ccccccc}
N & \xrightarrow{\iota=\binom{1^N}{0}} & \operatorname{Cone}\alpha & \xrightarrow{(0\ 1^{\Sigma M})} & \Sigma M & \xrightarrow{-\Sigma\alpha} & \Sigma N \\
\| & & \| & \vartheta=(0\ 1^{\Sigma M}\ 0) & \Big\uparrow\Big\downarrow & \varphi=\begin{pmatrix}0\\1^{\Sigma M}\\-\Sigma\alpha\end{pmatrix} & \| \\
N & \xrightarrow[\iota=\binom{1^N}{0}]{} & \operatorname{Cone}\alpha & \xrightarrow[\begin{pmatrix}1^N & 0\\0 & 1^{\Sigma M}\\0 & 0\end{pmatrix}]{} & \operatorname{Cone}\iota & \xrightarrow[(0\ 0\ 1^{\Sigma N})]{} & \Sigma N
\end{array}$$

$$(\ddagger)\qquad \begin{array}{ccccccc}
\Sigma^{-1}\operatorname{Cone}\alpha & \xrightarrow{\pi=(0\ -1^M)} & M & \xrightarrow{\ \alpha\ } & N & \xrightarrow{\binom{1^N}{0}} & \operatorname{Cone}\alpha \\
\| & & \| & \xi=(\alpha\ 1^N\ 0) & \Big\uparrow\Big\downarrow & \psi=\begin{pmatrix}0\\1^N\\0\end{pmatrix} & \| \\
\Sigma^{-1}\operatorname{Cone}\alpha & \xrightarrow[\pi=(0\ -1^M)]{} & M & \xrightarrow[\begin{pmatrix}1^M\\0\\0\end{pmatrix}]{} & \operatorname{Cone}\pi & \xrightarrow[\begin{pmatrix}0 & 1^N & 0\\0 & 0 & 1^{\Sigma M}\end{pmatrix}]{} & \operatorname{Cone}\alpha
\end{array}$$

First consider the diagram $(\star)$. Note that φ and ϑ are morphisms, as one has

$$\partial^{\operatorname{Cone}\iota}\varphi = \begin{pmatrix}\partial^N & \Sigma\alpha & 1^{\Sigma N}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix}\begin{pmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{pmatrix} = \begin{pmatrix}0\\ 1^{\Sigma M}\\ -\Sigma\alpha\end{pmatrix}\partial^{\Sigma M} = \varphi\partial^{\Sigma M}$$

and

$$\partial^{\Sigma M}\vartheta = \partial^{\Sigma M}\big(0\ 1^{\Sigma M}\ 0\big) = \big(0\ 1^{\Sigma M}\ 0\big)\begin{pmatrix}\partial^N & \Sigma\alpha & 1^{\Sigma N}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix} = \vartheta\partial^{\operatorname{Cone}\iota}\,.$$

To make sense of these computations, recall from 4.1.10 that in the matrix $\partial^{\operatorname{Cone}\iota}$, the maps $\Sigma\alpha$ and $1^{\Sigma N}$ are viewed as a degree -1 chain maps $\Sigma M \to N$ and $\Sigma N \to N$.

We claim that φ is a homotopy equivalence with homotopy inverse ϑ. It is clear that $\vartheta\varphi = 1^{\Sigma M}$ holds, so it remains to show that the morphism

$$1^{\operatorname{Cone}\iota} - \varphi\vartheta = \begin{pmatrix} 1^N & 0 & 0 \\ 0 & 1^{\Sigma M} & 0 \\ 0 & 0 & 1^{\Sigma N} \end{pmatrix} - \begin{pmatrix} 0 \\ 1^{\Sigma M} \\ -\Sigma\alpha \end{pmatrix} \begin{pmatrix} 0 & 1^{\Sigma M} & 0 \end{pmatrix} = \begin{pmatrix} 1^N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma\alpha & 1^{\Sigma N} \end{pmatrix}$$

is null-homotopic. Consider the map $\sigma\colon \operatorname{Cone}\iota \to \operatorname{Cone}\iota$ given by

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1^N & 0 & 0 \end{pmatrix} ;$$

here 1^N is viewed as a degree 1 chain map $N \to \Sigma N$, and thus σ is a degree 1 homomorphism; cf. 2.2.3. It is elementary to verify that one has

$$\begin{pmatrix} \partial^N & \Sigma\alpha & 1^{\Sigma N} \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & 0 & \partial^{\Sigma N} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1^N & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1^N & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial^N & \Sigma\alpha & 1^{\Sigma N} \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & 0 & \partial^{\Sigma N} \end{pmatrix} = \begin{pmatrix} 1^N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma\alpha & 1^{\Sigma N} \end{pmatrix} ;$$

that is, $\partial^{\operatorname{Cone}\iota}\sigma + \sigma\partial^{\operatorname{Cone}\iota} = 1^{\operatorname{Cone}\iota} - \varphi\vartheta$ holds. Thus ϑ is a homotopy inverse of φ.

Now we turn to the issue of commutativity of $(\star)$. The left- and right-hand squares in $(\star)$ are even commutative in $\mathcal{C}(R)$. For the commutativity, up to homotopy, of the middle square, it must be proved that the morphism $\beta\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\iota$, defined by

$$\beta = \begin{pmatrix} 1^N & 0 \\ 0 & 1^{\Sigma M} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1^{\Sigma M} \\ -\Sigma\alpha \end{pmatrix} \begin{pmatrix} 0 & 1^{\Sigma M} \end{pmatrix} = \begin{pmatrix} 1^N & 0 \\ 0 & 0 \\ 0 & \Sigma\alpha \end{pmatrix} ,$$

is null-homotopic. Let $\tau\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\iota$ be the map

$$\tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1^N & 0 \end{pmatrix} ,$$

where 1^N is viewed as a degree 1 chain map $N \to \Sigma N$. Thus τ is a degree 1 homomorphism. It is straightforward to verify the equality

$$\begin{pmatrix} \partial^N & \Sigma\alpha & 1^{\Sigma N} \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & 0 & \partial^{\Sigma N} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1^N & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1^N & 0 \end{pmatrix} \begin{pmatrix} \partial^N & \Sigma\alpha \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \begin{pmatrix} 1^N & 0 \\ 0 & 0 \\ 0 & \Sigma\alpha \end{pmatrix} ;$$

that is, $\partial^{\operatorname{Cone}\iota}\tau + \tau\partial^{\operatorname{Cone}\alpha} = \beta$ holds, and hence β is null-homotopic.

For the second diagram (‡), arguments similar to the ones given above show that ψ is a homotopy equivalence with homotopy inverse ξ, and that (‡) is commutative up to homotopy.

(TR4'): Consider the following diagram in $\mathcal{C}(R)$, where the rows and the morphisms φ and ψ are given, and the left-hand square is commutative up to homotopy,

$$(\diamond)\qquad \begin{array}{ccccccc} M & \xrightarrow{\ \alpha\ } & N & \xrightarrow{\left(\begin{smallmatrix}1^N\\ 0\end{smallmatrix}\right)} & \operatorname{Cone}\alpha & \xrightarrow{(0\ 1^{\Sigma M})} & \Sigma M \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \vdots\ {\scriptstyle\chi=\left(\begin{smallmatrix}\chi^{11} & \chi^{12}\\ \chi^{21} & \chi^{22}\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\Sigma\varphi} \\ M' & \xrightarrow[\ \alpha'\ ]{} & N' & \xrightarrow[\left(\begin{smallmatrix}1^{N'}\\ 0\end{smallmatrix}\right)]{} & \operatorname{Cone}\alpha' & \xrightarrow[(0\ 1^{\Sigma M'})]{} & \Sigma M'\,. \end{array}$$

To verify that (TR4') holds we are, in view of the definition of distinguished triangles in $\mathcal{K}(R)$, required to prove that there exists a morphism $\chi\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\alpha'$ with the following two properties: First of all, χ must make $(\diamond)$ commutative up to homotopy. In this case, $(\mathrm{Q}(\varphi),\mathrm{Q}(\psi),\mathrm{Q}(\chi))$ is a morphism of distinguished triangles in $\mathcal{K}(R)$, and its mapping cone candidate triangle is given by application of the functor Q to the following diagram in $\mathcal{C}(R)$,

$$(\S)\qquad \begin{matrix} M' \\ \oplus \\ N \end{matrix} \xrightarrow{\begin{pmatrix} \alpha' & \psi \\ 0 & -1^N \\ 0 & 0 \end{pmatrix}} \begin{matrix} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{matrix} \xrightarrow{\begin{pmatrix} 1^{N'} & \chi^{11} & \chi^{12} \\ 0 & \chi^{21} & \chi^{22} \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}} \begin{matrix} \operatorname{Cone}\alpha' \\ \oplus \\ \Sigma M \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix}} \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix}\,.$$

Secondly, Q applied to (§) must yield a distinguished triangle in $\mathcal{K}(R)$.

We start by constructing a morphism χ that makes $(\diamond)$ commutative up to homotopy. By assumption, there is a degree 1 homomorphism $\sigma\colon M \to N'$ such that the equality $\psi\alpha - \alpha'\varphi = \partial^{N'}\sigma + \sigma\partial^M$ holds. Define $\chi\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\alpha'$ to be the degree 0 homomorphism

$$\chi = \begin{pmatrix} \psi & \Sigma\sigma \\ 0 & \Sigma\varphi \end{pmatrix}\,,$$

where $\Sigma\sigma$ is viewed as a degree 0 homomorphism $\Sigma M \to N'$. It is elementary to verify that χ is a morphism; that is,

$$\partial^{\operatorname{Cone}\alpha'}\chi = \begin{pmatrix} \partial^{N'} & \Sigma\alpha' \\ 0 & \partial^{\Sigma M'} \end{pmatrix}\begin{pmatrix} \psi & \Sigma\sigma \\ 0 & \Sigma\varphi \end{pmatrix} = \begin{pmatrix} \psi & \Sigma\sigma \\ 0 & \Sigma\varphi \end{pmatrix}\begin{pmatrix} \partial^{N} & \Sigma\alpha \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \chi\partial^{\operatorname{Cone}\alpha}\,.$$

Notice that χ makes the middle and right-hand squares in $(\diamond)$ commutative in $\mathcal{C}(R)$.

Finally, to see that application of the functor Q to (§) yields a distinguished triangle in $\mathcal{K}(R)$, note that (§) is the top row in following diagram, and that the bottom row yields a strict triangle in $\mathcal{K}(R)$ when Q is applied. Thus, it suffices to argue that the diagram below is commutative up to homotopy, and that all vertical morphisms are homotopy equivalences.

$$
\begin{array}{ccccccc}
\begin{matrix} M' \\ \oplus \\ N \end{matrix} & \xrightarrow{\theta=\begin{pmatrix}\alpha' & \psi \\ 0 & -1^N \\ 0 & 0\end{pmatrix}} & \begin{matrix} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{matrix} & \xrightarrow{\begin{pmatrix}1^{N'} & \psi & \Sigma\sigma \\ 0 & 0 & \Sigma\varphi \\ 0 & 0 & -1^{\Sigma M}\end{pmatrix}} & \begin{matrix} \operatorname{Cone}\alpha' \\ \oplus \\ \Sigma M \end{matrix} & \xrightarrow{\begin{pmatrix}0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha\end{pmatrix}} & \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix} \\
\Vert & & \Vert & \eta=\begin{pmatrix}1^{N'} & \psi & \Sigma\sigma & 0 & 0 \\ 0 & 0 & \Sigma\varphi & 1^{\Sigma M'} & 0 \\ 0 & 0 & -1^{\Sigma M} & 0 & 0\end{pmatrix} & \Big\uparrow\Big\downarrow & \xi=\begin{pmatrix}1^{N'} & 0 & \Sigma\sigma \\ 0 & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \\ 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha\end{pmatrix} & \Vert \\
\begin{matrix} M' \\ \oplus \\ N \end{matrix} & \xrightarrow[\theta=\begin{pmatrix}\alpha' & \psi \\ 0 & -1^N \\ 0 & 0\end{pmatrix}]{} & \begin{matrix} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{matrix} & \xrightarrow[\begin{pmatrix}1^{N'} & 0 & 0 \\ 0 & 1^N & 0 \\ 0 & 0 & 1^{\Sigma M} \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}]{} & \operatorname{Cone}\theta & \xrightarrow[\begin{pmatrix}0 & 0 & 0 & 1^{\Sigma M'} & 0 \\ 0 & 0 & 0 & 0 & 1^{\Sigma N}\end{pmatrix}]{} & \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix}
\end{array}
$$

The differentials on the complexes $\operatorname{Cone}\alpha' \oplus \Sigma M$ and $\operatorname{Cone}\theta$ are given by

$$
\partial^{\operatorname{Cone}\alpha' \oplus \Sigma M} = \begin{pmatrix} \partial^{N'} & \Sigma\alpha' & 0 \\ 0 & \partial^{\Sigma M'} & 0 \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} \quad \text{and} \quad \partial^{\operatorname{Cone}\theta} = \begin{pmatrix} \partial^{N'} & 0 & 0 & \Sigma\alpha' & \Sigma\psi \\ 0 & \partial^{N} & \Sigma\alpha & 0 & -1^{\Sigma N} \\ 0 & 0 & \partial^{\Sigma M} & 0 & 0 \\ 0 & 0 & 0 & \partial^{\Sigma M'} & 0 \\ 0 & 0 & 0 & 0 & \partial^{\Sigma N} \end{pmatrix}.
$$

It is straightforward to verify that ξ and η are morphisms. Evidently there is an equality $\eta\xi = 1^{\operatorname{Cone}\alpha' \oplus \Sigma M}$. Furthermore, the morphism $\xi\eta - 1^{\operatorname{Cone}\theta}$ is null-homotopic, as the degree 1 homomorphism $\tau \colon \operatorname{Cone}\theta \to \operatorname{Cone}\theta$ given by

$$
\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1^N & 0 & 0 & 0 \end{pmatrix}
$$

satisfies the identity $\partial^{\operatorname{Cone}\theta}\tau + \tau\partial^{\operatorname{Cone}\theta} = \xi\eta - 1^{\operatorname{Cone}\theta}$. Hence ξ is a homotopy equivalence with homotopy inverse η.

The left-hand and right-hand squares in the diagram are commutative in $\mathcal{C}(R)$. The diagram's middle square is commutative up to homotopy, indeed, the morphism $\gamma \colon N' \oplus \operatorname{Cone}\alpha \to \operatorname{Cone}\theta$, defined by

$$
\gamma = \begin{pmatrix} 1^{N'} & 0 & \Sigma\sigma \\ 0 & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \\ 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix} \begin{pmatrix} 1^{N'} & \psi & \Sigma\sigma \\ 0 & 0 & \Sigma\varphi \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix} - \begin{pmatrix} 1^{N'} & 0 & 0 \\ 0 & 1^N & 0 \\ 0 & 0 & 1^{\Sigma M} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \psi & 0 \\ 0 & -1^N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma\alpha \end{pmatrix},
$$

is null-homotopic. This follows as $\varrho \colon N' \oplus \operatorname{Cone}\alpha \to \operatorname{Cone}\theta$, given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1^N & 0 \end{pmatrix},$$

is a degree 1 homomorphism with $\partial^{\operatorname{Cone}\theta}\varrho + \varrho\partial^{N'\oplus\operatorname{Cone}\alpha} = \gamma$. □

THE UNIVERSAL PROPERTY

6.1.12 Definition. Let $(\mathcal{U},\Sigma_{\mathcal{U}})$ be a triangulated category. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ is called *quasi-triangulated* if there is a natural isomorphism $\varphi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$ such that

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \mathrm{F}(\operatorname{Cone}\alpha) \xrightarrow{\varphi_M\circ\mathrm{F}(0\ 1^{\Sigma M})} \Sigma_{\mathcal{U}}\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{U}$ for every morphism of R-complexes $\alpha\colon M \to N$.

The canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ has the following universal property.

6.1.13 Theorem. *If* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ *is a functor that maps homotopy equivalences to isomorphisms, then there exists a unique functor* $\mathrm{F}'\colon \mathcal{K}(R) \to \mathcal{U}$ *that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R) & \xrightarrow{\ \mathrm{Q}\ } & \mathcal{K}(R) \\ {\scriptstyle \mathrm{F}}\downarrow & \swarrow{\scriptstyle \mathrm{F}'} & \\ \mathcal{U}\,; & & \end{array}$$

here Q *is the canonical functor from* 6.1.4. *Furthermore, the next assertions hold.*

(a) *Assume that* $\mathcal{U}$ *is* $\Bbbk$*-prelinear; then* F *is* $\Bbbk$*-linear if and only if* F' *is* $\Bbbk$*-linear.*

(b) *Assume that* $\mathcal{U}$ *has (co)products; then* F *preserves (co)products if and only if* F' *preserves (co)products.*

(c) *If* $\mathcal{U}$ *is triangulated and* F *is quasi-triangulated, then* F' *is triangulated.*

PROOF. Uniqueness of F' follows as Q is the identity on objects and it is full.

For existence of F', set $\mathrm{F}'(M) = \mathrm{F}(M)$ for every R-complex M and $\mathrm{F}'([\alpha]) = \mathrm{F}(\alpha)$ for every morphism α of R-complexes. To see that this makes sense—in which case the the identity $\mathrm{F}'\mathrm{Q} = \mathrm{F}$ evidently holds—let $\alpha,\beta\colon M \to N$ be homotopic morphisms of R-complexes. It must be shown that one has $\mathrm{F}(\alpha) = \mathrm{F}(\beta)$. To this end, define an R-complex C as follows,

$$C^\natural = \begin{matrix} M^\natural \\ \oplus \\ M^\natural \\ \oplus \\ \Sigma M^\natural \end{matrix} \qquad \text{and} \qquad \partial^C = \begin{pmatrix} \partial^M & 0 & -1^{\Sigma M} \\ 0 & \partial^M & 1^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} ,$$

where the maps $\pm 1^{\Sigma M}$ in the right-hand column of the matrix ∂^C are viewed as degree -1 chain maps $\Sigma M \to M$. The following three maps are morphisms in $\mathcal{C}(R)$,

$$M \underset{\iota=\left(\begin{smallmatrix} 0 \\ 1^M \\ 0 \end{smallmatrix}\right)}{\overset{\varepsilon=\left(\begin{smallmatrix} 1^M \\ 0 \\ 0 \end{smallmatrix}\right)}{\rightrightarrows}} C \xrightarrow{\pi=(1^M\ 1^M\ 0)} M .$$

Moreover, ι is a homotopy equivalence with homotopy inverse π, indeed, the equality $\pi\iota = 1^M$ evidently holds. To prove that the morphism

$$\iota\pi - 1^C = \begin{pmatrix} 0 \\ 1^M \\ 0 \end{pmatrix} \begin{pmatrix} 1^M & 1^M & 0 \end{pmatrix} - \begin{pmatrix} 1^M & 0 & 0 \\ 0 & 1^M & 0 \\ 0 & 0 & 1^{\Sigma M} \end{pmatrix} = \begin{pmatrix} -1^M & 0 & 0 \\ 1^M & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}$$

is null-homotopic, consider the degree 1 homomorphism $\sigma\colon C \to C$ given by

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1^M & 0 & 0 \end{pmatrix} ,$$

where 1^M is viewed as degree 1 chain map $M \to \Sigma M$. One readily verifies the equality $\partial^C\sigma + \sigma\partial^C = \iota\pi - 1^C$, that is,

$$\begin{pmatrix} \partial^M & 0 & -1^{\Sigma M} \\ 0 & \partial^M & 1^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1^M & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1^M & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial^M & 0 & -1^{\Sigma M} \\ 0 & \partial^M & 1^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} = \begin{pmatrix} -1^M & 0 & 0 \\ 1^M & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix} .$$

These arguments show that ι is a homotopy equivalence with homotopy inverse π. It follows that $\mathrm{F}(\iota)$ is an isomorphism in $\mathcal{U}$ with inverse $\mathrm{F}(\pi)$. As $\pi\varepsilon = 1^M$, and hence $\mathrm{F}(\pi)\mathrm{F}(\varepsilon) = 1^{\mathrm{F}(M)}$, holds it follows that one has $\mathrm{F}(\varepsilon) = \mathrm{F}(\iota)$. Since the morphisms $\alpha,\beta\colon M \to N$ are homotopic, there exists a degree 1 homomorphism $\varrho\colon M \to N$ with $\beta - \alpha = \partial^N\varrho + \varrho\partial^M$. Viewing $\Sigma\varrho$ as a degree 0 homomorphism $\Sigma M \to N$, the degree 0 homomorphism $\gamma = \begin{pmatrix} \alpha & \beta & \Sigma\varrho \end{pmatrix} : C \to N$ is a morphism, as one has

$$\partial^N\gamma = \partial^N \begin{pmatrix} \alpha & \beta & \Sigma\varrho \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \Sigma\varrho \end{pmatrix} \begin{pmatrix} \partial^M & 0 & -1^{\Sigma M} \\ 0 & \partial^M & 1^{\Sigma M} \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} = \gamma\partial^C .$$

From the equalities $\alpha = \gamma\varepsilon$ and $\gamma\iota = \beta$ one gets $\mathrm{F}(\alpha) = \mathrm{F}(\gamma)\mathrm{F}(\varepsilon) = \mathrm{F}(\gamma)\mathrm{F}(\iota) = \mathrm{F}(\beta)$.

It remains to prove the assertions (a), (b), and (c).

(a): If F' is $\Bbbk$-linear then so is $\mathrm{F} = \mathrm{F}'\mathrm{Q}$, as a composition of two $\Bbbk$-linear functors. Conversely, if F is $\Bbbk$-linear then so is F', since the equalities

$$\mathrm{F}'(x[\alpha]+[\beta]) = \mathrm{F}'([x\alpha+\beta]) = \mathrm{F}(x\alpha+\beta) = x\mathrm{F}(\alpha)+\mathrm{F}(\beta) = x\mathrm{F}'([\alpha])+\mathrm{F}'([\beta])$$

hold for every pair α,β of parallel morphisms in $\mathcal{C}(R)$ and every element x in $\Bbbk$.

(b): Let $\{M^u\}_{u\in U}$ be a family of R-complexes. Since the functor Q preserves coproducts; see 6.1.6, the canonical morphism

$$\coprod_{u\in U} M^u = \coprod_{u\in U} \mathrm{Q}(M^u) \xrightarrow{\;\psi\;} \mathrm{Q}(\coprod_{u\in U} M^u)$$

in $\mathcal{K}(R)$ is an isomorphism; cf. 3.1.9. Application of F' yields an isomorphism

$$\mathrm{F}'(\coprod_{u\in U} M^u) \xrightarrow{\mathrm{F}'(\psi)} \mathrm{F}'\mathrm{Q}(\coprod_{u\in U} M^u) = \mathrm{F}(\coprod_{u\in U} M^u)$$

in $\mathcal{U}$ such that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{F}'(\coprod_{u\in U} M^u) & \xrightarrow[\cong]{\mathrm{F}'(\psi)} & \mathrm{F}(\coprod_{u\in U} M^u) \\ \Big\downarrow{\scriptstyle\varphi'} & & \Big\downarrow{\scriptstyle\varphi} \\ \coprod_{u\in U} \mathrm{F}'(M^u) & = & \coprod_{u\in U} \mathrm{F}(M^u) \end{array},$$

where φ and φ' are the canonical morphisms. It follows that φ is and isomorphism if and only φ' is an isomorphism; and hence the functor F preserves coproducts if and only if F' preserves coproducts.

The assertion about products is proved similarly.

(c): Let $\varphi\colon \mathrm{F}\Sigma_{\mathcal{C}} \to \Sigma_{\mathcal{U}}\mathrm{F}$ be a natural isomorphism as in 6.1.12. By 6.1.8 there are equalities $\mathrm{F}\Sigma_{\mathcal{C}} = \mathrm{F}'\mathrm{Q}\Sigma_{\mathcal{C}} = \mathrm{F}'\Sigma_{\mathcal{K}}\mathrm{Q}$, and one has $\Sigma_{\mathcal{U}}\mathrm{F} = \Sigma_{\mathcal{U}}\mathrm{F}'\mathrm{Q}$. Since Q is the identity on objects, φ can be viewed as a natural isomorphism $\varphi'\colon \mathrm{F}'\Sigma_{\mathcal{K}} \to \Sigma_{\mathcal{U}}\mathrm{F}'$. We verify that the functor $\mathrm{F}'\colon \mathcal{K}(R) \to \mathcal{U}$ with the isomorphism φ', is triangulated. By definition, every distinguished triangle in $\mathcal{K}(R)$ is isomorphic to a strict triangle, that is, to a diagram of the form

$$M \xrightarrow{\;\mathrm{Q}(\alpha)\;} N \xrightarrow{\;\mathrm{Q}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)\;} \operatorname{Cone}\alpha \xrightarrow{\;\mathrm{Q}(0\ 1^{\Sigma M})\;} \Sigma_{\mathcal{K}}M\,,$$

where $\alpha\colon M \to N$ is a morphism in $\mathcal{C}(R)$. Thus, it must be shown that the following candidate triangle in $\mathcal{U}$ is distinguished,

$$\mathrm{F}'(M) \xrightarrow{\;\mathrm{F}'\mathrm{Q}(\alpha)\;} \mathrm{F}'(N) \xrightarrow{\;\mathrm{F}'\mathrm{Q}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)\;} \mathrm{F}'(\operatorname{Cone}\alpha) \xrightarrow{\;\varphi'_M\circ\mathrm{F}'\mathrm{Q}(0\ 1^{\Sigma M})\;} \Sigma_{\mathcal{U}}\mathrm{F}'(M)\,.$$

However, this diagram is identical to the one in 6.1.12, which is distinguished triangle in $\mathcal{U}$ since F is assumed to be quasi-triangulated. □

REMARK. The main assertion in 6.1.13 is that the homotopy category $\mathcal{K}(R)$ is the localization of $\mathcal{C}(R)$ with respect to the collection of homotopy equivalences. In the next section, we focus on yet another important example of a localization, namely the localization of $\mathcal{K}(R)$ with respect to the collection of quasi-isomorphisms; this gives rise to the derived category.

6.1.14 Definition. Let $(\mathcal{V}, \Sigma_{\mathcal{V}})$ be a triangulated category. A functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ is called *quasi-triangulated* if the functor G^{op} from $\mathcal{C}(R)$ to the triangulated category $(\mathcal{V}^{\mathrm{op}}, \Sigma_{\mathcal{V}}^{-1})$, see A.5, is quasi-triangulated in the sense of 6.1.12. Explicitly, this means that there exists a natural isomorphism $\psi\colon \Sigma_{\mathcal{V}}^{-1}\mathrm{G} \to \mathrm{G}\Sigma$ of functors $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$, such that

$$\Sigma_{\mathcal{V}}^{-1}\mathrm{G}(M) \xrightarrow{\mathrm{G}(0\ 1^{\Sigma M})\circ\psi_M} \mathrm{G}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{G}\binom{1^N}{0}} \mathrm{G}(N) \xrightarrow{\mathrm{G}(\alpha)} \mathrm{G}(M)$$

is a distinguished triangle in $\mathcal{V}$ for every morphism of R-complexes $\alpha\colon M \to N$.

A morphism in $\mathcal{C}(R)^{\mathrm{op}}$ is called a homotopy equivalence if the corresponding morphism in $\mathcal{C}(R)$ is a homotopy equivalence in the sense of 2.2.25. To parse and prove the next result, recall further that if $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is a functor between categories with products (coproducts), then $\mathrm{F}^{\mathrm{op}}\colon \mathcal{U}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ is a functor between categories with coproducts (products), and F preserves products (coproducts) if and only if F^{op} preserves coproducts (products).

6.1.15 Theorem. *If $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ is a functor that maps homotopy equivalences to isomorphisms, then there exists a unique functor $\mathrm{G}'\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$ that makes the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{C}(R)^{\mathrm{op}} & \xrightarrow{\mathrm{Q}^{\mathrm{op}}} & \mathcal{K}(R)^{\mathrm{op}} \\ {\scriptstyle \mathrm{G}}\downarrow & \swarrow {\scriptstyle \mathrm{G}'} & \\ \mathcal{V}\,; & & \end{array}$$

here Q is the canonical functor from 6.1.4. Furthermore, the next assertions hold.

(a) *Assume that $\mathcal{V}$ is $\Bbbk$-prelinear; then G is $\Bbbk$-linear if and only if G' is $\Bbbk$-linear.*

(b) *Assume that $\mathcal{V}$ has (co)products; then G preserves (co)products if and only if G' preserves (co)products.*

(c) *If $\mathcal{V}$ is triangulated and G is quasi-triangulated, then G' is triangulated.*

PROOF. Apply 6.1.13 to the functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{C}(R) \to \mathcal{V}^{\mathrm{op}}$. □

UNIQUE LIFTING PROPERTIES

Throughout the rest of this chapter, we use Greek letters for morphisms in $\mathcal{K}(R)$; that is, $\alpha, \beta, \gamma, \ldots$ denote homotopy equivalence classes of morphisms in $\mathcal{C}(R)$.

6.1.16 Definition. A morphism α in $\mathcal{K}(R)$ is called a *quasi-isomorphism* if some, equivalently every, morphism in $\mathcal{C}(R)$ that represents the homotopy equivalence class α is a quasi-isomorphism.

Next we reformulate 5.2.16 and 5.3.21 in the language of the homotopy category.

6.1.17. Let P be a semi-projective R-complex. If $\alpha\colon P \to N$ is a morphism and $\beta\colon M \to N$ is a quasi-isomorphism in $\mathcal{K}(R)$, then there exists a unique morphism γ that makes the following diagram in $\mathcal{K}(R)$ commutative,

$$\begin{array}{ccc} & & P \\ & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle \alpha} \\ M & \underset{\beta}{\xrightarrow{\ \simeq\ }} & N\,. \end{array}$$

6.1.18. Let I be a semi-injective R-complex. If $\alpha\colon M \to I$ is a morphism and $\beta\colon M \to N$ is a quasi-isomorphism in $\mathcal{K}(R)$, then there exists a unique morphism γ that makes the following diagram in $\mathcal{K}(R)$ commutative,

$$\begin{array}{ccc} M & \overset{\beta}{\underset{\simeq}{\longrightarrow}} & N \\ {\scriptstyle \alpha}\downarrow & \underset{\gamma}{\swarrow} & \\ I\,. & & \end{array}$$

EXERCISES

E 6.1.1 Show that a morphism in a triangulated category is a monomorphism (epimorphism) if and only if it has a left (right) inverse. Conclude that every object in a triangulated category is both injective and projective.

E 6.1.2 Show that the homotopy category $\mathcal{K}(\mathbb{Z})$ is not Abelian.

E 6.1.3 Show that the homotopy category $\mathcal{K}(R)$ is Abelian if R is semi-simple.

E 6.1.4 Two homomorphisms of R-modules $\alpha, \beta\colon M \to N$ are called stably equivalent if $\alpha - \beta$ factors through a projective R-module. The *stable module category* $\underline{\mathcal{M}}(R)$ has as objects all R-modules. The hom-set $\underline{\mathcal{M}}(R)(M,N)$ in this category, often written as $\underline{\mathrm{Hom}}_R(M,N)$, is the set of classes of stably equivalent homomorphisms $M \to N$. (a) Show that $\underline{\mathcal{M}}(R)$ is a $\Bbbk$-linear category with coproducts. (b) For an R-module M, let $\Omega(M)$ be the kernel of any surjective homomorphism $\alpha\colon P \to M$ where P is projective. Show that Ω is a well-defined endofunctor on $\underline{\mathcal{M}}(R)$. (c) Show that the category $(\underline{\mathcal{M}}(R), \Omega)$ is triangulated if R is quasi-Frobenius.

E 6.1.5 Show that the stable module category $\underline{\mathcal{M}}(\mathbb{Z})$ is not Abelian.

E 6.1.6 (Cf. A.5) Let $(\mathcal{T},\Sigma)$ be a triangulated category. Show that $(\mathcal{T}^{\mathrm{op}},\Sigma^{-1})$ is triangulated in the canonical way: A candidate triangle $M \to N \to X \to \Sigma^{-1}M$ in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the candidate triangle $\Sigma^{-1}M \to X \to N \to M$ is distinguished in $\mathcal{T}$.

E 6.1.7 Let $(\mathcal{T},\Sigma)$ be a triangulated category and let $\mathcal{S}$ be a subcategory of $\mathcal{T}$ that is closed under isomorphisms. Show that $\mathcal{S}$ is a triangulated subcategory if and only if it is a triangulated category and the embedding functor $\mathcal{S} \to \mathcal{T}$ is full and triangulated.

E 6.1.8 Let $\mathcal{S}$ be a triangulated subcategory of $(\mathcal{T},\Sigma)$ and let $M \to N \to X \to \Sigma M$ be a distinguished triangle in $\mathcal{T}$. Show that if two of the objects M, N, and X are in $\mathcal{S}$, then the third object is also in $\mathcal{S}$.

E 6.1.9 Show that the full subcategory of $\mathcal{K}(R)$ whose objects are all acyclic R-complexes is triangulated.

E 6.1.10 Show that the full subcategories of $\mathcal{K}(R)$ defined by specifying their objects as follows:

$$\mathcal{K}_{\sqsubset}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded above complex } M' \text{ with } M \cong M' \text{ in } \mathcal{K}(R)\}\ ,$$
$$\mathcal{K}_{\square}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded complex } M' \text{ with } M \cong M' \text{ in } \mathcal{K}(R)\}\ , \text{ and}$$
$$\mathcal{K}_{\sqsupset}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded below complex } M' \text{ with } M \cong M' \text{ in } \mathcal{K}(R)\}$$

are triangulated subcategories of $\mathcal{K}(R)$.

E 6.1.11 Show that the full subcategory of $\mathcal{K}(R)$ defined by specifying its objects,

$$\mathcal{K}(\mathrm{Prj}\,R) = \{P \in \mathcal{K}(R) \mid P \text{ is a complex of projective modules}\}\ ,$$

is a triangulated category but, in general, not a triangulated subcategory of $\mathcal{K}(R)$.

E 6.1.12 Show that the full subcategory of $\mathcal{K}_{\mathrm{prj}}(R)$ defined by specifying its objects,

$$\mathcal{K}_{\mathrm{prj}}(R) = \{P \in \mathcal{K}(\mathrm{Prj}\,R) \mid P \text{ is semi-projective}\}\ ,$$

is a triangulated subcategory of $\mathcal{K}(\mathrm{Prj}\,R)$.

E 6.1.13 Show that the full subcategory of $\mathcal{K}(R)$ defined by specifying its objects,

$$\mathcal{K}(\mathrm{Inj}\,R) = \{I \in \mathcal{K}(R) \mid I \text{ is a complex of injective modules}\}\ ,$$

is a triangulated category but, in general, not a triangulated subcategory of $\mathcal{K}(R)$.

E 6.1.14 Show that the full subcategory of $\mathcal{K}(R)$ defined by specifying its objects,

$$\mathcal{K}_{\mathrm{inj}}(R) = \{I \in \mathcal{K}(\mathrm{Inj}\,R) \mid \ \text{is semi-injective}\}\ ,$$

is a triangulated subcategory of $\mathcal{K}(\mathrm{Inj}\,R)$.

6.2 The Derived Category

SYNOPSIS. Derived category; (co)product; triangulation; universal property.

OBJECTS AND MORPHISMS

Recall that we use Greek letters for morphisms in $\mathcal{K}(R)$, that is, $\alpha,\beta,\gamma,\ldots$ denote homotopy equivalence classes of morphisms in $\mathcal{C}(R)$.