To Christian U. Jensen,
who started it all
H.B.F.

To Betina and Sofie
H.H.
Preface

Homological algebra originated in late 19th century topology. Homological studies of algebraic objects, such as rings and modules only got under way in the middle of the 20th century—Cartan and Eilenberg’s classic text “Homological Algebra” serves as a historic marker. The utility of homological methods in commutative algebra was firmly established in the mid 1950s through the solutions to Krull’s conjectures on regular local rings, which were achieved, independently, by Auslander and Buchsbaum and by Serre.

The technically more advanced methods of derived categories came into commutative algebra through the work of Grothendieck and his school; this happened only some ten years after the initial successes of classic homological algebra. Applications of derived categories in commutative algebra have grown steadily, albeit slowly, since the mid 1960s. One reason for the modest pace is, without doubt, the absence of a coherent introduction to the topic. There are several excellent textbooks from which one can learn about classic homological algebra and its applications, but to become an efficient practitioner of derived category methods in commutative algebra one must be well-versed in a train of research articles and lecture notes, including unpublished ones.

With this book, we aim to provide an accessible and coherent introduction to derived category methods—sometimes called hyperhomological algebra—and their applications in commutative algebra. We want to make the case that these methods, compared to those of classic homological algebra, provide broader and stronger results with cleaner proofs—this to an extent that outweighs the effort it requires to master them. Moreover, there are important results in commutative algebra whose natural habitat is the derived category. The Local Duality Theorem, for example, has an elegant formulation in the derived category, but only for certain rings does it have a satisfactory one within classic homological algebra.

The book is intended to double as a graduate textbook and a work of reference. It is organized in three parts “Foundations”, “Tools”, and “Applications”. In the first part, we introduce the fundamental homological machinery and construct the derived category over a ring. The second part continues with a systematic study
of functors and invariants of utility in ring theory. In the third part we assemble
textbook applications of derived category methods in commutative algebra into a
treatise on the homological aspects of that theory.

This organization serves several purposes. Readers familiar with derived cate-
gories may skip “Foundations”. The tools are kept separate from their applications
in order to develop them in higher generality. This should make “Foundations” and
“Tools” useful, not only to commutative algebraists, but also to readers from neigh-
boring fields. The third part “Applications” is intended to be a high-level primer to
homological aspects of commutative algebra.

We have learned the material in this book from our teachers, our collaborators, and
other colleagues—we hope that they will find that we have done it justice.

The forerunners of this book are two collections of lecture notes by H.-B.F., that
were used at the University of Copenhagen from 1982. Indeed, L.W.C. and H.H.
were first introduced to the subject through these notes. Preliminary versions of this
book have been used in graduate classes and lectures at Beijing Normal University,
Texas Tech University, and at the University of Nebraska-Lincoln. We thank the
many students who gave their comments on these early versions of the manuscript.

Great strides were made on the book during sojourns at Banff International Re-
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                              H.H.
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Introduction

The year 1956 saw the first major applications of homological algebra in commutative ring theory. In seminal papers by Auslander and Buchsbaum [2] and by Serre [46], homological algebra was used to characterize regular local rings in a way that opened to proofs of two conjectures by Krull [30, 31]. This breakthrough established homological algebra as a powerful research tool in commutative ring theory and, in the words of Kaplansky [28], “marked a turning point of the subject.” It coincided with the appearance of the magnum opus “Homological Algebra” [15], by Cartan and Eilenberg, which made the tools of classic homological algebra generally available. The present book expounds the more advanced methods of derived categories and their applications in commutative algebra; applications to other areas of mathematics are beyond the scope of the book.

Theme and Goal

Under the heading “Hyperhomology”, a predecessor to the framework of derived categories was briefly treated by Cartan and Eilenberg in the final chapter of [15]. Yet, it was the work of Grothendieck—in particular, Hartshorne’s notes “Residues and Duality” [24] from 1966—that truly brought derived category methods into commutative algebra. In the 1970s, works by Iversen [26] and Roberts [43] emphasized the utility of these methods, and since then their importance has grown steadily.

THE THEME OF THIS BOOK, as stated in the Preface, is that derived category methods have significant advantages over those of classic homological algebra, which they extend. To facilitate a discussion of their similarities and differences, we sketch elements of the two pieces of machinery.

Classic homological algebra studies modules through the behavior—notably vanishing—of derived functors on modules. Let $T: \mathcal{M}(R) \to \mathcal{M}(R)$ be an additive functor on the category of modules over a ring $R$. The value of the $n^{th}$ left derived functor $L_n T$ on an $R$-module $M$ is computed follows:
(1) Choose any free resolution $F_\bullet$ of $M$

(2) Apply the functor $T$ to this resolution to get a complex $T(F_\bullet)$ of $R$-modules

(3) By definition, $L_n T(M)$ is the homology module $H_n(T(F_\bullet))$

This procedure determines $L_n T(M)$ up to isomorphism in $\mathcal{M}(R)$, and a similar one for homomorphisms establishes $L_n T$ as a functor. The purpose of step (3) is to return an object in $\mathcal{M}(R)$; this is not only esthetically pleasing but also useful, as it allows the procedure to be iterated. However, the honest output is the complex $T(F_\bullet)$.

Derived category methods are used to study $R$-modules and complexes of such through the behavior—notably boundedness—of derived functors on complexes. Let $T: \mathcal{C}(R) \to \mathcal{C}(R)$ be an additive functor on the category of complexes of $R$-modules. The value of the left derived functor $L T$ on a complex $M_\bullet$ of $R$-modules is computed as follows:

(1) Choose any semi-free resolution $F_\bullet$ of the complex $M_\bullet$

(2) By definition, $L T(M_\bullet)$ is the complex $T(F_\bullet)$ of $R$-modules

This procedure only determines $L T(M_\bullet)$ up to homology isomorphism in $\mathcal{C}(R)$. Thus, the homology $H(L T(M_\bullet))$ is a well-defined object in $\mathcal{C}(R)$, but $L T(M_\bullet)$ itself is not. This is overcome by passage to the derived category $\mathcal{D}(R)$, which has the same objects as $\mathcal{C}(R)$ but more morphisms—enough to make homology isomorphisms invertible. In this category, $L T(M_\bullet)$ is a well-defined object and $L T$ is a functor. The motivation for focusing on the object $L T(M_\bullet)$ over $H(L T(M_\bullet))$ is evident: the former contains more information than the latter as one cannot, in general, retrieve a complex from its homology.

A module $M$ can be viewed as a complex, and in that perspective $H(L T(M))$ is nothing but the assembly of all the modules $L_n T(M)$. The theme of the book, in a nutshell, is that working with functors $L T$ on $\mathcal{D}(R)$, as opposed to $L_n T$ on $\mathcal{M}(R)$, yields broader and stronger results, even for modules. An early example of the case in point is Roberts’ “No Holes Theorem” [43] in local algebra, which gave new insight into the structure of injective resolutions of modules. Another example comes from algebraic geometry. Grothendieck’s localization problem for flat local homomorphisms was stated in EGA [23], but a solution was only obtained 30 years later, when derived category methods were applied to study a wider class of ring homomorphisms [6].

The goal of this book is to make derived category methods in ring theory generally available and to illustrate their utility. For commutative rings, the goal is further to collect textbook applications of these methods into a compact, yet comprehensive, introduction to homological aspects of commutative algebra. The book is intended to serve as a graduate textbook as well as a work of reference for professional mathematicians.
Contents and Organization

The facets of the goal, as described above, are reflected in the organization of the book. First we build the framework, that is, the derived category over a ring. Next we develop a set of efficacious tools, such as local cohomology, dualizing complexes, fundamental isomorphisms, Künneth type formulas, and homological dimensions. Finally, we apply these tools to prove classic and modern results in commutative algebra. The body of the book consists of three parts.

PART I “FOUNDATIONS” presents the primary machinery of homological algebra. Chapter 1 is focused on the category of modules over a ring. For ease of reference, its first section recounts a few basic notions and results that we expect the reader to be familiar with, such as bimodule structures and diagram lemmas. Chapters 2–4 treat the category of complexes of modules and introduce homotopy and homology. Resolutions play a key role in homological algebra; they are the topic of Chap. 5. The derived category is constructed in Chap. 6, and Chap. 7 treats derived functors.

PART II “TOOLS” is devoted to homological invariants of complexes and studies of special complexes, morphisms, and functors in the derived category. Chapter 8 introduces, among other functors, the derived tensor product and the derived Hom; together with the standard isomorphisms that link them, these functors are key players in the rest of the book. Homological dimensions are developed in Chaps. 9 and 11, where they are related to a group of canonical morphisms in the derived category, which we call evaluation morphisms. Together with the standard isomorphisms from Chap. 8, these morphisms play a central role in our approach to the material in Part III. Another powerful technical tool, dualizing complexes, is presented in Chap. 10, where also Grothendieck Duality and Foxby Equivalence are treated. Čech complexes, Koszul complexes, and their relations to local (co)homology are the topics of Chap. 12.

PART III “APPLICATIONS” focuses on the homological theory of commutative Noetherian rings. It opens with a “Brief for Commutative Ring Theorists” (Chap. 13), which recapitulates central results from Chaps. 5 and 8–10 in the facile form they take for commutative Noetherian rings. In the remaining chapters, we refer to the statements in Chap. 13; as such, Part III is essentially self-contained.

Chapter 14 extends standard notions from commutative algebra, such as support and Krull dimension, to objects in the derived category. With these notions in place, the tools from Part II are applied to prove time-honored and recent theorems on commutative rings and their modules. Chapter 15 treats numerical invariants of modules over local rings, and Grothendieck’s Local Duality Theorem is proved in Chap. 16. Chapters 17 and 18 focus on regular rings and on Gorenstein rings, respectively. Intersection theorems and their applications to the study of Cohen–Macaulay rings are the topic of Chap. 19. The final two chapters, 20–21, cover the theory of local ring homomorphisms and modules over such.

THE CHOICE OF TOPICS is detailed in the synopses that open each section; they are also embedded in the table of contents.
A few topics that one might expect to find in this book—for example Yoneda Ext and spectral sequences—are absent. The simplistic reason is that we manage without them. For example, standard arguments based on collapsing spectral sequences coming from double complexes are naturally replaced by isomorphisms in the derived category. More to the point, our rationale is that, within the scope of this book, we see no way to improve on existing expositions of these topics, and since we do not need them, we have decided to avoid them all together.

At the same time, the book covers two topics that could be considered non-standard: Gorenstein homological dimensions and local ring homomorphisms. The motivation for including each of these topics is two-fold. Significant progress has been made within the last two decades and derived category methods have been crucial for the successes of this research. The literature is difficult to penetrate, and the existing surveys [16, 17] and [3, 4] are not up to date or present no proofs.

The derived category over a ring is a particular example of a triangulated category. Other such categories—stable module categories, for example—also play significant roles in ring theory, but they are not treated in this book.

Exposition

We have striven to make the book coherent and self-contained. Assuming some background knowledge—as specified in the Reader’s Guide below—we provide proofs of all results in the book with two exceptions. We use Cohen’s structure theorem for complete local rings [18]; its proof is beyond the scope of this book but can be found in [37]. We use the New Intersection Theorem for arbitrary local rings, but we only include Peskine and Szpiro’s proof [41] for equicharacteristic rings. Roberts’ proof for rings of mixed characteristic makes a book of its own [44].

Constructions and results that are required to keep the book self-contained, but might otherwise disrupt the flow of the material, have been relegated to appendices. To keep the text accessible to a wide audience, we have resisted temptations to increase the level of generality beyond what is justified by the goal of the book; even when it would come at low or no cost. For example, in the construction of the homotopy category over a ring, one could easily replace the module category with any other abelian category. Similarly, parts of the material on fundamental isomorphisms could be developed in the more general context of a closed monoidal category. In the same vein, we emphasize explicit constructions over axiomatic approaches. For example, we use resolutions to establish certain properties of the derived category over a ring, though they could be deduced from formal properties of triangulated categories.

The material in this textbook is not new, and the fact that references are sparse in the main text should not be interpreted as the authors’ subtle claim for credit. In fact, every significant statement in the text is either folklore or can be traced to the papers and books listed in the Bibliography. Among the texts that have inspired
the contents, we emphasize three that have also influenced the exposition: “Homological Algebra” [15] by Cartan and Eilenberg, “Triangulated Categories” [40] by Neeman, and “Differential Graded Homological Algebra” [7] by Avramov, Foxby, and Halperin.

**Reader’s Guide**

This book is written for researchers and advanced graduate students, so the reader should possess the mathematical maturity and background knowledge of a doctoral student of algebra.

The prerequisites include familiarity with basic notions from set theory, category theory, module theory and, for Part III, commutative algebra. Notice, though, that we assume no prerequisites in homological algebra.

Commutative algebraists who want to dive into Part III will find an extract of essential results from Parts I and II in Chap. 13.

The chapters of the book depend on each other, conceptually, as depicted by the following diagram.
THE CONVENTIONS  we employ are explained right after this introduction.

A GLOSSARY is provided towards the end of the book. It lists terms that are used but not defined in the text along with their definition and/or a reference to a textbook. We consistently refer to “Categories for the Working Mathematician” [35] by MacLane for notions in category theory, to Lam’s “A First Course in Noncommutative Rings” [33] and “Lectures on Modules and Rings” [32] for notions in ring theory, and to Eisenbud’s “Commutative Algebra with a View Toward Algebraic Geometry” [19] for notions in commutative algebra.

A LIST OF SYMBOLS follows the glossary; it includes most symbols used—and certainly all those defined—in the book.

EXERCISES at the end of each section supplement the text in three ways.

• Examples in the text are kept as simple as reasonably possible; exercises are used to develop more interesting ones.
• To some results there are parallel or dual versions that are not needed in the exposition; these are relegated to exercises.
• Exercises are used to explore directions beyond the scope of the book; some of these come with references to the literature to facilitate further study. A few exercises deal with problems that are open at the time of writing.

We emphasize that the main text does not depend on the exercises.

Teacher’s Guide

For pedagogical reasons, we have included several exercises that ask the student to perform elementary verifications and computations that are omitted from the text. The details examined in these exercises are ones that a professional mathematician can fill in on the fly, so we stand by the claim made right above that the main text does not depend on the exercises.

Here are some suggestions for how to use the book as a graduate textbook.

A FIRST COURSE IN HOMOLOGICAL ALGEBRA could consist of Chaps. 1–8 and parts of Chap. 9. A sequel course with emphasis on commutative algebra could be based on Chaps. 9–10, 14–17 and, possibly, the first few sections of Chaps. 18–19.

A COURSE ON DERIVED CATEGORY METHODS IN COMMUTATIVE ALGEBRA for an audience with prior knowledge of classic homological algebra could be based on Chaps. 2–8 and 13–16 and, even, selected sections of Chaps. 17–19.

A TOPICS COURSE on Gorenstein dimensions can be built around Chaps. 10–11 and 18; one on local ring homomorphisms could pivot on Chaps. 20–21.
Conventions and Notation

Throughout the book, the symbols \( Q, R, S, \) and \( T \) denote associative unital rings; they are assumed to be algebras over a common commutative unital ring \( k \). Homomorphisms between these rings are tacitly assumed to be a \( k \)-algebra homomorphisms. The generic choice of \( k \) is the ring \( \mathbb{Z} \) of integers, but in concrete settings other choices may be useful. For example, in studies of algebras over a field \( k \), the natural choice is \( k = k \); and in studies of Artin \( A \)-algebras, the commutative Artinian ring \( A \) is the natural candidate for \( k \). In a different direction, the center of \( R \) is a possible choice for \( k \), when one studies ring homomorphisms \( R \to S \).

Ideals in a ring are subsets that are both left ideals and right ideals. Similarly, a ring is called Artinian, Noetherian, perfect etc. if it is both left and right Artinian, Noetherian, perfect etc. In Part III, the rings \( Q, R, S, \) and \( T \) are assumed to be commutative and such left/right distinctions conveniently disappear.

Modules are assumed to be unitary, and by convention the ring acts on the left. That is, an \( R \)-module is a left \( R \)-module. Right \( R \)-modules are, consequently, considered to be modules (i.e. left modules) over \( R^{\text{op}} \), the opposite ring of \( R \). Thus, a left ideal in \( R \) is a submodule of the \( R \)-module \( R \), while a right ideal in \( R \) is a submodule of the \( R^{\text{op}} \)-module \( R \). When needed, \( R \) is adorned with a subscript to show which module structure is used; that is, \( _R R \) is the \( R \)-module \( R \) and \( R_R \) is the \( R^{\text{op}} \)-module \( R \).

Functors are by convention covariant. A contravariant functor \( F: \mathcal{U} \to \mathcal{V} \) is hence considered to be a functor (i.e. a covariant functor), still denoted \( F \), from \( \mathcal{U}^{\text{op}} \) to \( \mathcal{V} \), where \( \mathcal{U}^{\text{op}} \) is the opposite category of \( \mathcal{U} \). The notation \( F^{\text{op}} \) is reserved for the opposite functor \( \mathcal{U}^{\text{op}} \to \mathcal{V}^{\text{op}} \) induced by \( F: \mathcal{U} \to \mathcal{V} \).

The notation is either standard or explained in the text, starting here:

| (\( \ll 0 \)) | \( \gg 0 \) | sufficiently (small) large | \( \subseteq \) | complex numbers |
| (\( \to \)) | \rightarrow | injective map | \( \subseteq \subseteq \) | field with \( q \) elements |
| (\( \twoheadrightarrow \)) | \twoheadrightarrow | surjective map | \( \subseteq \subseteq \) | natural numbers |
| (\( \subset \)) | \subset | (proper) subset | \( \subseteq \subseteq \) | \( \mathbb{Q} \) rational numbers |
| (\( \setminus \)) | \setminus | difference of sets | \( \subseteq \subseteq \) | \( \mathbb{R} \) real numbers |
| (\( \bigcup \)) | \bigcup | disjoint union of sets | \( \subseteq \subseteq \) | \( \text{Id} \) Identity functor |
Chapter 1
Modules

The collection of all $R$-modules and all homomorphisms of $R$-modules forms an
Abelian category with set indexed products and coproducts (also, but not here,
known as direct sums); it is denoted $\mathcal{M}(R)$. We take this for granted, and the first
section of this chapter sums up the key properties of this category that we will build
on throughout the book.

1.1 Prerequisites

SYNOPSIS. Module category; (split) exact sequence; Five Lemma; Snake Lemma; Hom functor;
tensor product functor; linear category; linear functor; biproduct; product; coproduct; direct sum;
bimodule; (half/left/right) exact functor; faithful functor.

The primary purpose of this section is to remind the reader of some basic material
and, in that process, to introduce the accompanying symbols and nomenclature. The
material in this section is used throughout the book and usually without reference.

1.1.1 Definition. A sequence of $R$-modules is a, possibly infinite, diagram in $\mathcal{M}(R)$,

\[
\cdots \longrightarrow M^0 \xrightarrow{a_0} M^1 \xrightarrow{a_1} M^2 \xrightarrow{a_2} \cdots;
\]

(1.1.1)

it is called exact if one has $\text{Im}a_{n-1} = \text{Ker}a_n$ for all $n$. Notice that (1.1.1.1) is exact
if and only if every sequence $0 \rightarrow \text{Im}a_{n-1} \rightarrow M^n \rightarrow \text{Im}a^n \rightarrow 0$ is exact. An exact
sequence of the form $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is called a short exact sequence.

Two sequences \( \{ \alpha^n : M^n \rightarrow M^{n+1} \}_{n \in \mathbb{Z}} \) and \( \{ \beta^n : N^n \rightarrow N^{n+1} \}_{n \in \mathbb{Z}} \) of $R$-modules
are called isomorphic if there exists a family of isomorphisms $\{ \varphi^n \}_{n \in \mathbb{Z}}$ such that
the diagram

\[
\begin{array}{ccc}
M^n & \xrightarrow{\alpha^n} & M^{n+1} \\
\downarrow{\varphi^n} & & \downarrow{\varphi^{n+1}} \\
N^n & \xrightarrow{\beta^n} & N^{n+1}
\end{array}
\]
is commutative for every $n \in \mathbb{Z}$.

1.1.2 Five Lemma. Let

$$
\begin{array}{cccccc}
M^1 & \to & M^2 & \to & M^3 & \to & M^4 & \to & M^5 \\
\phi^1 & & \phi^2 & & \phi^3 & & \phi^4 & & \phi^5 \\
N^1 & \to & N^2 & \to & N^3 & \to & N^4 & \to & N^5
\end{array}
$$

be a commutative diagram in $\mathbb{M}(R)$ with exact rows. The following assertions hold.

(a) If $\phi^1$ is surjective, and $\phi^2$ and $\phi^4$ are injective, then $\phi^3$ is injective.

(b) If $\phi^5$ is injective, and $\phi^2$ and $\phi^4$ are surjective, then $\phi^3$ is surjective.

(c) If $\phi^1, \phi^2, \phi^4,$ and $\phi^5$ are isomorphisms, then $\phi^3$ is an isomorphism.

1.1.3 Construction. Let

$$
\begin{array}{cccccc}
M' & \to & M & \to & M'' & \to & 0 \\
\phi' & & \phi & & \phi'' & & \\
0 & \to & N' & \to & N & \to & N''
\end{array}
$$

be a commutative diagram in $\mathbb{M}(R)$ with exact rows. Given an element $x'' \in \text{Ker } \phi''$ choose, by surjectivity of $\alpha'$, a preimage $x$ in $M$. Set $y = \phi(x)$ and note that $y$ belongs to $\text{Ker } \beta$, by commutativity of the right-hand square. By exactness at $N$ choose $y' \in N'$ with $\beta'(y') = y$. It is straightforward to verify that the element $[y']_{\text{Im } \phi'}$ in $\text{Coker } \phi'$ does not depend on the choices of $x$ and $y'$, so this procedure defines a map $\delta: \text{Ker } \phi'' \to \text{Coker } \phi'$.

1.1.4 Snake Lemma. The map $\delta: \text{Ker } \phi'' \to \text{Coker } \phi'$ defined in 1.1.3 is a homomorphism of $R$-modules, called the connecting homomorphism, and there is an exact sequence in $\mathbb{M}(R)$,

$$\text{Ker } \phi' \xrightarrow{\alpha'} \text{Ker } \phi \xrightarrow{\alpha} \text{Ker } \phi'' \xrightarrow{\delta} \text{Coker } \phi' \xrightarrow{\beta'} \text{Coker } \phi \xrightarrow{\beta} \text{Coker } \phi''.$$

Moreover, if $\alpha'$ is injective then so is the restriction $\alpha': \text{Ker } \phi' \to \text{Ker } \phi$, and if $\beta$ is surjective, then so is the induced homomorphism $\beta: \text{Coker } \phi \to \text{Coker } \phi''$.

1.1.5 Homomorphisms. From the hom-sets in $\mathbb{M}(R)$ one can construct a functor

$$\text{Hom}_R(-,-): \mathbb{M}(R)^{\text{op}} \times \mathbb{M}(R) \to \mathbb{M}(k).$$

For homomorphisms $\alpha: M' \to M$ and $\beta: N \to N'$ of $R$-modules, the functor acts as follows,

$$\text{Hom}_R(\alpha, \beta): \text{Hom}_R(M,N) \to \text{Hom}_R(M',N') \text{ is given by } \theta \mapsto \beta \theta \alpha.$$
1.1 Prerequisites

1.1.6 Example. Let \( a \) be a left ideal in \( R \) and let \( M \) be an \( R \)-module. There is an isomorphism of \( k \)-modules \( \text{Hom}_R(R/\alpha, M) \to (0 :_M \alpha) \) given by \( \alpha \mapsto \alpha(1_\alpha) \).

1.1.7 Tensor Products. The tensor product of modules yields a functor

\[- \otimes_R - : \mathcal{M}(R^o) \times \mathcal{M}(R) \to \mathcal{M}(k)\]

For a homomorphism \( \alpha : M \to M' \) of \( R^o \)-modules and a homomorphism \( \beta : N \to N' \) of \( R \)-modules, the functor acts as follows,

\[\alpha \otimes_R \beta : M \otimes_R N \to M' \otimes_R N'\]

is given by \( m \otimes n \mapsto \alpha(m) \otimes \beta(n) \).

1.1.8 Example. Let \( b \) be a right ideal in \( R \) and let \( M \) be an \( R \)-module. There is an isomorphism of \( k \)-modules \( R/b \otimes_R M \to M/bM \) given by \( [r]_b \otimes m \mapsto [rm]_{bM} \).

LINEARITY

Because \( R \) is assumed to be a \( k \)-algebra, the module category \( \mathcal{M}(R) \) and its opposite category \( \mathcal{M}(R)^{op} \) are \( k \)-linear in the following sense.

1.1.9 Definition. A category \( \mathcal{U} \) is called \( \mathcal{k} \)-linear if it satisfies the next conditions.

1. There is a \( \text{zero object}, 0, \) in \( \mathcal{U} \). That is, for each object \( M \) in \( \mathcal{U} \) there is a unique morphism \( M \to 0 \) and a unique morphism \( 0 \to M \).

2. For each pair of objects \( M \) and \( N \) in \( \mathcal{U} \) there is a \( \text{biproduct}, M \oplus N, \) in \( \mathcal{U} \). That is, given \( M \) and \( N \) there is a diagram in \( \mathcal{U} \),

\[
\begin{array}{ccc}
M & \xrightarrow{\pi^M} & M \oplus N \\
\downarrow{\iota^M} & & \downarrow{\iota^N} \\
0 & & N
\end{array}
\]

such that \( \pi^M \iota^M = 1_M, \pi^N \iota^N = 1_N, \) and \( \iota^M \pi^M + \iota^N \pi^N = 1_{M \oplus N} \) hold. Here \( 1_X \) denotes the \textit{identity morphism} of the object \( X \).

A category that satisfies (1) is called \( \mathcal{k} \)-prelinear.

Remark. The module category \( \mathcal{M}(R) \) has additional structure; indeed, it is Abelian, and so is the category \( \mathcal{C}(R) \) of \( R \)-complexes, which is the topic of the next chapter. However, the homotopy category \( \mathcal{K}(R) \) and the derived category \( \mathcal{D}(R) \), which are constructed in Chap. 6, are not Abelian but triangulated. All four categories are \( \mathcal{k} \)-linear, hence the focus on that notion.

1.1.10. Let \( \mathcal{U} \) be a \( \mathcal{k} \)-linear category. The biproduct \( \oplus \) is both a product and a coproduct, and it is elementary to verify that it is associative. For a finite set \( U \) and a family \( \{M^u\}_{u \in U} \) of objects in \( \mathcal{U} \) the notation \( \bigoplus_{u \in U} M^u \) is, therefore, unambiguous.

The homomorphism functor 1.1.5 and the tensor product functor 1.1.7 are both \( \mathcal{k} \)-multilinear in the following sense.
1.1.11 Definition. A functor $F: \mathcal{U} \to \mathcal{V}$ between $k$-prelinear categories is called $k$-linear if it satisfies the following conditions:

1. $F(\alpha + \beta) = F(\alpha) + F(\beta)$ for all parallel morphisms $\alpha$ and $\beta$ in $\mathcal{U}$.
2. $F(x\alpha) = xF(\alpha)$ for all morphisms $\alpha$ in $\mathcal{U}$ and all $x \in k$.

Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ and $\mathcal{V}$ be $k$-prelinear categories. A functor $F: \mathcal{U}_1 \times \cdots \times \mathcal{U}_n \to \mathcal{V}$ is called $k$-multilinear if it is $k$-linear in each variable.

1.1.12. There is a unique ring homomorphism $\mathbb{Z} \to k$; therefore, every $k$-linear category/functor is $\mathbb{Z}$-linear in a canonical way. A $\mathbb{Z}$-linear category/functor is also called additive. A $\mathbb{Z}$-prelinear category is also called preadditive.

1.1.13. Let $M$ and $N$ be objects in an additive category $\mathcal{U}$. The zero morphism from $M$ to $N$ is the composite of the unique morphisms $M \to 0$ and $0 \to N$; this morphism is denoted by $0$.

Let $F: \mathcal{U} \to \mathcal{V}$ be an additive functor between additive categories. Then $F$ takes zero morphisms in $\mathcal{U}$ to zero morphisms in $\mathcal{V}$. In particular, $F$ takes the zero object in $\mathcal{U}$ to the zero object in $\mathcal{V}$.

**Products, Coproducts, and Direct Sums of Modules**

1.1.14. For a family $\{M^u\}_{u \in U}$ in $\mathcal{M}(R)$, the product $\prod_{u \in U} M^u$ in $\mathcal{M}(R)$ is the Cartesian product of the underlying sets, with the $R$-module structure given by coordinatewise operations, together with the projections $\pi^u: \prod_{u \in U} M^u \to M^u$. Given a family of homomorphisms $\{\alpha^u: N \to M^u\}_{u \in U}$, the map $\alpha: N \to \prod_{u \in U} M^u$ defined by $n \mapsto (\alpha^u(n))_{u \in U}$ is the unique homomorphism that makes the diagram

\[
\begin{array}{ccc}
\prod_{u \in U} M^u & \xrightarrow{\pi^u} & M^u \\
\alpha \downarrow & & \downarrow \\
N & \xrightarrow{\alpha^u} & M^u
\end{array}
\]

commutative for every $u \in U$. For a family $\{\alpha^u: M^u \to N^u\}_{u \in U}$ of homomorphisms, the product $\prod_{u \in U} \alpha^u: \prod_{u \in U} M^u \to \prod_{u \in U} N^u$ is given by $(m^u)_{u \in U} \mapsto (\alpha^u(m^u))_{u \in U}$.

If one has $M^u = M$ for every $u \in U$, then the product $\prod_{u \in U} M^u$ is denoted $M^U$ and called the $U$-fold product of $M$.

1.1.15. For a family $\{M^u\}_{u \in U}$ in $\mathcal{M}(R)$, the coproduct $\coprod_{u \in U} M^u$ is the submodule

\[
\{ (x^u)_{u \in U} \in \prod_{u \in U} M^u \mid x^u = 0 \text{ for all but finitely many } u \in U \}
\]
of the product, together with the embeddings \( \iota^u : M^u \to \coprod_{u \in U} M^u \). Given a family of homomorphisms \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \), the map \( \alpha : \coprod_{u \in U} M^u \to N \) given by \( (m^u)_{u \in U} \mapsto \sum_{u \in U} \alpha^u(m^u) \) is the unique homomorphism that makes the following diagram commutative

\[
\begin{array}{ccc}
M^u & \xrightarrow{\iota^u} & \coprod_{u \in U} M^u \\
\downarrow{\alpha^u} & & \downarrow{\alpha}
\end{array}
\]

for every \( u \in U \). For a family \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) of homomorphisms the coproduct \( \coprod_{u \in U} M^u = \coprod_{u \in U} N^u \) is given by \( (m^u)_{u \in U} \mapsto (\alpha^u(m^u))_{u \in U} \).

If one has \( M^u = M \) for every \( u \in U \), then the coproduct \( \coprod_{u \in U} M^u \) is denoted \( M^{(U)} \) and called the \( U \)-fold coproduct of \( M \).

### 1.1.16. If \( U \) is a set with two elements (or, more generally, a finite set) then the product and the coproduct of a family \( \{ M^u \}_{u \in U} \) in \( \mathcal{M}(R) \) coincide, and this module \( \coprod_{u \in U} M^u = \coprod_{u \in U} M^u \) is the (iterated) biproduct \( M = \bigoplus_{u \in U} M^u \) of the family. We refer to \( M \) as the direct sum of the family \( \{ M^u \}_{u \in U} \), and each module \( M^u \) is called a direct summand of \( M \).

### 1.1.17. Let \( \{ M^u \}_{u \in U} \) be a family of \( R \)-modules. Notice that for every finite subset \( U' \) of \( U \) there are canonical isomorphisms \( \prod_{u \in U} M^u \cong \left( \bigoplus_{u \in U} M^u \right) \oplus \prod_{u \in U \setminus U'} M^u \) and \( \coprod_{u \in U} M^u \cong \left( \bigoplus_{u \in U} M^u \right) \oplus \coprod_{u \in U \setminus U'} M^u \).

### Bimodules

An Abelian group \( M \) that is both an \( R \)-module and an \( S \)-module is called an \( R \)-\( S \)-bimodule if the two module structures are compatible, i.e. one has \( s(rm) = r(sm) \) for all \( r \in R, s \in S \), and \( m \in M \). A homomorphism of \( R \)-\( S \)-bimodules is a homomorphism of Abelian groups that is both \( R \)- and \( S \)-linear.

The convention to identify right modules with left modules over the opposite ring is highly efficient for abstract considerations. However, in concrete computations with elements in, say, an \( R \)-\( S \)-bimodule it often adds clarity to write the \( S \)-action on the right; the bimodule condition then reads \( (rm)s = r(ms) \).

An \( R \)-\( R^0 \)-bimodule is called symmetric if one has \( rm = mr \) for all \( r \in R \) and \( m \in M \). If \( R \) is commutative, then every \( R \)-module has a canonical structure of a symmetric \( R \)-\( R^0 \)-bimodule. Unless another convention is specified, we shall consider modules over commutative rings as symmetric bimodules.

As \( R \) and \( S \) are \( k \)-algebras, every \( R \)-\( S \)-bimodule is a \( k \)-\( k \)-bimodule. We assume that this structure is symmetric; i.e. we only consider \( k \)-symmetric \( R \)-\( S \)-bimodules. The category of all such modules and their homomorphisms is denoted \( \mathcal{M}(R \rightarrow S) \).

### 1.1.18. By the conventions above, an \( R \)-module is an \( R \)-\( k \)-bimodule, and an \( S^0 \)-module is a \( k \)-\( S^0 \)-bimodule.
A ring homomorphism \( R \to S \) induces \( R\text{-}S^0\text{-}\) and \( S\text{-}R^0\text{-}\) bimodule structures on every \( S\text{-}S^0\text{-}\) bimodule \( M \); in particular on \( S \).

**Remark.** Neither of the two \( R\text{-}R^0\text{-}\) bimodule structures induced on a ring \( R \) by an endomorphism \( \varphi: R \to R \) is symmetric, unless \( \varphi \) is the identity—not even if \( R \) is commutative.

**1.1.19 Example.** For integers \( m,n \geq 1 \) let \( M_{m \times n}(R) \) denote the \( R\text{-}R^0\text{-}\) bimodule of \( m \times n \)-matrices with entries from \( R \). Set \( M = M_{1 \times n}(R) \) and \( N = M_{n \times 1}(R) \), and let \( Q \) denote the ring \( M_{n \times n}(R) \). Then \( M \) is an \( R\text{-}Q^0\text{-}\) bimodule and \( N \) is a \( Q\text{-}R^0\text{-}\) bimodule.

**1.1.20.** There is an equivalence of \( \mathbb{k}\text{-}\) linear Abelian categories,

\[
\mathcal{M}(R\text{-}S^0) \xrightarrow{F} \mathcal{M}(R \otimes_{\mathbb{k}} S^0). 
\]

The functor \( F \) assigns to an \( R\text{-}S^0\text{-}\) bimodule \( M \) the \( R \otimes_{\mathbb{k}} S^0 \)-module with action given by \( (r \otimes s)m = rms \). Conversely, \( G \) assigns to an \( R \otimes_{\mathbb{k}} S^0 \)-module \( M \) the \( R\text{-}S^0\text{-}\) bimodule with \( R\)-action given by \( rm = (r \otimes 1)m \) and right \( S \)-action given by \( ms = (1 \otimes s)m \).

While the set \( \text{Hom}_{R}(M,N) \) of \( R\)-linear maps between \( R\)-modules is a \( \mathbb{k}\)-module it has, in general, no built-in \( R\)-module structure. The reason is, so to say, that the \( R\)-structures on \( M \) and \( N \) are taken up by the \( R\)-linearity of the maps. Similarly, the \( R\)-balancedness takes up the \( R^0\)- and \( R\)-structures on the factors in a tensor product \( M \otimes_{R} N \) and leaves only a \( \mathbb{k}\)-module. If \( R \) is commutative, then one can take \( \mathbb{k} = R \).

At work here is the tacit assumption that a module over a commutative ring \( R \) is a symmetric \( R\text{-}R^0\text{-}\) bimodule. Also in a non-commutative setting, access to richer module structures on hom-sets and tensor products is via bimodules.

**1.1.21.** If \( M \) is an \( R\text{-}Q^0\text{-}\) bimodule and \( N \) is an \( R\text{-}S^0\text{-}\) bimodule, then the \( \mathbb{k}\)-module \( \text{Hom}_{R}(M,N) \) has a \( Q\text{-}S^0\text{-}\) bimodule structure given by

\[
(q \varphi)(m) = \varphi(qm) \quad \text{and} \quad (\varphi s)(m) = (\varphi(m))s.
\]

Moreover, if \( \alpha: M \to M' \) is a homomorphism of \( R\text{-}Q^0\text{-}\) bimodules, and \( \beta: N \to N' \) is a homomorphism of \( R\text{-}S^0\text{-}\) bimodules, then \( \text{Hom}_{R}(\alpha,\beta) \), as defined in 1.1.5, is a homomorphism of \( Q\text{-}S^0\text{-}\) bimodules. Thus, there is an induced \( \mathbb{k}\text{-}\) bilinear functor,

\[
\text{Hom}_{R}(\cdot,\cdot): \mathcal{M}(R\text{-}Q^0)^{\text{op}} \times \mathcal{M}(R\text{-}S^0) \to \mathcal{M}(Q\text{-}S^0).
\]

**1.1.22 Example.** Let \( R \to S \) be a ring homomorphism, and consider \( S \) as an \( R\text{-}S^0\text{-}\) bimodule; see 1.1.18. Now \( \text{Hom}_{R}(S,\cdot) \) is a functor from \( \mathcal{M}(R) \to \mathcal{M}(S) \).

**1.1.23 Example.** Set \( Q = M_{n \times n}(R) \) and \( M = M_{1 \times n}(R) \) as in 1.1.19. An application of 1.1.21 with \( S = \mathbb{k} \) yields a functor \( \text{Hom}_{R}(M,\cdot): \mathcal{M}(R) \to \mathcal{M}(Q) \). Another application of 1.1.21, this time with \( S = M_{n}(R) = Q \), shows that \( \text{Hom}_{R}(M,\cdot) \) is a functor from \( \mathcal{M}(R\text{-}Q^0) \) to \( \mathcal{M}(Q\text{-}Q^0) \). In particular, \( \text{Hom}_{R}(M,M) \) has the structure of a \( Q\text{-}Q^0\text{-}\) bimodule.
1.1 Prerequisites

1.1.24. If $M$ is a $Q-R^{op}$-bimodule and $N$ is an $R-S^{op}$-bimodule, then the $\otimes$-module $M \otimes_R N$ has a $Q-S^{op}$-bimodule structure given by

$$q(m \otimes n) = (qm) \otimes n \quad \text{and} \quad (m \otimes n)s = m \otimes (ns).$$

Moreover, if $\alpha: M \to M'$ is a homomorphism of $Q-R^{op}$-bimodules, and $\beta: N \to N'$ is a homomorphism of $R-S^{op}$-bimodules, then $\alpha \otimes \beta$, as defined in 1.1.7, is a homomorphism of $Q-S^{op}$-bimodules. Thus, there is an induced $\otimes$-bilinear functor,

$$- \otimes_R - : \mathcal{M}(Q-R^{op}) \times \mathcal{M}(R-S^{op}) \to \mathcal{M}(Q-S^{op}).$$

1.1.25 Example. Let $R \to S$ be a ring homomorphism, and consider $S$ as an $S-R^{op}$-bimodule; see 1.1.18. Now $S \otimes_R -$ is a functor from $\mathcal{M}(R)$ to $\mathcal{M}(S)$.

1.1.26 Example. Set $Q = M_{n \times n}(R)$, $M = M_{1 \times n}(R)$, and $N = M_{n \times 1}(R)$ as in 1.1.19. An application of 1.1.24 with $S = \otimes$ yields a functor $N \otimes_R - : \mathcal{M}(R) \to \mathcal{M}(Q)$. Another application, this time with $S = M_{n \times n}(R) = Q$, shows that $N \otimes_R -$ is a functor from $\mathcal{M}(R-Q^{op})$ to $\mathcal{M}(Q-Q^{op})$. In particular, $N \otimes_R M$ is a $Q-Q^{op}$-bimodule.

Exactness

Though we are not concerned with abstract Abelian categories, the language is useful for the following reason. The convention that every functor is covariant forces one to consider, for example, $\text{Hom}_R(-,-)$ as a functor on the product category $\mathcal{M}(R)^{op} \times \mathcal{M}(R)$. While the category $\mathcal{M}(R)^{op}$, and thus $\mathcal{M}(R)^{op} \times \mathcal{M}(R)$, is Abelian, these categories are in general not module categories; that is, they are not equivalent to $\mathcal{M}(S)$ for a ring $S$.

Remark. By Freyd’s theorem [21], as strengthened by Mitchell [38], every Abelian category is, in fact, a full subcategory of the category of modules over some ring.

1.1.27 Split Exact Sequences. An exact sequence $0 \to M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \to 0$ in $\mathcal{M}(R)$ is called split if it satisfies the following equivalent conditions.

(i) There exist homomorphisms $\varphi: M \to M'$ and $\sigma: M'' \to M$ such that one has

$$\varphi \alpha' = 1^M, \quad \alpha' \varphi + \sigma \alpha = 1^M, \quad \text{and} \quad \alpha \sigma = 1^{M''}.$$

(ii) There exists a homomorphism $\varphi: M \to M'$ such that $\varphi \alpha' = 1^M$.

(iii) There exists a homomorphism $\sigma: M'' \to M$ such that $\alpha \sigma = 1^{M''}$.

(iv) The sequence is isomorphic to $0 \to M' \xrightarrow{\iota} M' \oplus M'' \xrightarrow{\pi} M'' \to 0$, where $\iota$ and $\pi$ are the embedding and the projection, respectively.

The definitions 1.1.1 and 1.1.27 of (split) exact sequences in $\mathcal{M}(R)$ make sense in any Abelian category. A (split) exact sequence in $\mathcal{M}(R)^{op}$ is just a (split) exact sequence in $\mathcal{M}(R)$ with the arrows reversed.
1.1.28. Let \( F : \mathcal{U} \to \mathcal{V} \) be an additive functor between Abelian categories. For every split exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{U} \) the induced sequence in \( \mathcal{V} \), \( 0 \to F(M') \to F(M) \to F(M'') \to 0 \), is split exact.

1.1.29 Example. The assignments \( M \mapsto k \oplus M \) and \( \alpha \mapsto 1^k \oplus \alpha \) define a functor \( F : \mathcal{M}(k) \to \mathcal{M}(k) \), which is not additive, as \( F(0) \) is non-zero; cf. 1.1.13.

A frequently used consequence of 1.1.28 is that additive functors preserve direct sums; that is, one has \( F(\bigoplus_{u \in U} M^u) = \bigoplus_{u \in U} F(M^u) \) for an additive functor \( F \) and a finite family of objects \( \{ M^u \}_{u \in U} \).

A functor between Abelian categories is additive, as a matter of course, if it is half exact in the following sense.

1.1.30 Half Exactness. A functor \( F : \mathcal{U} \to \mathcal{V} \) between Abelian categories is called half exact if for every short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{U} \), the sequence \( F(M') \to F(M) \to F(M'') \) in \( \mathcal{V} \) is exact.

The Hom functor 1.1.5 is left exact in the following sense.

1.1.31 Left Exactness. A functor \( F : \mathcal{U} \to \mathcal{V} \) between Abelian categories is called left exact if it satisfies the following equivalent conditions.

(i) For every short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{U} \), the sequence \( 0 \to F(M') \to F(M) \to F(M'') \) in \( \mathcal{V} \) is exact.

(ii) For every (left) exact sequence \( 0 \to M' \to M \to M'' \) in \( \mathcal{U} \), the sequence \( 0 \to F(M') \to F(M) \to F(M'') \) in \( \mathcal{V} \) is exact.

Remark. If \( G : \mathcal{V}(R) \to \mathcal{M}(S) \) is a left exact functor that preserves products, then there exists an \( R \to S \)-bi-module \( N \) and a natural isomorphism \( G \cong \text{Hom}_R(-, N) \). If \( F : \mathcal{M}(R) \to \mathcal{M}(S) \) is a left exact functor that preserves inverse limits, then there exists an \( R \)-module \( M \) and a natural isomorphism \( F \cong \text{Hom}_R(M, -) \); see Watts [52].

The tensor product functor 1.1.7 is right exact in the following sense.

1.1.32 Right Exactness. A functor \( F : \mathcal{U} \to \mathcal{V} \) between Abelian categories is called right exact if it satisfies the following equivalent conditions.

(i) For every short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{U} \), the sequence \( F(M') \to F(M) \to F(M'') \to 0 \) in \( \mathcal{V} \) is exact.

(ii) For every (right) exact sequence \( M' \to M \to M'' \) in \( \mathcal{U} \), the sequence \( F(M') \to F(M) \to F(M'') \) in \( \mathcal{V} \) is exact.

Remark. If \( F : \mathcal{M}(R) \to \mathcal{M}(S) \) is a right exact functor that preserves coproducts, then there exists an \( S \to R \)-bi-module \( M \) and a natural isomorphism \( F \cong M \otimes_R - \); see Watts [52].

Given a homomorphism of rings \( \varphi : R \to S \), the restriction of scalars functor \( \mathcal{M}(S) \to \mathcal{M}(R) \) assigns to each \( S \)-module \( M \) the \( R \)-module with the action induced by \( \varphi \). This functor is exact in the following sense.

1.1.33 Exactness. A functor \( F : \mathcal{U} \to \mathcal{V} \) between Abelian categories is called exact if it satisfies the following equivalent conditions.
1.1 Prerequisites

(i) For every short exact sequence \(0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0\) in \(\mathcal{U}\), the sequence \(0 \rightarrow F(M') \rightarrow F(M) \rightarrow F(M'') \rightarrow 0\) in \(\mathcal{V}\) is exact.

(ii) \(F\) preserves exactness of sequences.

(iii) \(F\) is left exact and right exact.

Faithfulness

1.1.34. Let \(F: \mathcal{U} \rightarrow \mathcal{V}\) be an additive functor between additive categories. The functor \(F\) is called faithful if for every morphism \(\alpha\) in \(\mathcal{U}\) one has \(F(\alpha) = 0\) in \(\mathcal{V}\) only if \(\alpha = 0\) in \(\mathcal{U}\). In that case it follows that for objects \(M\) in \(\mathcal{U}\) one has \(F(M) \cong 0\) in \(\mathcal{V}\) only if \(M \cong 0\) in \(\mathcal{U}\).

1.1.35 Faithful Exactness. An additive functor \(F: \mathcal{U} \rightarrow \mathcal{V}\) between Abelian categories is called faithfully exact if it satisfies the following equivalent conditions.

(i) \(F\) is exact and faithful.

(ii) \(F\) is exact and for every \(M\) in \(\mathcal{U}\) one has \(F(M) \cong 0\) in \(\mathcal{V}\) only if \(M \cong 0\) in \(\mathcal{U}\).

Faithful functors have convenient “cancellation” properties.

1.1.36. Let \(F: \mathcal{U} \rightarrow \mathcal{V}\) be an additive functor between Abelian categories. If \(F\) is faithfully exact and \(M' \rightarrow M \rightarrow M''\) is a sequence in \(\mathcal{U}\), then exactness of the induced sequence \(F(M') \rightarrow F(M) \rightarrow F(M'')\) in \(\mathcal{V}\) implies exactness of \(M' \rightarrow M \rightarrow M''\) in \(\mathcal{U}\).

Let \(G: \mathcal{V} \rightarrow \mathcal{W}\) be a faithfully exact functor between Abelian categories. The composite \(GF: \mathcal{U} \rightarrow \mathcal{W}\) is then (faithfully) exact if and only if \(F\) is (faithfully) exact.

Exercises

E 1.1.1 Let \(\varphi: M \rightarrow N\) be a homomorphism of \(R\)-modules. Show that if \(\varphi\) is bijective, then the inverse map \(N \rightarrow M\) is also a homomorphism of \(R\)-modules; that is, \(\varphi\) is an isomorphism.

E 1.1.2 Show that a morphism in \(\mathcal{M}(R)\) is a monomorphism if and only if it is injective, and that it is an epimorphism if and only if it is surjective.

E 1.1.3 Show that the category \(\mathcal{M}(R)\) is Abelian.

E 1.1.4 Show that in the category of unital rings the embedding \(Z \rightarrow Q\) is both a monomorphism and an epimorphism though not an isomorphism.

E 1.1.5 Consider commutative diagrams in \(\mathcal{M}(R)\) with exact rows

\[
\begin{array}{ccc}
M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' \\
\downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} \\
N' & \xrightarrow{\beta'} & N & \xrightarrow{\beta} & N''
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' \\
\downarrow{\psi'} & & \downarrow{\psi} & & \downarrow{\psi} \\
N' & \xrightarrow{\phi'} & N & \xrightarrow{\phi} & N''
\end{array}
\]

Show that there exist unique homomorphisms \(\varphi': M' \rightarrow N'\) and \(\phi''': M'' \rightarrow N''\), such that the diagrams remain commutative.
Let $F: E \to E''$ be a commutative diagram in $\mathcal{M}(R)$ with exact rows. (a) Show that if $\alpha$ is surjective, then the sequence $\text{Coker} \varphi' \to \text{Coker} \varphi \to \text{Coker} \varphi''$ is exact. (b) Show that if $\beta'$ is injective, then the sequence $\text{Ker} \varphi' \to \text{Ker} \varphi \to \text{Ker} \varphi''$ is exact.

E 1.1.17 Determine the inverse of the map $\text{Hom}_R(R/\alpha,M) \to (0:U \alpha)$ given in 1.1.6.

E 1.1.18 Determine the inverse of the map $R/b \otimes_R M \to M/bM$ given in 1.1.8.

E 1.1.9 Let $\{\alpha^a: M^a \to N^a\}_{a \in U}$ be a family of homomorphisms in $\mathcal{M}(R)$.

(a) Show that $\prod_{a \in U} \alpha^a$, as defined in 1.1.14, is the unique homomorphism $\alpha$ that makes the following diagram commutative for every $a \in U$.

\[
\begin{array}{ccc}
\prod_{a \in U} M^a & \longrightarrow & \prod_{a \in U} N^a \\
\prod_{a \in U} \alpha^a & \quad & \prod_{a \in U} \alpha^a \\
M^a & \longrightarrow & N^a
\end{array}
\]

(b) Show that $\prod_{a \in U} \alpha^a$, as defined in 1.1.15, is the unique homomorphism $\alpha$ that makes the following diagram commutative for every $a \in U$.

\[
\begin{array}{ccc}
M^a & \longrightarrow & \prod_{a \in U} M^a \\
\alpha^a & \quad & \prod_{a \in U} \alpha^a \\
N^a & \longrightarrow & \prod_{a \in U} N^a
\end{array}
\]

E 1.1.10 Show that a non-zero $R$-module is semi-simple if and only if it is isomorphic to a coproduct of simple $R$-modules.

E 1.1.11 For $Q$ and $M$ as in 1.1.23 decide if $\text{Hom}_R(M,M)$ and $Q$ are isomorphic $Q$--$Q^a$-bimodules.

E 1.1.12 For $Q$, $M$, and $N$ as in 1.1.26 decide if $N \otimes_R M$ and $Q$ are isomorphic $Q$--$Q^a$-bimodules.

E 1.1.13 Let $M$ be an $R$-module. (a) Show that $S = \text{Hom}_R(M,M)$ is a $k$-algebra with multiplication given by composition of homomorphisms. (b) Show that $M$ is an $S$-module.

E 1.1.14 Show that the endomorphism algebra $\text{Hom}_R(kR,kR)$, see E 1.1.13, is isomorphic to $R^a$.

E 1.1.15 Show that a half exact functor between Abelian categories is additive.

E 1.1.16 Show that a functor $F: U \to V$ between Abelian categories is exact if and only if the induced sequence $F(M') \to F(M) \to F(M'')$ in $V$ is exact for every exact sequence $M' \to M \to M''$ in $U$.

E 1.1.17 Show that a functor $F: U \to V$ between Abelian categories is right exact if and only if the opposite functor $F^{op}: U^{op} \to V^{op}$ is left exact.

E 1.1.18 Let $F: U \to V$ and $G: V \to W$ be functors between Abelian categories. (a) Show that if both functors are right exact, then $GF$ is right exact. (b) Show that if both functors are left exact, then $GF$ is left exact. (c) Show that if one functor is left exact and the other is right exact, then $GF$ need not be half exact.

E 1.1.19 Let $k$ be a field and set $(-)^* = \text{Hom}_k(-, k)$. Let $M$ be a $k$-vector space. (a) For $m \in M$, show that the map $\epsilon^m: M^* \to k$ given by $\varphi \mapsto \varphi(m)$ is an element in $M^{**} = (M^*)^*$. (b) Show that the map $\delta^M: M \to M^{**}$ given by $m \mapsto \epsilon^m$ is $k$-linear. (c) Show that $\delta: \text{Id}^M(k) \to (-)^*$ is a natural transformation of functors from $\mathcal{M}(k)$ to $\mathcal{M}(k)$.
1.2 Standard Isomorphisms

SYNOPSIS. Commutativity and associativity of tensor product; Hom-tensor adjunction; Hom swap.

Under suitable assumptions about bimodule structures, it makes sense to consider composites like \( \text{Hom}(X \otimes M, N) \) and \( \text{Hom}(M, \text{Hom}(X, N)) \), and they turn out to be isomorphic. At work here is adjunction, one of several natural isomorphisms of composites of Hom and tensor product functors, which are the focus of this section.

We start by recalling that both functors \( R \otimes_R - \) and \( \text{Hom}_R(R, -) \) are naturally isomorphic to the identity functor on \( \mathcal{M}(R) \).

IDENTITIES

1.2.1. For every \( R \)-module \( M \) there are isomorphisms of \( R \)-modules,

\[
\begin{align*}
(1.2.1.1) \quad R \otimes_R M & \cong M \quad \text{given by} \quad r \otimes m \mapsto rm \\
(1.2.1.2) \quad \text{Hom}_R(R, M) & \cong M \quad \text{given by} \quad \psi \mapsto \psi(1),
\end{align*}
\]

and they are natural in \( M \). If \( M \) is an \( R-S^0 \)-bimodule, then these maps are isomorphisms of \( R-S^0 \)-bimodules.

The tensor product behaves as one would expect of a "product". Being additive, the tensor product distributes over direct sums, and the first two standard isomorphisms below show that it is commutative and associative.

COMMUTATIVITY

1.2.2 Proposition. Let \( M \) be an \( R^0 \)-module and let \( N \) be an \( R \)-module. The (tensor) commutativity map

\[
\varpi^{MN} : M \otimes_R N \longrightarrow N \otimes_{R^0} M \quad \text{given by} \quad \varpi^{MN}(m \otimes n) = n \otimes m
\]

is an isomorphism of \( k \)-modules, and it is natural in \( M \) and \( N \). Moreover, if \( M \) is in \( \mathcal{M}(Q-R^0) \) and \( N \) is in \( \mathcal{M}(R-S^0) \), then \( \varpi^{MN} \) is an isomorphism in \( \mathcal{M}(Q-S^0) \).

PROOF. For every \( R^0 \)-module \( M \) and every \( R \)-module \( N \), the map \( \pi \) from \( M \times N \) to \( N \otimes_{R^0} M \) given by \( (m, n) \mapsto n \otimes m \) is \( k \)-bilinear and middle \( R \)-linear. Indeed, one has

\[
\begin{align*}
\pi(m, n + n') &= (n + n') \otimes m = n \otimes m + n' \otimes m = \pi(m, n) + \pi(m, n') \\
\pi(m + m', n) &= n \otimes (m + m') = n \otimes m + n \otimes m' = \pi(m, n) + \pi(m', n) \\
\pi(mr, n) &= n \otimes mr = rn \otimes m = \pi(m, rn) \\
\pi(mx, n) &= \pi(m, xn) = xn \otimes m = x(n \otimes m) = x\pi(m, n)
\end{align*}
\]
for all \( m \in M, n \in N, r \in R, \) and \( x \in k \). Thus, \( \sigma^{MN} \) is a homomorphism of \( k \)-modules.

Let \( \alpha : M \to M' \) be a homomorphism of \( R^e \)-modules and let \( \beta : N \to N' \) be a homomorphism of \( R \)-modules. The diagram

\[
\begin{array}{ccc}
M \otimes_R N & \xrightarrow{\sigma^{MN}} & N \otimes_R M \\
\downarrow{\alpha \otimes \beta} & & \downarrow{\beta \otimes \alpha} \\
M' \otimes_R N' & \xrightarrow{\sigma^{M'N'}} & N' \otimes_R M'
\end{array}
\]

is commutative, as one has

\[
(\sigma^{M'N'} \circ (\alpha \otimes_R \beta))(m \otimes n) = \sigma^{M'N'}(\alpha(m) \otimes \beta(n)) = \beta(n) \otimes \alpha(m) = (\beta \otimes_R \alpha)(n \otimes m) = ((\beta \otimes_R \alpha) \circ \sigma^{MN})(m \otimes n)
\]

for all \( m \in M \) and all \( n \in N \). Thus, \( \sigma \) is a natural transformation of functors from \( \mathcal{M}(R^e) \times \mathcal{M}(R) \) to \( \mathcal{M}(k) \).

For \( M \) in \( \mathcal{M}(R^e) \) and \( N \) in \( \mathcal{M}(R) \) the map from \( N \otimes_R M \) to \( M \otimes_R N \) given by \( n \otimes m \mapsto m \otimes n \) is an inverse of \( \sigma^{MN} \), so \( \sigma^{MN} \) is an isomorphism of \( k \)-modules.

If \( M \) is a \( Q-R^e \)-bimodule and \( N \) is an \( R-S^0 \)-bimodule, then \( M \otimes_R N \) and \( N \otimes_R M \) are \( Q-S^0 \)-bimodules. The computation

\[
\sigma^{MN}(q(m \otimes n)s) = \sigma^{MN}(qm \otimes ns) = ns \otimes qm = q(n \otimes m)s = q(\sigma^{MN}(m \otimes n))s,
\]

which holds for all \( q \in Q, s \in S, m \in M, \) and \( n \in N \), shows that the isomorphism \( \sigma^{MN} \) is \( Q \)- and \( S^0 \)-linear.

\[\Box\]

**Associativity**

**1.2.3 Proposition.** Let \( M \) be an \( R^e \)-module, \( X \) be an \( R-S^0 \)-bimodule, and \( N \) be an \( S \)-module. The (tensor) associativity map

\[
\omega^{MXN} : (M \otimes_R X) \otimes_S N \rightarrow M \otimes_R (X \otimes_S N)
\]

given by

\[
\omega^{MXN}((m \otimes x) \otimes n) = m \otimes (x \otimes n)
\]

is an isomorphism of \( k \)-modules, and it is natural in \( M, X, \) and \( N \). Moreover, if \( M \) is in \( \mathcal{M}(Q-R^e) \) and \( N \) is in \( \mathcal{M}(S-T^0) \), then \( \omega^{MXN} \) is an isomorphism in \( \mathcal{M}(Q-T^0) \).

**Proof.** Proceeding as in the proof of 1.2.2, it is straightforward to verify that \( \omega \) is a natural transformation of functors from \( \mathcal{M}(R^e) \times \mathcal{M}(R-S^0) \times \mathcal{M}(S) \) to \( \mathcal{M}(k) \). Further, for modules \( M, X, \) and \( N \) as in the statement, the assignment \( m \otimes (x \otimes n) \mapsto \)
$(m \otimes x) \otimes n$ defines a map $M \otimes_R (X \otimes_S N) \to (M \otimes_R X) \otimes_S N$; it is an inverse of $\omega^{M \otimes N}$ which, therefore, is an isomorphism of $k$-modules.

If $M$ is in $\mathcal{M}(Q-R^o)$ and $N$ is in $\mathcal{M}(S-T^o)$, then the modules $(M \otimes_R X) \otimes_S N$ and $M \otimes_R (X \otimes_S N)$ are in $\mathcal{M}(Q-T^o)$, and a computation similar to the one performed in the proof of 1.2.2 shows that $\omega^{M \otimes N}$ is $Q$- and $T^o$-linear.

ADJUNCTION

The next isomorphism expresses that Hom and tensor product are adjoint functors.

1.2.4 Proposition. Let $M$ be an $R$-module, $X$ be an $S-R^o$-bimodule, and $N$ be an $S$-module. The (Hom-tensor) adjunction map

$$\rho^{XMN} : \text{Hom}_S(X \otimes_R M, N) \to \text{Hom}_R(M, \text{Hom}_S(X, N))$$

given by

$$\rho^{XMN}(\psi)(m)(x) = \psi(x \otimes m)$$

is an isomorphism of $k$-modules, and it is natural in $X$, $M$, and $N$. Moreover, if $M$ is in $\mathcal{M}(R-Q^o)$ and $N$ is in $\mathcal{M}(S-T^o)$, then $\rho^{XMN}$ is an isomorphism in $\mathcal{M}(Q-T^o)$.

PROOF. It is straightforward to verify that $\rho$ is a natural transformation of functors from $\mathcal{M}(S-R^o)^{op} \times \mathcal{M}(R)^{op} \times \mathcal{M}(S)$ to $\mathcal{M}(k)$. Further, for modules $M$, $X$, and $N$ as in the statement, it is elementary to verify that the map

$$\kappa : \text{Hom}_R(M, \text{Hom}_S(X, N)) \to \text{Hom}_S(X \otimes_R M, N)$$

given by $\kappa(\varphi)(x \otimes m) = \varphi(m)(x)$ is an inverse of $\rho^{XMN}$.

If $M$ is in $\mathcal{M}(R-Q^o)$ and $N$ is in $\mathcal{M}(S-T^o)$, then $\text{Hom}_S(X \otimes_R M, N)$ is a $Q-T^o$-bimodule and so is $\text{Hom}_R(M, \text{Hom}_S(X, N))$. The computation

$$\rho^{XMN}(q\psi t)(m)(x) = (q\psi t)(x \otimes m)$$

$$= (\psi(x \otimes mq))t$$

$$= (\rho^{XMN}(\psi)(mq)(x))t$$

$$= ((q(\rho^{XMN}(\psi)))(m)(x))t$$

$$= (q(\rho^{XMN}(\psi))(x)(m)(x))$$

which holds for all $q \in Q$, $t \in T$, $\psi \in \text{Hom}_S(X \otimes_R M, N)$, $m \in M$, and $x \in X$, shows that the isomorphism $\rho^{XMN}$ is $Q$- and $T^o$-linear. □
1.2.5 Proposition. Let $M$ be an $R$-module, $X$ be an $R^{-S^o}$-bimodule, and $N$ be an $S^o$-module. The (Hom) swap map

$$\zeta^{MNX} : \text{Hom}_R(M, \text{Hom}_{S^o}(N, X)) \longrightarrow \text{Hom}_{S^o}(N, \text{Hom}_R(M, X))$$

given by

$$\zeta^{MNX}(\psi)(n)(m) = \psi(m)(n)$$

is an isomorphism of $\mathbb{k}$-modules, and it is natural in $M$, $N$, and $X$. Moreover, if $M$ is in $\mathcal{M}(R^{-Q^o})$ and $N$ is in $\mathcal{M}(T^{-S^o})$, then $\zeta^{MNX}$ is an isomorphism in $\mathcal{M}(Q^{-T^o})$.

**Proof.** It is straightforward to verify that $\zeta$ is a natural transformation of functors from $\mathcal{M}(R)^{op} \times \mathcal{M}(S)^{op} \times \mathcal{M}(R^{-S^o})$ to $\mathcal{M}(\mathbb{k})$. Further, for modules $M$, $N$, and $X$ as in the statement, it is immediate that the swap map $\zeta^{MNX}$ is an inverse of $\zeta^{MNX}$.

If $M$ is in $\mathcal{M}(R^{-Q^o})$ and $N$ is in $\mathcal{M}(T^{-S^o})$, then $\text{Hom}_R(M, \text{Hom}_{S^o}(N, X))$ and $\text{Hom}_{S^o}(N, \text{Hom}_R(M, X))$ are $Q^{-T^o}$-bimodules. The computation

$$\zeta^{MNX}(q\psi t)(n)(m) = (\psi q t)(m)(n)$$

$$= \psi(mq)(n)$$

$$= \zeta^{MNX}(\psi)(m)(mq)$$

$$= (\psi(\zeta^{MNX}(\psi)) t)(n)(m),$$

which holds for all $q \in Q$, $t \in T$, $\psi \in \text{Hom}_R(M, \text{Hom}_{S^o}(N, X))$, $m \in M$, and $n \in N$, shows that the isomorphism $\zeta^{MNX}$ is $Q$- and $T^o$-linear. \(\square\)

**Exercises**

**E 1.2.1** Determine the inverse maps of (1.2.1.1) and (1.2.1.2).

**E 1.2.2** Let $\mathbb{k}$ be a field and set $(\cdot)^* = \text{Hom}_{\mathbb{k}}(\cdot, \mathbb{k})$. Let $L$ be a $\mathbb{k}$-vector space with basis $\{e_u\}_{u \in U}$. For each $u \in U$ let $e^*_u : L \to \mathbb{k}$ be the functional given by $e^*_u(e_v) = \delta_{uv}$. Show that the assignment $e_u \mapsto e^*_u$ defines a homomorphism $\epsilon : L \to L^*$ of $\mathbb{k}$-vector spaces. Assume that $L$ has rank at least 2; show that there is an automorphism $\alpha : L \to L$ such that the following diagram is not commutative,

$$\begin{array}{ccc}
L & \xrightarrow{\epsilon} & L^* \\
\downarrow{\alpha} & & \downarrow{\alpha^*} \\
L & \xrightarrow{\epsilon} & L^*.
\end{array}$$

**E 1.2.3** Let $M$ be a $\mathbb{k}$-module. Show that the functor $M \otimes_{\mathbb{k}} \cdot$ is left adjoint to $\text{Hom}_{\mathbb{k}}(M, \cdot)$.

**E 1.2.4** Let $M$ be an $R$-module, $X$ be an $S^{-R^o}$-bimodule, and $N$ be an $S$-module. Without using 1.2.2–1.2.5, show that there is a natural isomorphism of $\mathbb{k}$-modules

$$\text{Hom}_S(M \otimes_{R^o} X, N) \longrightarrow \text{Hom}_R(M, \text{Hom}_S(X, N)).$$
1.3 Exact Functors and Classes of Modules

**Synopsis.** Basis; free module; extension property; projective module; injective module; lifting properties; semi-simple ring; Baer’s criterion; flat module; von Neumann regular ring.

We start by recalling the language of generators for modules and ideals.

1.3.1. Let \( M \) be an \( R \)-module, and let \( E = \{ e_u \}_{u \in U} \) be a subset of \( M \). It is simple to verify that the subset \( R\langle E \rangle = \{ \sum_{u \in U} r_u e_u \mid r_u \in R \text{ and } r_u = 0 \text{ for all but finitely many } u \in U \} \) of \( M \) is a submodule; it is called the **module of \( M \) generated by \( E \)**. By convention, \( R\langle / 0 \rangle \) is the zero module. If one has \( R\langle E \rangle = M \), then \( E \) is called a **set of generators** for \( M \). If \( M \) has a finite set of generators, then \( M \) is called **finitely generated**. If \( M \) is generated by one element, then \( M \) is called **cyclic**.

1.3.2. Left ideals and right ideals in \( R \) generated by elements \( x_1, \ldots, x_n \) are denoted \( R\langle x_1, \ldots, x_n \rangle \) and \( (x_1, \ldots, x_n)R \), respectively. The abridged notations \( Rx \) and \( xR \) are used for principal left and right ideals. If \( R \) is commutative, then the ideal \( R\langle x_1, \ldots, x_n \rangle = (x_1, \ldots, x_n)R \) is written \( (x_1, \ldots, x_n) \), and a principal ideal may be written using any of the notations \( Rx \), \( xR \), and \( (x) \).

**Remark.** Though left and right ideals in \( R \) are submodules of the \( R \)-module \( R \) and the \( R^0 \)-module \( R \), respectively, it would be awkward to insist on applying the notation from 1.3.1 to ideals. Indeed, a principal right ideal would be written \( R\langle x \rangle \); even worse, writing \( k\langle x_1, x_2 \rangle \) for the ideal in \( k \) generated by elements \( x_1 \) and \( x_2 \) would conflict with the standard notation for a free \( k \)-algebra.

**Free Modules**

1.3.3 **Definition.** Let \( L \) be an \( R \)-module and let \( E = \{ e_u \}_{u \in U} \) be a set of generators for \( L \). Every element in \( L \) can then be expressed on the form \( \sum_{u \in U} r_u e_u \); if this expression is unique, then \( E \) is called a **basis** for \( L \), and \( L \) is called **free**. By convention, the zero module is free with the empty set as basis.

For a set \( E \), not \textit{a priori} assumed to be a subset of a module, the free \( R \)-module with basis \( E \) is denoted \( R\langle E \rangle \).

1.3.4 **Example.** If \( R \) is a division ring, then every \( R \)-module is free; indeed, every module over a division ring has a basis.

The \( \mathbb{Z} \)-module \( M = \mathbb{Z}/2\mathbb{Z} \) is not free; indeed, a set of generators for \( M \) must include \( [1]_{\mathbb{Z}} \), and one has \( 0[1]_{\mathbb{Z}} = 2[1]_{\mathbb{Z}} \).
Bases of free modules have the following unique extension property.

1.3.5. Let $L$ be a free $R$-module with basis $E = \{e_u\}_{u \in U}$ and let $M$ be an $R$-module. For every map $\alpha : E \rightarrow M$ there is a unique $R$-module homomorphism $\overline{\alpha} : L \rightarrow M$, such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & L \\
\downarrow & & \downarrow \\
M & \xrightarrow{\overline{\alpha}} & M
\end{array}
\]

is commutative; the homomorphism is given by $\overline{\alpha}(\sum_{u \in U} r_u e_u) = \sum_{u \in U} r_u \alpha(e_u)$.

Remark. The unique extension property characterizes bases; see E 1.3.3.

1.3.6. Let $L$ be a free $R$-module with basis $\{e_u\}_{u \in U}$ and let $\{f_u\}_{u \in U}$ be the standard basis for $R^{(U)}$. There is an isomorphism of $R$-modules, $L \rightarrow R^{(U)}$, given by $e_u \mapsto f_u$.

Important families of rings—non-zero commutative rings, left Noetherian rings, and local rings included—have the invariant basis number property (IBN).

1.3.7 Definition. A free module is said to have finite rank if it is finitely generated; that is, it has a finite basis. A free module that is not finitely generated is said to have infinite rank. For a finitely generated free module $L$ over a ring $R$ that has IBN, the rank of $L$, written $\text{rank}_R L$, is the number of elements in a basis for $L$.

Remark. If $U$ is infinite, then one has $R^{(U)} \cong R^V$ only if the sets $U$ and $V$ have the same cardinality, even if $R$ does not have IBN; see [32, cor. (1.2)].

1.3.8. Let $\{L^u\}_{u \in U}$ be a family of free $R$-modules with bases $\{E^u\}_{u \in U}$. The co-product $\coprod_{u \in U} L^u$ is then a free $R$-module with basis $\bigcup_{u \in U} i^u(E^u)$, where $i^u$ is the embedding $L^u \rightarrow \bigcup_{u \in U} L^u$. Notice that if $U$ is a finite set, each module $L^u$ is finitely generated, and $R$ has IBN, then one has $\text{rank}_R (\bigoplus_{u \in U} L^u) = \sum_{u \in U} \text{rank}_R L^u$.

1.3.9. Let $L$ and $L'$ be free $k$-modules with bases $\{e_u\}_{u \in U}$ and $\{f_v\}_{v \in V}$. It is elementary to verify that the $k$-module $L \otimes_k L'$ is free with basis $\{e_u \otimes f_v\}_{u \in U, v \in V}$. Thus, if $L$ and $L'$ are finitely generated, then one has $\text{rank}_k(L \otimes_k L') = (\text{rank}_k L)(\text{rank}_k L')$.

1.3.10 Theorem. If $R$ is a principal left ideal domain, then every submodule of a free $R$-module is free.

Proof. Let $L$ be a free $R$-module with basis $\{e_u\}_{u \in U}$, and let $M$ be a submodule of $L$. Choose a well-ordering $\leq$ on $U$. For $u \in U$ define submodules of $L$ as follows:

$$L^<u = R\langle e_v \mid v < u \rangle \quad \text{and} \quad L^\leq u = R\langle e_v \mid v \leq u \rangle.$$ 

Let $u \in U$ be given. Every element $l$ in $L^\leq u$ has a unique decomposition $l = l' + re_u$ with $l' \in L^<u$ and $r \in R$, so there is a split exact sequence of $R$-modules,

$$0 \rightarrow L^<u \rightarrow L^\leq u \xrightarrow{\pi_u} Re_u \rightarrow 0,$$
Let $\pi u$ be the restriction of $\pi u$ to $M \cap L^{\leq u}$; one has $\text{Ker}\varphi_u = M \cap \text{Ker}\pi u = M \cap L^{< u}$. The image of $\varphi_u$ is a submodule of $R e_u$ and, hence, isomorphic to a left ideal in $R$. It follows from the assumption on $R$ that $\text{Im}\varphi_u$ is cyclic and free, so choose $\chi_u \in R$ with $\text{Im}\varphi_u = R \chi_u e_u$ and note that if $x_u$ is non-zero, then $\{x_u e_u\}$ is a basis for $\text{Im}\varphi_u$. Now there is a short exact sequence

\[ 0 \longrightarrow M \cap L^{< u} \longrightarrow M \cap L^{\leq u} \xrightarrow{\pi u} R \chi_u e_u \longrightarrow 0, \]

and it splits. Indeed, if $x_u \neq 0$ choose an element $f_u$ in $M \cap L^{\leq u}$ with $\varphi_u(f_u) = x_u e_u$, and if $x_u = 0$ set $f_u = 0$. The assignment $x_u e_u \mapsto f_u$ then defines a right inverse homomorphism to $\varphi_u$; cf. 1.3.5. Thus, one has $M \cap L^{\leq u} = (M \cap L^{< u}) \oplus R f_u$, that is, every element $m$ in $M \cap L^{\leq u}$ has a unique decomposition $m = m' + r f_u$ with $m' \in M \cap L^{< u}$ and $r \in R$.

Set $U' = \{u \in U \mid x_u \neq 0\}$; we will show that $\{f_u\}_{u \in U'}$ is a basis for $M$. To see that every linear combination of the elements $f_u$ is unique, suppose that one has a relation $r_1 f_{u_1} + \cdots + r_n f_{u_n} = 0$ with $r_i \in R$ and $u_i \in U'$. We may assume that the elements $u_i$ are ordered $u_1 < \cdots < u_n$, and then consider the relation in $M \cap F^{\leq u}$.

Applying $\varphi_u$ to the relation one gets $r_n x_{u_n} e_{u_n} = 0$. As $u_n$ is in $U'$, the singleton $\{x_{u_n} e_{u_n}\}$ is a basis for $\text{Im}\varphi_u$, whence one has $r_n = 0$. Continuing in this manner, one gets $r_n = \cdots = r_1 = 0$, as desired. If $F = R\{f_u\}_{u \in U'}$ were a proper submodule of $M$, then there would be a least $u$ in $U'$ such that $M \cap L^{\leq u}$ contains an element $m$ not in $F$. This element has a unique decomposition $m = m' + r f_u$ with $m' \in M \cap L^{< u}$ and $r \in R$. The element $m'$ is in $M \cap L^{\leq v}$ for some $v < u$ and hence in $F$ by minimality of $u$. However, $r f_u$ is also in $F$, so one has $m = m' + r f_u \in F$; a contradiction. \hfill \Box

**Remark.** A commutative ring is a principal ideal domain if (and only if) every submodule of a free module is free; see E 1.3.18.

The content of the next result is often phrased as: the category of $R$-modules has enough free modules.

**1.3.11 Lemma.** For every $R$-module $M$ there is a surjective homomorphism $L \longrightarrow M$ of $R$-modules where $L$ is free. Moreover, if $M$ is finitely generated, then $L$ can be chosen finitely generated.

**Proof.** Choose a set $G$ of generators for $M$ and let $E = \{e_g \mid g \in G\}$ be an abstract set. Consider the free $R$-module $L = R\langle E \rangle$ and define by 1.3.5 a homomorphism $\pi : L \longrightarrow M$ by $\pi(\sum_{g \in G} r_g e_g) = \sum_{g \in G} r_g g$; it is surjective by the assumption on $G$. The statement about finitely generated modules follows from the construction of $L$. \hfill \Box

**1.3.12.** Let $M$ be an $R$-module. By 1.3.11 there exist free $R$-modules $L$ and $L'$ such that there is an exact sequence

\[ (1.3.12.1) \quad L' \longrightarrow L \longrightarrow M \longrightarrow 0. \]

**1.3.13 Definition.** Let $M$ be an $R$-module. The exact sequence (1.3.12.1) is called a free presentation of $M$. If $M$ has a free presentation with $L$ and $L'$ finitely generated, then $M$ is called finitely presented.
Every finitely presented module is finitely generated; the converse holds over left Noetherian rings; in fact, it defines left coherent rings.

Under suitable assumptions on the ring, the Hom and tensor product functors restrict to the subcategories of finitely generated modules.

1.3.14 Proposition. Assume that \( k \) is Noetherian and that \( R \) is finitely generated as a \( k \)-module. If \( M \) and \( N \) are finitely generated \( R \)-modules, then the \( k \)-module \( \text{Hom}_R(M,N) \) is finitely generated.

Proof. Choose by 1.3.11 a surjective homomorphism of \( R \)-modules \( L \to M \), where \( L \) is free and finitely generated; say, \( L \cong R^n \) as \( R \)-modules. Apply the left exact functor \( \text{Hom}_R(-,N) \) to get an injective homomorphism \( \text{Hom}_R(M,N) \to \text{Hom}_R(L,N) \). By (1.2.1.2) and additivity of the Hom functor there is an isomorphism of \( k \)-modules, \( \text{Hom}_R(L,N) \cong N^n \). Thus, \( \text{Hom}_R(M,N) \) is a submodule of a finitely generated \( k \)-module and hence finitely generated, as \( k \) is Noetherian.

1.3.15 Proposition. Assume that \( R \) is finitely generated as a \( k \)-module. If \( M \) is a finitely generated \( R \)-module and \( N \) is a finitely generated \( R \)-module, then the \( k \)-module \( M \otimes_R N \) is finitely generated.

Proof. Since \( M \) and \( N \) are finitely generated, there are surjective homomorphisms \( \pi_M : L' \to M \) of \( R^n \)-modules and \( \pi_N : L'' \to N \) of \( R \)-modules, where \( L' \) and \( L'' \) are finitely generated and free; see 1.3.11. By right exactness of the tensor product, the homomorphism of \( k \)-modules \( \pi_M \otimes_R \pi_N : L' \otimes_R L'' \to M \otimes_R N \) is also surjective.

By the assumption on \( R \), there is a surjective homomorphism of \( k \)-modules \( F \to R \), where \( F \) is finitely generated and free. Hence, there are surjective homomorphisms of \( k \)-modules \( \pi_{L'} : F' \to L' \) and \( \pi_{L''} : F'' \to L'' \), where \( F' \) and \( F'' \) are finitely generated and free. As above, the homomorphism \( \pi_{L'} \otimes_k \pi_{L''} \) is surjective, and the composite \( (\pi_M \otimes_R \pi_N) \circ (\pi_{L'} \otimes_k \pi_{L''}) \) realizes \( M \otimes_R N \) as a homomorphic image of the finitely generated \( k \)-module \( F' \otimes_k F'' \); cf. 1.3.9.

Projective Modules

For an \( R \)-module \( M \), the functors \( \text{Hom}_R(M,-) \), \( \text{Hom}_R(-,M) \), and \( - \otimes_R M \) are, in general, not exact. Modules that make one or more of these functors exact are of particular interest and play a pivotal role in homological algebra.

1.3.16 Definition. An \( R \)-module \( P \) is called projective if the functor \( \text{Hom}_R(P,-) \) from \( M(R) \) to \( M(k) \) is exact.

Part (ii) below captures the lifting property of projective modules, which amounts to the definition of projective objects in a general category.

1.3.17 Proposition. For an \( R \)-module \( P \), the following conditions are equivalent.

(i) \( P \) is projective.
1.3.20 Example. In a product ring $R \times S$, the ideal $P = R \times 0$ is a projective module; if $S$ is non-zero then $P$ is not free.

1.3.21 Example. In the commutative ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$, the subset $a = \{a + b\sqrt{-5} \mid a = b \mod 2\}$ is an ideal. The elements $1 \pm \sqrt{-5}$ in $a$ have no common factor, so $a$ is not a principal ideal, whence it is not free. Yet, the map $\mathbb{Z}[\sqrt{-5}] \oplus \mathbb{Z}[\sqrt{-5}] \to a \oplus a$ given by $(r,s) \mapsto (2r + (1 + \sqrt{-5})s, 2s + (1 - \sqrt{-5})r)$ is an isomorphism of $\mathbb{Z}[\sqrt{-5}]$-modules, so $a$ is a direct summand of a free module and hence projective.
1.3.22 Proposition. Let \( \{ P^u \}_{u \in U} \) be a family of \( R \)-modules. The coproduct \( \bigsqcup_{u \in U} P^u \) is projective if and only if each module \( P^u \) is projective.

Proof. If the coproduct \( \bigsqcup_{u \in U} P^u \) is projective, then by 1.3.17 it is a direct summand of a free module, and hence so is each module \( P^u \). Conversely, if each module \( P^u \) is projective and hence a direct summand of a free module \( L^u \), then the coproduct \( \bigsqcup_{u \in U} P^u \) is a direct summand of the free module \( \bigsqcup_{u \in U} L^u \); cf. 1.3.8. \( \square \)

Remark. A product of projective modules need not be projective; see E 1.3.17.

Injective Modules

1.3.23 Definition. An \( R \)-module \( I \) is called injective if the functor \( \text{Hom}_R(-, I) \) from \( \mathcal{M}(R)^{\text{op}} \) to \( \mathcal{M}(k) \) is exact. If the functor \( \text{Hom}_R(-, I) \) is faithfully exact, then \( I \) is called faithfully injective.

1.3.24. An \( R \)-module \( I \) is injective if and only if it has the following lifting property, which amounts to the definition of an injective object in a general category. Given a homomorphism \( \alpha : K \rightarrow I \) and an injective homomorphism \( \beta : K \rightarrow M \), there exists a homomorphism \( \gamma : M \rightarrow I \) such that the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{\beta} & M \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
I & & \\
\end{array}
\]

in \( \mathcal{M}(R) \) is commutative; that is, there is an equality \( \gamma \beta = \alpha \).

Remark. The epimorphisms in the category \( \mathcal{M}(R) \) are exactly the surjective homomorphisms; see E 1.1.2. In view of this and 1.3.17, an \( R \)-module \( P \) is projective if and only if the functor \( \text{Hom}_R(P, -) : \mathcal{M}(R) \rightarrow \mathcal{M}(k) \) takes epimorphisms to epimorphisms.

The epimorphisms in the category \( \mathcal{M}(R)^{\text{op}} \) correspond to monomorphisms—which by E 1.1.2 are the injective homomorphisms—in \( \mathcal{M}(R) \). In view of this and 1.3.24, an \( R \)-module \( I \) is injective if and only if the functor \( \text{Hom}_R(-, I) : \mathcal{M}(R)^{\text{op}} \rightarrow \mathcal{M}(k) \) takes epimorphisms to epimorphisms.

1.3.25 Proposition. Let \( \{ I^u \}_{u \in U} \) be a family of \( R \)-modules. The product \( \prod_{u \in U} I^u \) is injective if and only if each module \( I^u \) is injective.

Proof. Let \( \beta : K \rightarrow M \) be an injective homomorphism of \( R \)-modules, let \( \{ I^u \}_{u \in U} \) be a family of \( R \)-modules, and set \( I = \prod_{u \in U} I^u \). Assume first that each module \( I^u \) is injective. Let a homomorphism \( \alpha : K \rightarrow I \) be given. For each \( u \in U \) set \( \alpha^u = \pi^u \alpha \), where \( \pi^u \) is the projection \( I \rightarrow I^u \). By assumption there exist homomorphisms \( \gamma^u : M \rightarrow I^u \), such that \( \gamma^u \beta = \alpha^u \) holds for every \( u \in U \). The unique homomorphism \( \gamma : M \rightarrow I \) with \( \gamma^u = \pi^u \gamma \) now satisfies \( \gamma \beta = \alpha \), so \( I \) is injective.

Assume now that \( I \) is injective, fix an element \( u \in U \), and let \( i^u \) denote the embedding \( I^u \rightarrow I \). Given a homomorphism \( \alpha : K \rightarrow I^u \), one has a homomorphism \( i^u \alpha \) from...
1.3 Exact Functors and Classes of Modules

K to I. By injectivity of I there is a homomorphism γ : M → I such that γβ = i^rα holds and, therefore, one has (π''γ)β = α. □

1.3.26 Theorem. The following conditions are equivalent.

(i) R is semi-simple.
(ii) Every short exact sequence of R-modules is split.
(iii) Every R-module is projective.
(iv) Every R-module is injective.

Proof. Recall that R being semi-simple means that every submodule M' of an R-module M is a direct summand of that module; conditions (i) and (ii) are, therefore, equivalent. It is evident that (ii) implies (iii) and (iv). The converse implications follow from 1.3.17 and 1.3.24. □

Remark. A commutative ring is semi-simple if and only if it is a finite product of fields [33, §3].

1.3.27 Example. Every module over a division ring is injective, as division rings are semi-simple.

The next result is known as Baer’s criterion.

1.3.28 Lemma. Let I be an R-module, let a be a left ideal in R, and let i : a → R be the embedding. The module I is injective if and only if for every R-module homomorphism φ : a → I there exists a homomorphism φ : R → I with φ i = φ.

Proof. The “only if” part of the statement follows from 1.3.24. To prove “if”, let α : K → I and β : K → M be homomorphisms of R-modules and assume that β is injective. Denote by G the set of all homomorphisms γ : M' → I with Im β ⊆ M' and γβ = α. Since αβ−1 : Im β → I belongs to G, this set is non-empty. By declaring (γ' : M' → I) ≤ (γ'' : M'' → I) if M' ⊆ M'' and γ''|M' = γ', the set G becomes inductively ordered. Hence Zorn’s lemma guarantees the existence of a maximal element γ'' : M'' → I. To prove the lemma, we show the equality M'' = M. Assume, towards a contradiction, that M'' is a proper submodule of M and choose an element m ∈ M \ M''. The set a = (M'' :R m) is a left ideal of R. The map φ : a → I given by φ(r) = γ''(rm) is an R-module homomorphism, so by assumption it has a lift φ : R → I. Now, set M' = M'' + Rm and notice that γ' : M' → I given by γ'(m'' + rm) = γ''(m'' + φ(r)) satisfies γ'β = α. Hence γ' belongs to G and satisfies γ' > γ'', which contradicts the maximality of γ''. □

1.3.29. Let R be a domain. Recall that an R-module M is called divisible if rM = M holds for all r ≠ 0 in R. Every injective R-module I is divisible. Indeed, right-multiplication by r ≠ 0 yields an injective homomorphism R → R of R-modules, so the induced homomorphism Hom_R(R, I) → Hom_R(R, I) of -modules is surjective, whence one has rI = I; cf. (1.2.1.2). The converse statement in 1.3.30 below, however, hinges crucially on the principal ideal hypothesis, as illustrated in 1.3.31.
1.3.30 Proposition. Let $R$ be a principal left ideal domain. An $R$-module $I$ is injective if and only if it is divisible. Moreover, every quotient of an injective $R$-module is injective.

**Proof.** Once the first claim is proved, the second one follows, as the divisibility property is inherited by quotient modules. Every injective $R$-module is divisible by 1.3.29. Assume that $I$ is divisible. Let $a$ be a left ideal in $R$; by assumption there exists an element $x \in R$ with $Rx = a$. A homomorphism of $R$-modules $\alpha : a \to I$ is determined by the value $\alpha(x) = i$, and it can be lifted to a homomorphism $R \to I$ as there exists an element $x' \in I$ with $xi' = i$. Thus, it follows from Baer’s criterion 1.3.28 that $I$ is injective. \(\square\)

1.3.31 Example. Let $k$ be a field and consider the polynomial ring $R = k[x,y]$; its field of fractions is the field $Q = k(x,y)$ of rational functions. Evidently, $Q$ and, therefore, $Q/R$ are divisible $R$-modules. However, $Q/R$ is not injective. Indeed, let $M$ be the ideal of polynomials with zero constant term (called the irrelevant maximal ideal). The homomorphism $M \to Q/R$ given by $f \mapsto \frac{f(x,0)}{xy}$ does not extend to a homomorphism $\varphi : R \to Q/R$. If it did, then $\varphi(1)$, which has the form $[gh^{-1}]_R$ for some $g$ and $h$ in $R$, would satisfy $y\varphi(1) = \varphi(y) = [0]_R$ and $x\varphi(1) = \varphi(x) = [y^{-1}]_R$. The first equation shows that there would be a $k \in R$ with $ygh^{-1} = k$ and, therefore, $\varphi(1) = [gh^{-1}]_R = [ky^{-1}]_R$. To satisfy the second equation there would be an $l$ in $R$ with $xky^{-1} = y^{-1} + l$, that is, $xk = 1 + ly$, which is absurd.

**Faithful Injectivity**

It follows from 1.3.30 that $Q$ and $Q/\mathbb{Z}$ are injective $\mathbb{Z}$-modules; one can say even more about the latter module.

1.3.32 Lemma. The $\mathbb{Z}$-module $Q/\mathbb{Z}$ is faithfully injective.

**Proof.** Let $G$ be a non-zero $\mathbb{Z}$-module, and choose an element $g \neq 0$ in $G$. Define a homomorphism $\xi$ from the cyclic submodule $\mathbb{Z}\langle g \rangle$ to $Q/\mathbb{Z}$ as follows. If $g$ is torsion set $\xi(g) = \frac{1}{n}g$, where $n$ be the least positive integer with $ng = 0$. If $g$ is not torsion, set $\xi(g) = \frac{1}{2}g$. By the lifting property of injective modules 1.3.24, there is a homomorphism $G \to Q/\mathbb{Z}$ that restricts to $\xi$ on $\mathbb{Z}\langle g \rangle$, whence $\text{Hom}_{\mathbb{Z}}(G, Q/\mathbb{Z})$ is non-zero. \(\square\)

1.3.33 Lemma. Let $E$ be a $k$-module. If $E$ is (faithfully) injective then the $R$-module $\text{Hom}_k(R, E)$ is (faithfully) injective.
1.3 Exact Functors and Classes of Modules

PROOF. By adjunction 1.2.4 and (1.2.1.1) there are natural isomorphisms,
\[
\text{Hom}_R(-, \text{Hom}_E(R,E)) \cong \text{Hom}_E(R \otimes_R -, E) \cong \text{Hom}_E(-, E),
\]
of functors from \(M(R)\) to \(M(k)\). By assumption, the functor \(\text{Hom}_E(R \otimes_R -, E)\) is (faithfully) exact.

1.3.34 Definition. The faithfully injective \(R\)-module \(E^R = \text{Hom}_Z(R_R, \mathbb{Q}/\mathbb{Z})\) is called the character module of \(R\). The abbreviated notation \(E\) is used for \(E^k\).

Flat Modules

1.3.35 Definition. An \(R\)-module \(F\) is called flat if the functor \(- \otimes_R F\) from \(M(R)\) to \(M(k)\) is exact. If \(- \otimes_R F\) is faithfully exact, then \(F\) is called faithfully flat.

1.3.36. As the tensor product is right exact, an \(R\)-module \(F\) is flat if and only if the homomorphism \(\alpha \otimes_R F\) is injective for every injective homomorphism \(\alpha\) in \(M(R)\).

1.3.37 Example. It is elementary to verify that every free module is faithfully flat. Hence, by additivity of the tensor product, every projective module is flat; see 1.3.17.

1.3.38 Lemma. Let \(M\) be an \(R\)-module and \(K\) be a submodule of \(M\). If the quotient module \(M/K\) is flat, then one has \(bK = bM \cap K\) for every right ideal \(b\) in \(R\).

PROOF. Let \(\iota\) denote the embedding \(b \rightarrow R\). By assumption, the homomorphism \(\iota \otimes_R M/K\) is injective. In the commutative square of \(\mathbb{Z}\)-modules

\[
\begin{array}{ccc}
\text{b} \otimes_R M/K & \xrightarrow{\iota \otimes M/K} & R \otimes_R M/K \\
| & & | \\
\text{b} \otimes_R M/K & \xrightarrow{\mu_M/K} & M/K \\
\end{array}
\]

the right-hand vertical isomorphism is (1.2.1.1), and \(\mu_M/K\) is the homomorphism given by \(\mu_M/K(r \otimes [m]_K) = [rm]_K\). It follows from commutativity of the square that \(\mu_M/K\) is injective. There is a commutative diagram with exact rows

\[
\begin{array}{cccc}
b \otimes_R K & \xrightarrow{\mu_K} & b \otimes_R M & \xrightarrow{\mu_M} & b \otimes_R M/K & \rightarrow 0 \\
0 & \xrightarrow{\mu_M} & K & \xrightarrow{\mu_M/K} & M/K & \rightarrow 0 \\
\end{array}
\]

where the upper row is obtained by application of \(b \otimes_R -\) to the lower row. One has \(\text{Im} \mu_K = bK\) and \(\text{Im} \mu_M = bM\), so it follows from the Snake Lemma 1.1.4 that the canonical homomorphism \(K/bK \rightarrow M/bM\) is injective. This yields the containment \(K \cap bM \subseteq bK\); the opposite containment is trivial, so equality holds.
1.3.39 Proposition. If every $R$-module is flat, then $R$ is von Neumann regular.

**Proof.** Let $x$ be an element in $R$ and set $M = R$, $K = Rx$, and $b = xR$; then one has $x \in bM \cap K$. If every $R$-module is flat, then 1.3.38 yields $x \in bK = (xR)(Rx)$, so $R$ is von Neumann regular. 

**Remark.** The von Neumann regular rings are, in fact, precisely the rings over which every module is flat; see E 3.2.13.

1.3.40 Theorem. For an $R$-module $F$, the following conditions are equivalent.

(i) $F$ is a finitely presented flat $R$-module.

(ii) $F$ is a finitely generated projective $R$-module.

(iii) $F$ is a direct summand of a finitely generated free $R$-module.

**Proof.** (i)$\implies$(ii): Let $L' \to L \to F \to 0$ be a presentation with $L$ and $L'$ finitely generated free $R$-modules. It follows that $F$ and the kernel $K$ of the surjection $L \to F$ are finitely generated, so one has a short exact sequence $0 \to K \to L \to F \to 0$ of finitely generated $R$-modules. Showing that $F$ is projective is by 1.3.17 tantamount to showing that this sequence splits. To this end we construct a homomorphism $\varphi: L \to K$ with $\varphi|_K = 1$.

Let $\{e_1, \ldots, e_m\}$ be a basis for $L$ and let $\{k_1, \ldots, k_n\}$ be a set of generators for the submodule $K$. Let $k \in K$ be given; we start by constructing a homomorphism $\varphi_k: L \to K$ with $\varphi_k(k) = k$. Write $k$ in terms of the basis: $k = r_1e_1 + \cdots + r_me_m$. Let $b$ be the right ideal in $R$ generated by $r_1, \ldots, r_m$, then one has $k \in b \cap bL$, whence $k$ is in $b^2 = b^1$ by 1.3.38. It follows that there are elements $b_i \in b$ and $x_{ji} \in R$ such that the following equalities hold:

$$k = \sum_{i=1}^{n} b_i k_i = \sum_{i=1}^{n} (\sum_{j=1}^{m} r_j x_{ji}) k_i = \sum_{j=1}^{m} r_j (\sum_{i=1}^{n} x_{ji} k_i).$$

Define $\varphi_k$ by $e_j \mapsto \sum_{i=1}^{n} x_{ji} k_i$, then $\varphi_k(k) = \varphi_k(\sum_{i=1}^{n} r_i e_j) = k$ holds by (*).

To construct a homomorphism $\varphi: L \to K$ whose restriction to $K$ is the identity, it suffices to construct $\varphi$ with $\varphi(k_i) = k_i$ for all the generators $k_1, \ldots, k_n$. Proceed by induction on $n$; the construction above settles the base case $n = 1$. For $n > 1$ there exists by the same construction a homomorphism $\varphi_{k_n}$ that fixes $k_n$. For $i < n$ set $k'_i = k_i - \varphi_{k_n}(k_i)$. By the induction hypothesis, there is a homomorphism $\varphi': L \to K$ with $\varphi'(k'_i) = k'_i$ for $i < n$. Now, set $\varphi = \varphi' - \varphi' \varphi_{k_n} + \varphi_{k_n}$, then one has

$$\varphi(k_n) = \varphi'(k_n) - \varphi'(k_n) + \varphi_{k_n}(k_n) = \varphi'(k'_n) + \varphi_{k_n}(k_n) = \varphi_{k_n}(k_n) = k_n,$$

and for $i < n$ one has

$$\varphi(k_i) = \varphi'(k_i) - \varphi' \varphi_{k_n}(k_i) + \varphi_{k_n}(k_i) = \varphi'(k'_i) + \varphi_{k_n}(k_i) = k'_i + \varphi_{k_n}(k_i) = k_i.$$

(ii)$\implies$(iii): By 1.3.11 there is a short exact sequence $0 \to K \to L \to F \to 0$, where $L$ is a finitely generated free $R$-module, and by 1.3.17 the sequence splits.
(iii) $\implies$ (i): There is a split exact sequence $0 \to K \to L \to F \to 0$ of $R$-modules, where $L$ is finitely generated and free. The module $K$ is also a homomorphic image of $L$, in particular it is finitely generated, so by 1.3.11 there is a finitely generated free $R$-modul $L'$ and a surjective homomorphism $L' \to K$. Thus, one has a finite presentation $L' \to L \to F \to 0$, and $F$ is flat by 1.3.37.

\[\text{\textbf{Flat-Injective Duality}}\]

Projective and injective objects are categorically dual. In module categories there is another important duality: one between flat and injective modules which is rooted in the adjointness of Hom and tensor product.

1.3.41 Proposition. For an $R$-module $F$, the following conditions are equivalent.

(i) $F$ is flat.
(ii) For every right ideal $b$ in $R$, the homomorphism $\iota \otimes_R F$, induced by the embedding $\iota : b \to R$, is injective.
(iii) The $R^\op$-module $\text{Hom}_k(F, E)$ is injective for every injective $k$-module $E$.
(iv) The $R^\op$-module $\text{Hom}_k(F, E)$ is injective for a faithfully injective $k$-module $E$.

Moreover, if $E$ is a faithfully injective $k$-module, then the $R^\op$-module $\text{Hom}_k(F, E)$ is faithfully injective if and only if $F$ is a faithfully flat $R$-module.

Proof. Condition (iv) clearly follows from (iii), and it follows from 1.3.36 that (i) implies (ii).

(ii) $\implies$ (iii): Apply the exact functors $- \otimes_R F$ followed by $(\cdot)^\vee = \text{Hom}_k(\cdot, E)$ to the short exact sequence $0 \to b \to R \to R/b \to 0$. This yields the upper row in the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & (R/b \otimes_R F)^\vee & \to & (R \otimes_R F)^\vee & \to & (b \otimes_R F)^\vee & \to & 0 \\
& & \cong & \cong & \cong & & & \\
0 & \to & \text{Hom}_{R^\op}(R/b, F^\vee) & \to & \text{Hom}_{R^\op}(R, F^\vee) & \to & \text{Hom}_{R^\op}(b, F^\vee) & \to & 0
\end{array}
\]

The vertical isomorphisms follow from commutativity 1.2.2 and adjunction 1.2.4.

By commutativity of the diagram, the lower row is an exact sequence. Thus, the $R^\op$-module $F^\vee = \text{Hom}_k(F, E)$ is injective by Baer’s criterion 1.3.28.

(iv) $\implies$ (i): By adjunction 1.2.4 and commutativity 1.2.2 there is a natural isomorphism of functors from $\mathcal{M}(R)$ to $\mathcal{M}(k)$,

\[\text{Hom}_{R^\op}(\cdot, \text{Hom}_k(F, E)) \cong \text{Hom}_k(\cdot \otimes_R F, E).\]

By assumption, the left-hand functor is exact and $\text{Hom}_k(\cdot, E)$ is faithfully exact; it follows that $- \otimes_R F$ is exact, whence $F$ is flat.
Assuming that $E$ is faithfully injective, it also follows from (**) that the functor \( \text{Hom}_R(\cdot, \text{Hom}_R(F, E)) \) is faithfully exact if and only if \( - \otimes_R F \) is faithfully exact.

\[ \square \]

In Sect. 5.3 we shall make extensive use of the following special case.

**1.3.42 Corollary.** For every free \( R^a \)-module \( L \), the \( R \)-module \( \text{Hom}_R(L, \mathbb{B}) \) is faithfully injective.

\[ \square \]

**Exercises**

**E 1.3.1** Show that \( \mathbb{Q} \) is not a finitely generated (f.g.) \( \mathbb{Z} \)-module and that \( \mathbb{R} \) is not a f.g. \( \mathbb{Q} \)-module.

**E 1.3.2** Let \( L \) be an \( R \)-module and let \( E = \{ e_u \}_{u \in U} \) be a set of generators for \( L \). Show that every element in \( L \) can be expressed uniquely on the form \( \sum_{u \in U} r_u e_u \) if and only if some element in \( L \) can be expressed uniquely on that form.

**E 1.3.3** Let \( L \) be an \( R \)-module and let \( E \) be a subset of \( L \). Show that if every map from \( E \) to an \( R \)-module \( M \) extends uniquely to a homomorphism \( L \to M \) of \( R \)-modules, then \( E \) is a basis for \( L \); in particular, \( L \) is free.

**E 1.3.4** Let \( k \) be a field and let \( M \) be a \( k \)-vector space of infinite rank. Show that the endomorphism ring \( \text{Hom}_k(M, M) \) does not have IBN.

**E 1.3.5** Show that every division ring has IBN.

**E 1.3.6** Let \( L \) and \( L' \) be finitely generated free \( k \)-modules. Show that the \( k \)-module \( \text{Hom}_k(L, L') \) is free and find its rank as a function of the ranks of \( L \) and \( L' \).

**E 1.3.7** Denote by \( S \) the category of sets. Show that the functor \( S \to \text{M}(R) \) given by \( U \mapsto R^{|U|} \) is a left adjoint to the forgetful functor \( \text{M}(R) \to S \).

**E 1.3.8** Let \( M \) be an \( R^a \)-module generated by elements \( x_1, \ldots, x_n \), and let \( N \) be an \( R \)-module generated by \( y_1, \ldots, y_m \). Assume that \( R \) is generated as a \( k \)-module by \( r_1, \ldots, r_l \). Show that the elements in the set \( \{ x_i \otimes y_1 \} | 1 \leq h \leq l, 1 \leq i \leq m \} \) generate the \( k \)-module \( M \otimes_R N \).

**E 1.3.9** Let \( L \) be a free \( k \)-module with basis \( \{ e_1, \ldots, e_n \} \). Let \( l_1, \ldots, l_n \) be elements in \( L \) and write \( l_j = \sum_{i=1}^{n} a_{ij} e_j \). Show that \( \{ l_1, \ldots, l_n \} \) is a basis for \( L \) if and only if the matrix \( (a_{ij})_{1 \leq i, j \leq n} \) is invertible.

**E 1.3.10** Set \( R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1) \) and consider the homomorphism \( \varphi: R \to R^3 \) given by \( r \mapsto (rx, ry, rz) \) where, by abuse of notation, \( x, y, \) and \( z \) now denote the residue classes of the indeterminates in \( R \). (a) Show that the sequence \( 0 \to R \xrightarrow{\varphi} R^3 \to \text{Coker} \varphi \to 0 \) is split exact, and conclude that \( P = \text{Coker} \varphi \) is projective. (b) Show that \( P \) is not free.

**Hint:** See [19, 19.17].

**E 1.3.11** Let \( P \) be a projective \( R \)-module. Show that there exists a free \( R \)-module \( L \) with \( P \oplus \cong L \).

This is known as Eilenberg’s swindle.

**E 1.3.12** Show that a direct summand of a projective/injective/flat module is projective/injective/flat.

**E 1.3.13** Dualize the proof of 1.3.25 to show that a coproduct of projective modules is projective. This provides an alternative proof of 1.3.22.

**E 1.3.14** (Cf. 1.3.37) Show that every free module is faithfully flat.

**E 1.3.15** Show that \( \mathbb{Q} \) is not a projective \( \mathbb{Z} \)-module.

**E 1.3.16** Show that \( \mathbb{Q} \) is a flat (but not a faithfully flat) \( \mathbb{Z} \)-module.

**E 1.3.17** Show that \( \mathbb{Z}^N \) is not a projective \( \mathbb{Z} \)-module.
1.4 Evaluation Homomorphisms

Show that if every submodule of a free \( k \)-module is free, then \( k \) is a principal ideal domain.

Let \( R \) be left hereditary. Show that every submodule of a projective \( R \)-module is projective.

Show that if \( P \) is a free/projective \( R \)-module and \( P' \) is a free/projective \( k \)-module, then the \( R \)-module \( P \otimes_k P' \) is free/projective.

Let \( R \) be left hereditary. Show that every submodule of a projective \( R \)-module is projective.

Let \( P \) be a free/projective \( R \)-module and \( P' \) be a free/projective \( k \)-module, then the \( R \)-module \( P \otimes_k P' \) is free/projective.

Let \( F \) be an \( R \)-module and let \( F' \) be a flat \( k \)-module. Show: (a) If \( F \) is flat, then the \( R \)-module \( F \otimes_k F' \) is flat. (b) If \( F' \) is faithfully flat, then the \( R \)-module \( F \otimes_k F' \) is (faithfully) flat if and only if \( F \) is (faithfully) flat.

Let \( 0 \to K \to L \to M \to 0 \) be an exact sequence of \( R \)-modules with \( L \) free. If \( K \) is finitely generated, then \( M \) is said to be finitely related. Show that a finitely related flat module is projective. (A countably related flat module is almost projective; see D.10.)

Let \( F \) be an \( R \)-module and let \( F' \) be a flat \( k \)-module. Show: (a) If \( F \) is flat, then the \( R \)-module \( F \otimes_k F' \) is flat. (b) If \( F' \) is faithfully flat, then the \( R \)-module \( F \otimes_k F' \) is (faithfully) flat if and only if \( F \) is (faithfully) flat.

Let \( R \to S \) be a ring homomorphism. Show that if \( P \) is a free/projective \( R \)-module, then the \( S \)-module \( S \otimes_R P \) is free/projective.

Let \( R \to S \) be a ring homomorphism. Show that \( R \) has IBN if \( S \) has IBN.

Show that if \( R \) is local or commutative, then it has IBN.

Let \( R \to S \) be a ring homomorphism. Show that if \( I \) is a (faithfully) injective \( R \)-module, then the \( S \)-module \( \text{Hom}_R(S,I) \) is (faithfully) injective.

Let \( M \) be an \( R \)-module and \( E \) be a faithfully injective \( R \)-module. Show that for every \( m \neq 0 \) in \( M \) there exists a homomorphism \( \varphi \in \text{Hom}_k(M,E) \) with \( \varphi(m) \neq 0 \).

Give a proof of 1.3.33 that does not use adjunction 1.2.4. Hint: Let \( \bar{\gamma} : M \to E \) be a non-zero homomorphism of \( k \)-modules. For \( m \in M \) consider the map \( \gamma_m : R \to E \) defined by \( \gamma_m(r) = \bar{\gamma}(rm) \). Show that \( m \mapsto \gamma_m \) is a non-zero homomorphism of \( R \)-modules. To prove injectivity, turn a map to \( \text{Hom}_k(R,E) \) into a map to \( E \) by evaluation at 1.

Show that every flat \( k \)-module is torsion-free.

Let \( \{R^e\}_{e \in U} \) be a family of von Neumann regular rings. Show that \( \bigotimes_{e \in U} R^e \) is a von Neumann regular ring.

1.4 Evaluation Homomorphisms

SYNOPSIS. Tensor evaluation; homomorphism evaluation; biduality.

With projective, injective, and flat modules now available, we continue the comparisons, started in Sect. 1.2, of composites of Hom and tensor product functors.

TENSOR EVALUATION

1.4.1 Lemma. Let \( M \) be an \( R \)-module, \( X \) be an \( R\text{-}S^0 \)-bimodule, and \( N \) be an \( S \)-module. The tensor evaluation map,
\[ \theta^MNX : \text{Hom}_R(M, X) \otimes_SN \to \text{Hom}_R(M, X \otimes_SN) \]

given by
\[ \theta^MNX(\psi \otimes n)(m) = \psi(m) \otimes n, \]
is a homomorphism of \( k \)-modules, and it is natural in \( M, X, \) and \( N \). Moreover, if \( M \) is in \( \mathcal{M}(R-Q^0) \) and \( N \) is in \( \mathcal{M}(S-T^0) \), then \( \theta^MNX \) is a homomorphism in \( \mathcal{M}(Q-T^0) \).

**Proof.** It is straightforward to verify that \( \theta \) is a natural transformation of functors from \( \mathcal{M}(R)^\text{op} \times \mathcal{M}(R-S^0) \times \mathcal{M}(S) \) to \( \mathcal{M}(k) \); see the proof of 1.2.2.

If \( M \) is in \( \mathcal{M}(R-Q^0) \) and \( N \) is in \( \mathcal{M}(S-T^0) \), then \( \text{Hom}_R(M, X) \otimes_SN \) is a \( Q-T^0 \)-bimodule and so is \( \text{Hom}_R(M, X \otimes_SN) \). The computation
\[
\theta^MNX(q(\psi \otimes nt))(m) = \theta^MNX(q(\psi \otimes n))(m) \\
= (q\psi)(m) \otimes nt \\
= \psi(mq) \otimes nt \\
= (\psi(mq) \otimes n)t \\
= (\theta^MNX(\psi \otimes n)mq)t \\
= (\theta^MNX(\psi \otimes n))t(m),
\]
which holds for all \( q \in Q, t \in T, \psi \in \text{Hom}_R(M, X), m \in M, \) and \( n \in N \), shows that the homomorphism \( \theta^MNX \) is \( Q- \) and \( T^0 \)-linear.

**1.4.2 Example.** Set \( R = S = k = \mathbb{Z} \). For the \( \mathbb{Z} \)-modules \( M = \mathbb{Z}/2\mathbb{Z} = N \) and \( X = \mathbb{Z} \), the homomorphism \( \theta^MNX \) maps from 0 to \( \mathbb{Z}/2\mathbb{Z} \), so it is not an isomorphism.

**1.4.3 Proposition.** Let \( M \) be an \( R \)-module, \( X \) be an \( R-S^0 \)-bimodule, and \( N \) be an \( S \)-module. The tensor evaluation homomorphism \( \theta^MNX \) is an isomorphism under any one of the following conditions.

(a) \( M \) or \( N \) is finitely generated and projective.

(b) \( M \) is projective and \( N \) is finitely presented.

(c) \( M \) is finitely presented and \( N \) is flat.

**Proof.** (a): For an \( R \)-module \( Y \), let \( \varphi^Y \) denote the isomorphism (1.2.1.2). For every \( R-S^0 \)-bimodule \( X \) and every \( S \)-module \( N \), the commutative diagram,

\[
\text{Hom}_R(R, X) \otimes_SN \xrightarrow{\theta^RXN} \text{Hom}_R(R, X \otimes_SN) \\
\varphi^X \otimes_N \downarrow \cong \downarrow \varphi^X \otimes_N \\
X \otimes_SN \cong X \otimes_SN
\]

shows that \( \theta^RXN \) is an isomorphism. Similarly, it follows from (1.2.1.1) that \( \theta^MXS \) is an isomorphism for all \( R \)-modules \( M \) and all \( R-S^0 \)-bimodules \( X \). By additivity of the involved functors, it now follows that \( \theta^MNX \) is an isomorphism if \( M \) or \( N \) is finitely generated and projective.
(b): Choose a presentation of $N$ by finitely generated free $S$-modules
\[(\star) \quad L' \longrightarrow L \longrightarrow N \longrightarrow 0.\]

Consider the following diagram, which is commutative since the tensor evaluation homomorphism is natural; see 1.4.1.

\[
\begin{array}{ccc}
\text{Hom}_R(M, X) \otimes_S L' & \longrightarrow & \text{Hom}_R(M, X) \otimes_S L \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_R(M, X \otimes_S L') & \longrightarrow & \text{Hom}_R(M, X \otimes_S L) \longrightarrow \text{Hom}_R(M, X \otimes_S N) \longrightarrow 0.
\end{array}
\]

Either row in this diagram is exact. Indeed, they are obtained by applying the right exact functors $\text{Hom}_R(M, X) \otimes -$ and $\text{Hom}_R(M, X \otimes -$ to $(\star)$. Right exactness of the latter functor hinges on the assumption that $M$ is projective. The maps $\theta M_{XL'}$ and $\theta M_{XL}$ are isomorphisms by part (a), and it follows from the Five Lemma that $\theta M_{XN}$ is an isomorphism.

(c): Choose a presentation $L' \rightarrow L \rightarrow M \rightarrow 0$ of $M$ by finitely generated free $R$-modules. As in the proof of (b), we get the following commutative diagram with exact rows, since the functors $\text{Hom}_R(-, X) \otimes_S -$ and $\text{Hom}_R(-, X \otimes_S -$ are left exact.

Left exactness of the former functor hinges on the assumption that $N$ is flat.

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_R(M, X) \otimes_S N \\
& & \downarrow \cong \\
& & \text{Hom}_R(L, X) \otimes_S N \\
& & \downarrow \cong \\
& & \text{Hom}_R(L', X) \otimes_S N \\
0 & \longrightarrow & \text{Hom}_R(M, X \otimes_S N) \\
& & \downarrow \cong \\
& & \text{Hom}_R(L, X \otimes_S N) \\
& & \downarrow \cong \\
& & \text{Hom}_R(L', X \otimes_S N).
\end{array}
\]

The maps $\theta L_{XN}$ and $\theta L'_{XN}$ are isomorphisms by part (a), and it follows from the Five Lemma that $\theta M_{XN}$ is an isomorphism.

HOMOMORPHISM EVALUATION

1.4.4 Lemma. Let $M$ be an $R^o$-module, $X$ be an $S$-$R^o$-bimodule, and $N$ be an $S$-module. The homomorphism evaluation map

\[
\eta^{XNM} : \text{Hom}_S(X, N) \otimes_{R^o} M \longrightarrow \text{Hom}_S(\text{Hom}_{R^o}(M, X), N)
\]

given by

\[
\eta^{XNM}(\psi \otimes m)(\theta) = \psi \theta(m)
\]

is a homomorphism of $k$-modules, and it is natural in $X$, $N$, and $M$. Moreover, if $M$ is in $\mathcal{M}(Q-R^o)$ and $N$ is in $\mathcal{M}(S-T^o)$, then $\eta^{XNM}$ is a homomorphism in $\mathcal{M}(Q-T^o)$. 


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PROOF. It is straightforward to verify that $\eta$ is a natural transformation of functors from $\mathcal{M}(S^{-R^0})^\circ \times \mathcal{M}((S) \times \mathcal{M}(R^0))$ to $\mathcal{M}(\mathbb{Z})$. See the proof of 1.2.2.

If $M$ is in $\mathcal{M}(Q^{-R^0})$ and $N$ is in $\mathcal{M}(S^{-T^0})$, then $\text{Hom}_S(X, N) \otimes_{R^0} M$ is a $Q^{-T^0}$-bimodule, and so is $\text{Hom}_S(\text{Hom}_{R^0}(M, X), N)$. The computation

$$
\eta^{XNM}(q(\psi \otimes m)t)(\theta) = \eta^{XNM}(\psi t \otimes qm)(\theta)
= (\psi t)(qm)
= (\psi \theta(qm))t
= (\psi(\theta q)(m))t
= (\eta^{XNM}(\psi \otimes m)(\theta q))t
= (q(\eta^{XNM}(\psi \otimes m)t)(\theta))
,$$

which holds for all $q \in Q, t \in T, \psi \in \text{Hom}_S(X, M), m \in M$, and $\theta \in \text{Hom}_{R^0}(M, X)$, shows that the homomorphism $\eta^{XNM}$ is $Q$- and $T^0$-linear. \hfill \Box

1.4.5 Example. Set $R = S = \mathbb{Z} = \mathbb{Z}$. For the $\mathbb{Z}$-modules $M = \mathbb{Z}/2\mathbb{Z} = N$ and $X = \mathbb{Z}$, the homomorphism $\eta^{XNM}$ maps from $\mathbb{Z}/2\mathbb{Z}$ to 0, so it is not an isomorphism.

1.4.6 Proposition. Let $M$ be an $R^0$-module, $X$ be an $S^{-R^0}$-bimodule, and $N$ be an $S$-module. The evaluation homomorphism $\eta^{XNM}$ is an isomorphism under either one of the following conditions.

(a) $M$ is finitely generated and projective.
(b) $M$ is finitely presented and $N$ is injective.

PROOF. (a): In view of (1.2.1.1) and (1.2.1.2) it is elementary to verify that $\eta^{XNR}$ is an isomorphism for all $S^{-R^0}$-bimodules $X$ and all $S$-modules $N$. The claim then follows by additivity of the involved functors.

(b): Choose a presentation of $M$ by finitely generated free $R^0$-modules

$$(*) \quad L' \longrightarrow L \longrightarrow M \longrightarrow 0.$$

Consider the following diagram, which is commutative since the homomorphism evaluation map is natural; see 1.4.4.

$$
\begin{array}{ccc}
\text{Hom}_S(X, N) \otimes_{R^0} L' & \longrightarrow & \text{Hom}_S(X, N) \otimes_{R^0} L \\
\eta^{XNL'} & \cong & \eta^{XNL} \\
\text{Hom}_S(\text{Hom}_{R^0}(L', X), N) & \longrightarrow & \text{Hom}_S(\text{Hom}_{R^0}(L, X), N) \longrightarrow \text{Hom}_S(\text{Hom}_{R^0}(M, X), N) \longrightarrow 0.
\end{array}
$$

Either row in this diagram is exact. Indeed, they are obtained by applying the right exact functors $\text{Hom}_S(X, N) \otimes_{R^0} -$ and $\text{Hom}_S(\text{Hom}_{R^0}(-, X), N)$ to $(*)$. Right exactness of the latter functor hinges on the assumption that $N$ is injective. The maps $\eta^{XNL'}$ and $\eta^{XNL}$ are isomorphisms by part (a), and it follows from the Five Lemma that $\eta^{XNM}$ is an isomorphism. \hfill \Box

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1.4 Evaluation Homomorphisms

Biduality

1.4.7 Lemma. Let $M$ be an $R$-module and $X$ be an $R$-S-bimodule. The biduality map for $M$ with respect to $X$,

$$
\delta^M_X : M \longrightarrow \text{Hom}_S(\text{Hom}_R(M, X), X)
$$

given by

$$
\delta^M_X (m)(\psi) = \psi(m),
$$
is a homomorphism of $R$-modules, and it is natural in $M$ and $X$. Moreover, if $M$ is in $M(R-T^o)$, then $\delta^M_X$ is a homomorphism in $M(R-T^o)$.

Proof. It is straightforward to verify that $\delta$ is a natural transformation of endofunctors on $M(R)$; see the proof of 1.2.2.

If $M$ is an $R$-T-bimodule, then $\text{Hom}_S(\text{Hom}_R(M, X), X)$ is an $R$-T-bimodule as well. For $t \in T$, $m \in M$, and $\psi \in \text{Hom}_R(M, X)$ one has

$$
\delta^M_X (mt)(\psi) = \psi(mt) = (t\psi)(m) = \delta^M_X (m)(t\psi) = (\delta^M_X (m)t)(\psi);
$$

that is, the homomorphism $\delta^M_X$ is $T^o$-linear.

Exercises

E 1.4.1 Assume that $k$ is a field, and let $M$ and $X \neq 0$ and $N$ be $k$-vector spaces. Show that tensor evaluation $\theta^{MN} : \text{Hom}_k(M, X) \otimes_k N \rightarrow \text{Hom}_k(M, X \otimes_k N)$ is an isomorphism if and only if $M$ or $N$ has finite rank.

E 1.4.2 Assume that $k$ is a field, and let $M$ and $X \neq 0$ and $N \neq 0$ be $k$-vector spaces. Show that homomorphism evaluation $\eta^{XM} : \text{Hom}_k(X, N) \otimes_k M \rightarrow \text{Hom}_k(\text{Hom}_k(M, X), N)$ is an isomorphism if and only if $M$ has finite rank.

E 1.4.3 Use homomorphism evaluation 1.4.4 to show that a finitely presented flat module is projective.

E 1.4.4 Show that the biduality homomorphism $\delta^M_X$ is injective for every $R$-module $M$.

E 1.4.5 (a) Show that the biduality homomorphism need not be injective. (b) Show that the biduality homomorphism need not be surjective.

E 1.4.6 Show that for every $R$-module $M$ there is an injective homomorphism $M \rightarrow I$ where $I$ is injective.

E 1.4.7 Show that an $R$-module is injective if and only if it is a direct summand of a character module $\text{Hom}_R(L, \mathbb{Z})$, where $L$ is a free $R^*$-module.

E 1.4.8 For an $R$-module $I$, show that the following conditions are equivalent. (i) $I$ is injective. (ii) For every left ideal $a$ in $R$, the homomorphism $\text{Hom}_R(i, I)$, induced by the embedding $i : a \rightarrow R$, is surjective. (iii) Every exact sequence $0 \rightarrow I \rightarrow M \rightarrow M^o \rightarrow 0$ splits.
Chapter 2
Complexes

2.1 Definitions and Examples

SYNOPSIS. Graded module; graded homomorphism; graded basis; graded-free module; complex; chain map; (split) exact sequence; Five Lemma; Snake Lemma.

Many modules carry an intrinsic structure: a grading. By imposing an additional structure, a square zero endomap that respects the grading, one arrives at the notion of a complex. The zero map respects any grading, so one can always consider a graded module as a complex. After a short opening discussion of graded modules, we move on to complexes, which abound in mathematics. We illustrate the concept with examples from algebra, geometry, and topology.

GRADED MODULES

2.1.1 Definition. Let $U$ be a set. An $R$-module $M$ is called $U$-graded if there exists a family $\{M_u\}_{u \in U}$ of submodules of $M$ such that $M = \bigsqcup_{u \in U} M_u$. A $\mathbb{Z}$-graded module is simply called a graded module.

2.1.2 Example. Considered as an $R$-module, the ring of polynomials $M = R[\{x\}]$ is graded with $M_v = 0$ for $v < 0$ and $M_v = \{ m \in M \mid m \text{ is a monomial of degree } v \}$ for $v \geq 0$, with the convention that 0 is a monomial of every degree.

2.1.3 Definition. Let $M = \bigsqcup_{v \in \mathbb{Z}} M_v$ be a graded $R$-module. The submodule $M_v$ is called the module in degree $v$. An element $m$ in $M$ that belongs to a submodule $M_v$ is said to be homogeneous of degree $v$. Thus, the zero element is homogeneous of every degree. A homogeneous element $m \neq 0$ is homogeneous of exactly one degree, which is called the degree of $m$ and denoted $|m|$; that is

$$|m| = v \iff m \in M_v.$$
All subsequent formulas that involve the degree $|m|$ of an arbitrary homogeneous element $m$ will be valid no matter what value one assigns to $|0|$.  

2.1.4 Definition. Let $L$ be a graded $R$-module. A set $E = \{ e_{\nu} \}_{\nu \in I}$ of generators for $L$, a basis for $L$ in particular, is called graded if each element $e_{\nu}$ is homogeneous. The module $L$ is called graded-free if has a graded basis.

For a graded set $E$, not a priori assumed to be a subset of a module, the graded-free $R$-module with graded basis $E$ is denoted $R(E)$.  

2.1.5. Let $M = \prod_{v \in \mathbb{Z}} M_v$ be a graded $R$-module and let $N$ be an $R$-module. A homomorphism $\alpha$ from $M$ to $N$ is identified with the family $(\alpha_v)_{v \in \mathbb{Z}}$ in $\prod_{v \in \mathbb{Z}} \text{Hom}_R(M_v, N)$, given by $\alpha_v = \iota_v \circ \alpha$ where $\iota_v$ is the embedding $M_v \rightarrow M$.  

2.1.6 Definition. Let $M$ be a graded $R$-module, then the $R$-module defined in 2.1.6. The two interpretations do, in general, not agree, but all subsequent formulas that involve the degree $|m|$ of an arbitrary homogeneous element $m$ will be valid no matter what value one assigns to $|0|$.  

2.1.7 Example. Set $M = R[x]$ as in 2.1.2. The derivative $\frac{d}{dx}$ yields a graded homomorphism $M \rightarrow M$ of degree $-1$.  

2.1.8. If $M$ is a graded $R$-$Q^0$-bimodule and $N$ is a graded $R$-$S^0$-bimodule, then the graded $R$-module $\text{Hom}_R(M, N)$ has an induced graded $Q$-$S^0$-bimodule structure.  

2.1.9. Let $L$, $M$, and $N$ be graded $R$-modules, and let $p$ and $q$ be integers. There is a $k$-bilinear composition rule for graded homomorphisms: 

$$\text{Hom}_R(M, N)_p \times \text{Hom}_R(L, M)_q \rightarrow \text{Hom}_R(L, N)_{p+q}.$$  

For $\alpha = (\alpha_v)_{v \in \mathbb{Z}}$ and $\beta = (\beta_v)_{v \in \mathbb{Z}}$ it is given by $(\alpha, \beta) \mapsto \alpha \beta = (\alpha_v \beta_v)_{v \in \mathbb{Z}}$.  

Remark. It follows from 2.1.9 that if $M$ is a graded $R$-module, then $\text{Hom}_R(M, M)$ has a graded $k$-algebra structure with multiplication given by composition of homomorphisms.  

2.1.10 Example. Let $M$ be a $k$-module. Consider the graded $k$-module $T^k_p(M)$ with 

$$T^k_p(M) = 0 \text{ for } p < 0, \quad T^k_0(M) = k, \quad \text{and} \quad T^k_p(M) = M^{\otimes p} \text{ for } p > 0;$$  

where $M^{\otimes p} = M \otimes_k \cdots \otimes_k M$ is the $p$-fold tensor product of $M$. The module $T^k_p(M)$ is called the $p$th tensor power of $M$. With multiplication given by concatenation of
2.1 Definitions and Examples

elementary tensors, \( T^k(M) \) is a graded \( k \)-algebra called the tensor algebra of \( M \). To be precise, the product of elements \( x_1 \otimes \cdots \otimes x_p \) in \( T^k(M) \) and \( y_1 \otimes \cdots \otimes y_q \) in \( T^k_q(M) \) is the element \( x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q \) in \( T^k_{p+q}(M) \).

Denote by \( \mathfrak{J} \) the ideal in \( T^k(M) \) generated by the set \( \{ x \otimes x \mid x \in M \} \) of homogeneous elements of degree 2; the graded quotient algebra \( \wedge^k(M) = T^k(M)/\mathfrak{J} \) is called the exterior algebra of \( M \), and the module \( \wedge^k_p(M) \) in degree \( p \) is called the \( p \)th exterior power of \( M \). One writes \( x_1 \wedge \cdots \wedge x_p \) for the coset \([x_1 \otimes \cdots \otimes x_p]_\mathfrak{J} \) in \( \wedge^k(M) \) and it is called the wedge product of the elements \( x_1, \ldots, x_p \). As elements of the form \( x \otimes y + y \otimes x = (x+y) \otimes (x+y) - x \otimes x - y \otimes y \) belong to \( \mathfrak{J} \), one has \( x_1 \wedge \cdots \wedge x_p = -x_1 \wedge \cdots \wedge x_{i+1} \wedge x_i \wedge \cdots \wedge x_p \).

2.1.11 Definition. Let \( M \) be a graded \( R^o \)-module and let \( N \) be a graded \( R \)-module. Denote by \( M \otimes_R N \) the graded \( k \)-module with

\[
(M \otimes_R N)_p = \bigoplus_{v \in \mathbb{Z}} M_v \otimes_R N_{p-v};
\]

it is called the graded tensor product of \( M \) and \( N \). Notice that if \( m \) and \( n \) are homogeneous elements in \( M \) and \( N \), then \( m \otimes n \) is homogeneous in \( M \otimes_R N \) of degree \( |m \otimes n| = |m| + |n| \).

2.1.12. If \( M \) is a graded \( Q\otimes R^o \)-bimodule and \( N \) is a graded \( R\otimes S^o \)-bimodule, then the graded \( k \)-module \( M \otimes_R N \) has an induced graded \( Q\otimes S^o \)-bimodule structure.

The graded tensor product has the expected universal property.

2.1.13 Lemma. Let \( M \) be a graded \( R^o \)-module, let \( N \) be a graded \( R \)-module, let \( X \) be a graded \( k \)-module, and let \( p \) be an integer. For every family of \( k \)-bilinear and middle \( R \)-linear maps

\[
\{ \Phi_v : \bigcup_{i \in \mathbb{Z}} M_i \times N_{v-i} \rightarrow X_{v+p} \}_{v \in \mathbb{Z}}
\]

there is a unique degree \( p \) homomorphism of graded \( k \)-modules

\[
\varphi : M \otimes_R N \rightarrow X
\]

with \( \varphi(m \otimes n) = \Phi_v(m, n) \) for all elements \( m \in M \) and \( n \in N \) with \( |m \otimes n| = v \).

PROOF. It is evident that such a morphism is unique if it exists. For existence, it is sufficient to verify that for each integer \( v \) there is a homomorphism of \( k \)-modules \( \varphi_v : (M \otimes_R N)_v = \bigcup_{i \in \mathbb{Z}} M_i \otimes_R N_{v-i} \rightarrow X_{v+p} \) with \( \varphi_v(m \otimes n) = \Phi_v(m, n) \). Fix \( v \); for each \( i \in \mathbb{Z} \) it follows from the universal property of tensor products that there is a homomorphism \( \varphi'_i : M_i \otimes_R N_{v-i} \rightarrow X_{v+p} \) with \( \varphi'_i(m \otimes n) = \Phi_v(m, n) \). Now the universal property of coproducts yields the desired homomorphism \( \varphi_v \).

2.1.14 Addendum. If \( M \) is in \( \mathcal{M}_{gr}(Q\otimes R^o) \) and \( N \) is in \( \mathcal{M}_{gr}(R\otimes S^o) \), then the tensor product \( M \otimes_R N \) is a graded \( Q\otimes S^o \)-bimodule; see 2.1.12. If \( X \) is also in \( \mathcal{M}_{gr}(Q\otimes S^o) \)

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and \( \{ \Phi_v : \bigoplus_{i \in \mathbb{Z}} M_i \times N_{i-1} \to X_{v+p} \}_{v \in \mathbb{Z}} \) is a family of \( Q \)- and \( S^0 \)-linear maps that satisfy \( \Phi_v(m, n) = \Phi_v(m, rn) \), then \( \varphi : M \otimes_R N \to X \) is a morphism in \( \mathcal{M}_{gr}(Q-S^0) \).

### 2.1.15 Definition

Let \( M \) be a graded \( R \)-module. A graded submodule of \( M \) is a graded \( R \)-module \( K \), that is a submodule of \( M \) as an \( R \)-module such that the embedding \( K \to M \) is a morphism of graded \( R \)-modules.

### 2.1.16

Given a graded submodule \( K \) of a graded \( R \)-module \( M \), the quotient \( M/K \) is a graded module \( M/K = \bigsqcup_{i \in \mathbb{Z}} M_i/K_i \), and the canonical surjection \( M \to M/K \) is a morphism of graded \( R \)-modules.

### 2.1.17

As \( \mathcal{M}(R) \) is a \( k \)-linear Abelian category, it is elementary to see that \( \mathcal{M}_{gr}(R) \), and hence \( \mathcal{M}_{gr}(R)^{op} \), is \( k \)-linear and Abelian. In particular, the biproduct \( M \oplus N \) in \( \mathcal{M}_{gr}(R) \) of \( M = \bigsqcup_{i \in \mathbb{Z}} M_i \) and \( N = \bigsqcup_{i \in \mathbb{Z}} N_i \) is the graded module \( \bigsqcup_{i \in \mathbb{Z}} (M_i \oplus N_i) \). In this situation, one refers to \( M \oplus N \) as a graded direct sum and to \( M \) and \( N \) as graded direct summands.

### 2.1.18

There are exact \( k \)-linear functors between Abelian categories

\[
\mathcal{M}(R) \xrightarrow{\text{forgetful functor}} \mathcal{M}_{gr}(R) .
\]

The functor from \( \mathcal{M}(R) \) to \( \mathcal{M}_{gr}(R) \) is a full embedding; it equips an \( R \)-module \( M \) with the trivial grading \( M_0 = M \) and \( M_v = 0 \) for \( v \neq 0 \), and it acts analogously on homomorphisms. The functor from \( \mathcal{M}_{gr}(R) \) to \( \mathcal{M}(R) \) forgets the grading.

At the level of symbols, applications of these functors is suppressed. However, when we write e.g. “as an \( R \)-module” about a graded \( R \)-module, it means that we apply the forgetful functor.

**Remark.** The category \( \mathcal{M}_{gr}(R) \) also has products and coproducts (and limits and colimits); they will all be treated within the context of complexes; see Chap. 3.

### Complexes

#### 2.1.19 Definition

An \( R \)-complex is a graded \( R \)-module \( M \) equipped with a homomorphism \( \partial : M \to M \) of degree \(-1\) that satisfies \( \partial \partial = 0 \). It can be visualized as follows:

\[
\cdots \to M_{v+1} \overset{\partial \partial}{\longrightarrow} M_v \overset{\partial}{\longrightarrow} M_{v-1} \longrightarrow \cdots .
\]

The module \( M_v \) is the module in degree \( v \); the homomorphism \( \partial : M_v \to M_{v-1} \) is called the \( v \)th differential, and one has \( \partial \partial = 0 \) for all \( v \in \mathbb{Z} \).

Given an \( R \)-complex \( M \), the underlying graded \( R \)-module is denoted \( M^\natural \).

**Remark.** Another word for complex is differential graded module. In the notation for complexes introduced in 2.1.19, degrees are written as subscripts and descend in the direction of the arrows; this is called homological notation. In cohomological notation, degrees are written as superscripts and ascend in the direction of the arrows; in this notation, a complex \( M \) is visualized as follows:
In the literature, there is a strong tradition for employing homological (cohomological) notation for complexes that are bounded below (above) in the sense of 2.5.2; such complexes are often referred to as chain (cochain) complexes. Switching between homological and cohomological notation, it is standard to set \( M_0 = M^{-1} \). We have no proclivity for complexes bounded on either side, but we settle on homological notation and only deviate from it in a few examples, such as 2.1.23 below.

2.1.20 Example. Over the ring \( \mathbb{Z}/4\mathbb{Z} \) consider the graded module with \( \mathbb{Z}/4\mathbb{Z} \) in each degree. Endowed with the degree \(-1\) homomorphism that in each degree is multiplication by 2, it is a \( \mathbb{Z}/4\mathbb{Z} \)-complex,

\[
\cdots \rightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \rightarrow \cdots,
\]
called the Dold complex.

2.1.21 Example. Let \( M \) be a \( k \)-module and let \( \varepsilon : M \rightarrow k \) be a homomorphism. Recall from 2.1.10 that \( T^k(M) \) and \( \wedge^k(M) \) are the tensor algebra and the exterior algebra of \( M \). Consider the degree \(-1\) homomorphism \( \delta : T^k(M) \rightarrow \wedge^k(M) \) given by

\[
\delta(x_1 \otimes \cdots \otimes x_p) = \sum_{i=1}^{p} (-1)^{i+1} \varepsilon(x_i)x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p.
\]

Every element in the ideal \( \mathfrak{I} \subseteq T^k(M) \) from 2.1.10 is a \( k \)-linear combination of elements of the form \( y_1 \otimes \cdots \otimes y_p \otimes x \otimes x \otimes z_1 \otimes \cdots \otimes z_q \); since \( \delta \) vanishes on such elements, it factors through the exterior algebra, yielding a degree \(-1\) homomorphism \( \partial : \wedge^k(M) \rightarrow \wedge^k(M) \) given by

\[
\partial(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i+1} \varepsilon(x_i)x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p.
\]

This homomorphism is square zero; indeed, one has

\[
\partial \partial(x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i+1} \varepsilon(x_i) \partial(x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p)
\]

\[
= \sum_{j<k} (-1)^{(i+j+1)} \varepsilon(x_j)x_i x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_j \wedge x_{j+1} \wedge \cdots \wedge x_k \wedge x_{k+1} \wedge \cdots \wedge x_p
\]

\[
+ \sum_{i<k} (-1)^{(i+k)} \varepsilon(x_k)x_i x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{k-1} \wedge x_{k+1} \wedge \cdots \wedge x_p
\]

\[
= 0.
\]

The \( k \)-complex with underlying graded module \( \wedge^k(M) \) and differential \( \partial \) is called the Koszul complex over \( \varepsilon \) and denoted \( K^k(\varepsilon) \). For a sequence \( x_1, \ldots, x_n \) in \( k \), one writes \( K^k(x_1, \ldots, x_n) \) for the Koszul complex over the canonical homomorphism from the free module \( k(\varepsilon_1, \ldots, \varepsilon_n) \) to the ideal \( \langle x_1, \ldots, x_n \rangle \). Notice that in this important special case, the differential is given by

\[
\partial^{K^k(x_1, \ldots, x_n)}(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^{p} (-1)^{i+1} x_i e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_p.
\]

2.1.22 Example. Consider for each integer \( n \geq 0 \) the standard \( n \)-simplex,
$\Delta_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^{n} t_j = 1 \text{ and } t_0, \ldots, t_n \geq 0\}.
$

For $n \geq 1$ and $0 \leq i \leq n$ the $i$th face map $\partial_i^n : \Delta_{n-1} \to \Delta_n$ is given by

$$(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Let $X$ be a topological space. Denote by $C(\Delta_n, X)$ the set of all continuous maps from $\Delta_n$ to $X$, and consider the free Abelian group $S_n(X) = \mathbb{Z}(C(\Delta_n, X))$ on this set. The elements of $S_n(X)$ are called singular $n$-chains. For $n \geq 1$, the map

$$C(\Delta_n, X) \to S_{n-1}(X)$$

given by $\sigma \mapsto \sum_{i=0}^{n} (-1)^i \sigma \varepsilon_i^n$

extends uniquely by 1.3.5 to a group homomorphism $\partial^n_0 : S_n(X) \to S_{n-1}(X)$. Set $\partial^n_0 = 0$, then one has $\partial^n_0 \partial^n_1 = 0$; for $n \geq 1$ and a basis element $\sigma$ in $S_{n+1}(X)$ one has

$$\partial^n_n \partial^n_{n+1}(\sigma) = \partial^n_n (\sum_{i=0}^{n+1} (-1)^i \sigma \varepsilon_i^{n+1})
= \sum_{j=0}^{n} (-1)^j (\sum_{i=0}^{n+1} (-1)^i \sigma \varepsilon_i^{n+1}) \varepsilon_j^n
= \sum_{j<i} (-1)^{i+j} \sigma \varepsilon_i^{n+1} \varepsilon_j^n + \sum_{i<j} (-1)^{i+j} \sigma \varepsilon_i^{n+1} \varepsilon_j^n.
$$

For $0 \leq j < i \leq n + 1$ one has $\varepsilon_j^{n+1} \varepsilon_j^n = \varepsilon_{i+1}^{n+1} \varepsilon_{i+1}^n$ and, therefore,

$$(\sum_{j<i} (-1)^{i+j} \sigma \varepsilon_i^{n+1} \varepsilon_j^n) + (\sum_{i<j} (-1)^{i+j} \sigma \varepsilon_i^{n+1} \varepsilon_j^n) = (\sum_{i=j} (-1)^{i+j} \sigma \varepsilon_i^{n+1} \varepsilon_i^n).
$$

In combination, these displays yield $\partial^n_n \partial^n_{n+1} (\sigma) = 0$, and thus one has a $\mathbb{Z}$-complex,

$$S(X) = \cdots \longrightarrow S_2(X) \xrightarrow{\partial^n_2} S_1(X) \xrightarrow{\partial^n_1} S_0(X) \longrightarrow 0.
$$

This complex is called the singular chain complex of the space $X$.

Let $A$ be an Abelian group. Application of the functors $- \otimes \mathbb{Z} A$ and $\text{Hom}_{\mathbb{Z}}(-, A)$ to the complex $S(X)$ in each degree yields complexes called the singular chain and singular cochain complex of $X$ with coefficients in $A$.

### 2.1.23 Example

Let $M$ be a smooth $d$-dimensional real manifold. For a point $x$ on $M$, denote by $C^\infty_x(M)$ the germ of smooth functions at $x$; it is an $\mathbb{R}$-algebra. A linear map $\nu : C^\infty_x(M) \to \mathbb{R}$ with $\nu(fg) = \nu(f)g(x) + f(x)\nu(g)$ for all $f, g \in C^\infty_x(M)$ is called a derivation at $x$. The set $M_x$ of all derivations is a $d$-dimensional vector space, called the tangent space at $x$. Given a chart $\varphi = (\varphi^1, \ldots, \varphi^d) : U \to \mathbb{R}^d$, where $U$ is a neighbourhood of $x$, one can form a basis $\frac{\partial}{\partial \varphi^1}|_x, \ldots, \frac{\partial}{\partial \varphi^d}|_x$ for $M_x$; for $f$ in $C^\infty_x(M)$ the derivation $\frac{\partial}{\partial \varphi^i}|_x(f) = D_i(f \circ \varphi^{-1})|_{\varphi(x)}$, where $D_i(\cdot)|_{\varphi(x)}$ is the $i$th derivative at the point $\varphi(x) \in \mathbb{R}^d$. The dual space $M^*_x$ is called the cotangent space at $x$, and the dual basis is denoted by $(d\varphi^i)_x, \ldots, (d\varphi^d)_x$.  

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More generally, for a smooth function \( f: U \to \mathbb{R} \) its differential at \( x \) is the functional \((df)_x \in T^*_x U\) given by \((df)_x(\nu) = \nu(f)\) for \( \nu \in T^*_x U\).

Fix \( 0 \leq k \leq d \) and denote by \( \Lambda^k(M^*_x)\) the \( k \)th exterior power of \( M^*_x\), which is a vector space of dimension \( q = \binom{d}{k}\). One can show that the family \( \{\Lambda^k(M^*_x)\}_{x \in M}\) can be assembled to a smooth vector bundle \( E^k\) on \( M\) of rank \( q\); the total space is \( E^k = \bigsqcup_{x \in M} \Lambda^k(M^*_x)\) and the bundle projection \( \pi: E^k \to M\) maps \( \Lambda^k(M^*_x)\) to \( x\).

A differential form of degree \( k \) is a smooth section of \( \pi: E^k \to M\), i.e. a smooth map \( \omega: M \to E^k\) with \( \pi \omega = 1^M\). The vector space of all such maps is denoted \( \Omega^k(M)\).

As one has \( \Lambda^0(M^*_x) = \mathbb{R}\), the bundle \( E^0 = M \times \mathbb{R}\) is trivial. Thus, an element in \( \Omega^0(M)\) is nothing but a smooth map of the form \((1^M, f): M \to M \times \mathbb{R}\). Hence one naturally identifies \( \Omega^0(M)\) with the set \( C^\infty(M)\) of smooth functions \( M \to \mathbb{R}\).

As one has \( \Lambda^1(M^*_x) = M^*_x\), an element in \( \Omega^1(M)\) is a smooth map \( \omega: M \to E^1\) such that \( \omega_x = \omega(x)\) belongs to \( M^*_x\) for every \( x \in M\). In particular, \( \omega = df\) is a differential form of degree 1 for every \( f \) in \( C^\infty(M)\). Thus, the differential yields a map \( \Omega^1(M) \to \Omega^2(M)\). This map \( d^0 = d\) is part of the so-called de Rham complex, which traditionally is written in cohomological notation,

\[
\Omega(M) = 0 \longrightarrow \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \Omega^2(M) \xrightarrow{d^2} \cdots.
\]

The differential on \( \Omega(M)\) is called the exterior derivative. To define it, notice that the wedge product \( \wedge: \Lambda^k(M^*_x) \times \Lambda^j(M^*_x) \to \Lambda^{k+j}(M^*_x)\) on the exterior algebra induces a pairing \( \wedge: \Omega^i(M) \times \Omega^j(M) \to \Omega^{i+j}(M)\), defined by \((\omega \wedge \varphi)_x = \omega_x \wedge \varphi_x\). The exterior derivative is defined recursively by the formula

\[
d^{i+j}(\omega \wedge \varphi) = d^i(\omega) \wedge \varphi + (-1)^i \omega \wedge d^j(\varphi)
\]

for \( \omega \in \Omega^i(M)\) and \( \varphi \in \Omega^j(M)\).

2.1.24 Definition. Let \( M\) and \( N\) be \( R\)-complexes. A graded homomorphism of the underlying graded modules \( \alpha: M^\bullet \to N^\bullet\) is called a chain map if the equality

\[
\partial^N \alpha = (-1)^{|\alpha|} \partial^M \alpha
\]

holds. This means that every square in the following diagram in \( \mathcal{M}(R)\) commutes up to the sign \((-1)^{|\alpha|}\).

\[
\begin{array}{ccccccc}
\cdots & \to & M_{v+1} & \xrightarrow{\partial^M_{v+1}} & M_v & \xrightarrow{\partial^M_v} & M_{v-1} & \to & \cdots \\
& | & \alpha_{v+1} (-1)^{|\alpha|} & | & \alpha_v (-1)^{|\alpha|} & | & \alpha_{v-1} & \\
\cdots & \to & N_{v+1+|\alpha|} & \xrightarrow{\partial^N_{v+1+|\alpha|}} & N_{v+|\alpha|} & \xrightarrow{\partial^N_{v+|\alpha|}} & N_{v-1+|\alpha|} & \to & \cdots,
\end{array}
\]

A chain map of degree 0 is called a morphism of \( R\)-complexes.

Remark. The sign above follows Koszul’s sign convention: a sign \((-1)^{mn}\) is introduced when elements of degree \( m\) and \( n\) are interchanged; in this case, \((-1)^{|\alpha||\beta|} = (-1)^{-|\alpha|} = (-1)^{|\alpha|}\).
2.1.25 Example. For an $R$-complex $M$, the differential $\partial^M$ is a chain map from $M$ to $M$ of degree $-1$.

For a central element $x \in R$ and an $R$-complex $M$, the homothety $x^M : M \to M$ is given by multiplication by $x$; it is a morphism of $R$-complexes. Notice that the notation $x^M$ is in line with the notation $1^M$ for the identity morphism.

2.1.26 Example. Let $a = (x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n, y_1, \ldots, y_m) = b$ be ideals in $R$. With the notation from 2.1.21 it is elementary to verify that the map $K^a \to K^b$ given by $x_1 \wedge \cdots \wedge x_p \mapsto x_1 \wedge \cdots \wedge x_p$ is a morphism of $R$-complexes.

2.1.27. It is straightforward to verify that a composite of chain maps is a chain map. In particular, a composite of morphisms is a morphism; cf. 2.1.9.

2.1.28 Definition. The collection of all $R$-complexes and all morphisms of $R$-complexes form a category, which is denoted by the symbol $\mathcal{C}(R)$.

2.1.29 Definition. Let $M$ be an $R$-complex. A graded submodule $K$ of $M^\bullet$, with the property that $\partial^M(K)$ is contained in $K$, is a complex when endowed with the differential $\partial^M|_K$; it is called a subcomplex of $M$. In this case, the embedding $K \hookrightarrow M$ is a morphism of $R$-complexes.

Given a subcomplex $K$ of $M$, the differential $\partial^M$ induces a differential on the graded module $M^\bullet/K^\bullet$. The resulting complex $M/K$ is called a quotient complex. Note that the canonical map $M \twoheadrightarrow M/K$ is a morphism of $R$-complexes.

2.1.30 Example. Let $M$ be a smooth real manifold. Recall from 2.1.22 that the modules in the singular chain complex $S(M)$ are $S_n(M) = \mathbb{Z}(C(\Delta_n, M))$ for $n \geq 0$ and $S_n(M) = 0$ for $n < 0$. It is immediate from the definitions that the graded submodule $S^n(M)$ with $S_n^m(M) = \mathbb{Z}(C^m(\Delta_n, M))$ for $n \geq 0$ is a subcomplex of $S(M)$.

With the notation from 2.1.22 and 2.1.23 there exists a morphism of $\mathbb{R}$-complexes $\Omega(M) \to \text{Hom}_{\mathbb{R}}(S^n(M), \mathbb{R})$ that maps a differential form $\omega$ of degree $n$ to the group homomorphism $S^n(M) \to \mathbb{R}$ given by $\sigma \mapsto \int_\sigma \omega = \int_{\Delta_n} \sigma^* \omega$.

2.1.31. Let $\alpha : M \to N$ be a chain map of $R$-complexes. It is simple to verify that the kernel $\text{Ker}\alpha = \{m \in M \mid \alpha(m) = 0\}$ and the image $\text{Im}\alpha = \{\alpha(m) \mid m \in M\}$ are subcomplexes of $M$ and $N$, respectively. In particular, one can form the quotient complex $\text{Coker}\alpha = N/\text{Im}\alpha$.

Thus, kernels, images, and cokernels of morphisms in $\mathcal{C}(R)$ are complexes. As $\mathcal{M}_{gr}(R)$ is a $\mathbb{R}$-linear Abelian category, it is simple to see that $\mathcal{C}(R)$, and hence $\mathcal{C}(R)^{op}$, is $\mathbb{R}$-linear and Abelian. In particular, the biproduct of complexes $M$ and $N$ is the graded module $M^\bullet \oplus N^\bullet$ endowed with the square zero homomorphism $\partial^M \oplus \partial^N$.

2.1.32. There are exact $\mathbb{R}$-linear functors between Abelian categories

$$
\mathcal{M}_{gr}(R) \xrightarrow{(-)^\bullet} \mathcal{C}(R).
$$

The functor from $\mathcal{M}_{gr}(R)$ to $\mathcal{C}(R)$ is a full embedding; it equips a graded module $M$ with the trivial differential $\partial^M = 0$, and it is the identity on morphisms. The
functor $(-)^{\flat}$ from $\mathcal{C}(R)$ to $\mathcal{M}_{gr}(R)$ forgets the differential on $R$-complexes, and it is the identity on morphisms. At the level of symbols, applications of these functors is often suppressed. When we write e.g. “as an $R$-complex” about a graded $R$-module, it means that we apply the full embedding.

The module category $\mathcal{M}(R)$ is a full subcategory of $\mathcal{C}(R)$ via the full embeddings $\mathcal{M}(R) \rightarrow \mathcal{M}_{gr}(R) \rightarrow \mathcal{C}(R)$; see 2.1.18.

2.1.33. It follows from 2.1.31 that graded $R$-modules are isomorphic if and only if they are isomorphic as $R$-complexes.

It is simple to verify that a morphism $M \rightarrow N$ of $R$-complexes is an isomorphism if and only if it is an isomorphism of the underlying graded modules.

2.1.34. Let $\mathcal{C}(R \otimes_k S^0)$ denote the $k$-linear Abelian category whose objects are com-
plexes of $R$-$S^0$-bimodules with $R$- and $S^0$-linear differentials, and whose morphisms are $R$- and $S^0$-linear chain maps of degree 0. It follows that there is an equivalence of the $k$-linear Abelian categories $\mathcal{C}(R \otimes_k S^0)$.

**Diagram Lemmas**

In the next several paragraphs, in 2.1.35–2.1.41 to be precise, the category of com-
plexes could be replaced by any Abelian category, and in that sense they repeat material from Sect. 1.1. We include them, nevertheless, for ease of reference, and we provide proofs, because the material is central.

2.1.35 Definition. A sequence of $R$-complexes is a, possibly infinite, diagram,

$$
\cdots \rightarrow M^0 \xrightarrow{\alpha^0} M^1 \xrightarrow{\alpha^1} M^2 \xrightarrow{\alpha^2} \cdots,
$$

in $\mathcal{C}(R)$; it is called *exact* if one has $\text{Im} \alpha^{n-1} = \text{Ker} \alpha^n$ for all $n$. Notice that (2.1.35.1) is exact if and only if every sequence $0 \rightarrow \text{Im} \alpha^{n-1} \rightarrow M^n \rightarrow \text{Im} \alpha^n \rightarrow 0$ is exact. An exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is called a *short exact sequence*.

Two sequences $\{a^n \colon M^n \rightarrow M^{n+1}\}_{n \in \mathbb{Z}}$ and $\{\beta^n \colon N^n \rightarrow N^{n+1}\}_{n \in \mathbb{Z}}$ of $R$-complexes are called *isomorphic* if there exists a family of isomorphisms $\{\varphi^n\}_{n \in \mathbb{Z}}$ such that the diagram

$$
\begin{array}{ccc}
M^n & \xrightarrow{a^n} & M^{n+1} \\
\downarrow \varphi^n & & \downarrow \varphi^{n+1} \\
N^n & \xrightarrow{\beta^n} & N^{n+1}
\end{array}
$$

is commutative for every $n \in \mathbb{Z}$.

2.1.36. Exactness of a sequence in $\mathcal{C}(R)$ does not depend on the differentials, so it can be detected degreewise. That is, (2.1.35.1) is exact if and only the sequence

$$
\cdots \rightarrow M^0 \xrightarrow{a^0} M^1 \xrightarrow{a^1} M^2 \xrightarrow{a^2} \cdots
$$
in \( M(R) \) is exact for every \( v \in \mathbb{Z} \).

The next result is known as the \textit{Five Lemma}.

\textbf{2.1.37 Lemma.} Let

\[
\begin{array}{cccc}
M^1 & \longrightarrow & M^2 & \longrightarrow & M^3 & \longrightarrow & M^4 & \longrightarrow & M^5 \\
\downarrow{\phi^1} & & \downarrow{\phi^2} & & \downarrow{\phi^3} & & \downarrow{\phi^4} & & \downarrow{\phi^5} \\
N^1 & \longrightarrow & N^2 & \longrightarrow & N^3 & \longrightarrow & N^4 & \longrightarrow & N^5
\end{array}
\]

be a commutative diagram in \( C(R) \) with exact rows. The following assertions hold.

(a) If \( \phi^1 \) is surjective, and \( \phi^2 \) and \( \phi^4 \) are injective, then \( \phi^3 \) is injective.

(b) If \( \phi^3 \) is injective, and \( \phi^2 \) and \( \phi^4 \) are surjective, then \( \phi^5 \) is surjective.

(c) If \( \phi^1, \phi^2, \phi^4, \) and \( \phi^5 \) are isomorphisms, then \( \phi^3 \) is an isomorphism.

\textbf{Proof.} (a): The assumptions imply for each \( v \in \mathbb{Z} \) that the homomorphism \( \phi^1_v \) is surjective, and the homomorphisms \( \phi^2_v \) and \( \phi^4_v \) are injective. Thus, the Five Lemma for modules 1.1.2 applied to the degree \( v \) part of the given diagram yields that \( \phi^3_v \) is injective. Since this holds for every \( v \in \mathbb{Z} \), it follows that \( \phi^3 \) is injective.

The proofs of parts (b) and (c) are similar.

\textbf{2.1.38 Construction.} Let

\[
\begin{array}{cccc}
M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow{\varphi'} & & \downarrow{\varphi} & & \downarrow{\varphi''} & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N''
\end{array}
\]

be a commutative diagram in \( C(R) \) with exact rows. In each degree \( v \) it yields a commutative diagram in \( M(R) \), to which 1.1.3 applies to give a homomorphism \( \delta_v : \ker \varphi'' \rightarrow \operatorname{coker} \varphi' \). Denote by \( \delta : (\ker \varphi'')^\wedge \rightarrow (\operatorname{coker} \varphi')^\wedge \) the morphism \( (\delta_v)_{v \in \mathbb{Z}} \) of graded \( R \)-modules.

The next result is known as the \textit{Snake Lemma}, and the morphism \( \delta \) is known as the \textit{connecting morphism}.

\textbf{2.1.39 Lemma.} The morphism \( \delta \) of graded \( R \)-modules, defined in 2.1.38, is a morphism of \( R \)-complexes, and there is an exact sequence in \( C(R) \),

\[
\ker \varphi' \longrightarrow \ker \varphi \longrightarrow \ker \varphi'' \xrightarrow{\delta} \operatorname{coker} \varphi' \longrightarrow \operatorname{coker} \varphi'' \xrightarrow{\beta} \operatorname{coker} \varphi \longrightarrow \operatorname{coker} \varphi''
\]

Moreover, if \( \alpha' \) is injective then so is the restricted morphism \( \alpha' : \ker \varphi' \rightarrow \ker \varphi \), and if \( \beta \) is surjective, then so is the induced morphism \( \beta : \operatorname{coker} \varphi' \rightarrow \operatorname{coker} \varphi'' \).

\textbf{Proof.} By 1.1.4 there is an exact sequence
The following conditions are equivalent.

- if there exist morphisms \( \delta' \in (\text{Coker} \varphi')_v \) and \( \delta \in (\text{Coker} \varphi)_v \), \( \delta' \rightarrow \delta \) is a morphism. It remains to show that \( \delta \) is a morphism. For a homogeneous element \( n'' \in \text{Ker} \varphi'' \) one has \( \delta(n'') = [n''|_{\text{Im} \varphi}] \) for a homogeneous element \( n' \in N' \) that satisfies \( \beta'(n') = \varphi(m) \) for some \( m \in M \) with \( \alpha(m) = m'' \); see 1.1.3. Now one has

\[
\partial \text{Coker} \varphi' \delta(n'') = \partial \text{Coker} \varphi \varphi(m) = \partial \varphi (\partial M(m)),
\]

The element \( \partial N'(n') \) satisfies \( \beta'(\partial N'(n')) = \partial N(\beta'(n')) = \partial N(\varphi(m)) = \varphi(\partial M(m)) \), and one has \( \alpha(\partial M(m)) = \partial M(\alpha(m)) = \partial M''(m'') \). Thus, by the definition of \( \delta \) one has \( \delta(\partial M''(m'')) = [\partial N'(n')|_{\text{Im} \varphi}] = \partial \text{Coker} \varphi' \delta(n'') \).

**Split Sequences**

**2.1.40 Definition.** An exact sequence \( 0 \rightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \rightarrow 0 \) in \( \mathbb{C}(R) \) is called **split** if there exist morphisms \( \varrho : M \rightarrow M' \) and \( \sigma : M'' \rightarrow M \) such that one has

\[
\varrho \alpha' = 1_{M'}, \quad \alpha' \varrho + \sigma \alpha = 1_{M}, \quad \text{and} \quad \alpha \sigma = 1_{M''}.
\]

**2.1.41 Proposition.** Let \( 0 \rightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \rightarrow 0 \) be an exact sequence in \( \mathbb{C}(R) \). The following conditions are equivalent.

1. The sequence is split exact.
2. There exists a morphism \( \varrho : M \rightarrow M' \) such that \( \varrho \alpha' = 1_{M'} \).
3. There exists a morphism \( \sigma : M'' \rightarrow M \) such that \( \alpha \sigma = 1_{M''} \).
4. The sequence is isomorphic to \( 0 \rightarrow M' \xrightarrow{\iota} M' \oplus M'' \xrightarrow{\pi} M'' \rightarrow 0 \), where \( \iota \) and \( \pi \) are the embedding and the projection, respectively.

**Proof.** Conditions (ii) and (iii) follow from (i). To see that (ii) implies (iv), let \( \varrho : M \rightarrow M' \) be a morphism with \( \varrho \alpha' = 1_{M'} \). There is then a commutative diagram,

\[
\begin{array}{cccccc}
0 & \rightarrow & M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' & \rightarrow & 0 \\
& & \uparrow{\varrho \alpha'} & & \rightarrow & & \varrho \alpha' & & \\
0 & \rightarrow & M' & \xrightarrow{\iota \alpha} & M' \oplus M'' & \xrightarrow{\pi} M'' & \rightarrow & 0 ,
\end{array}
\]

in \( \mathbb{C}(R) \), and it follows from the Five Lemma 2.1.37 that \( (\varrho \alpha) \) is an isomorphism. A parallel argument shows that (iii) implies (iv).

Given a commutative diagram

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Let $\mathcal{C}$ be a category. A sequence $\cdots \xrightarrow{\alpha_{i-1}} M_{i-1} \xrightarrow{\alpha_i} M_i \xrightarrow{\alpha_{i+1}} M_{i+1} \cdots$ is said to be degreewise split if for every degree $n$ there is a splitting of the sequences $M_n \xrightarrow{\alpha_n} M_{n+1}$ and $M_n \xrightarrow{\alpha_n} M_{n-1}$. If $\mathcal{C}$ is abelian and $\mathcal{C}$ has enough projectives and enough injectives, then $\mathcal{C}$ is equivalent to the category of chain complexes of $\mathcal{C}$.

**2.1.42 Lemma.** A short exact sequence in $\mathcal{C}(R)$,

$$0 \to M' \to M \to M'' \to 0,$$

is called degreewise split if the exact sequence $0 \to M'^\mathcal{g} \to M^\mathcal{g} \to M''^\mathcal{g} \to 0$ splits.

**2.1.43 Lemma.** Let $F: \mathcal{C}(R) \to \mathcal{C}(S)$ be an additive functor, such that the composites $(-)^3 \circ F$ and $(-)^3 \circ F \circ (-)^3$ are naturally isomorphic functors from $\mathcal{C}(R)$ to $\mathcal{M}_{gr}(S)$. For every degreewise split exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}(R)$ the induced sequence $0 \to F(M') \to F(M) \to F(M'') \to 0$ is degreewise split exact in $\mathcal{C}(S)$.

**PROOF.** It follows from the assumptions on $F$ that the sequence

$(\star) \quad 0 \to (F(M'))^3 \to (F(M))^3 \to (F(M''))^3 \to 0$

is isomorphic to

$(\ddagger) \quad 0 \to (F(M'^\mathcal{g}))^3 \to (F(M^\mathcal{g}))^3 \to (F(M''^\mathcal{g}))^3 \to 0$

in $\mathcal{M}_{gr}(S)$. The second sequence arises from application of the additive functor $(-)^3 \circ F: \mathcal{M}_{gr}(R) \to \mathcal{M}_{gr}(S)$ to the split exact sequence $0 \to M'^\mathcal{g} \to M^\mathcal{g} \to M''^\mathcal{g} \to 0$ in $\mathcal{M}_{gr}(R)$. Therefore, $(\ddagger)$ is split exact in $\mathcal{M}_{gr}(S)$ by 1.1.28, and hence so is the isomorphic sequence $(\star)$.

Similarly one proves the next result.

**2.1.44 Lemma.** Let $G: \mathcal{C}(R)^{op} \to \mathcal{C}(S)$ be an additive functor, such that the composites $(-)^3 \circ G$ and $(-)^3 \circ G \circ (-)^3$ are naturally isomorphic functors from $\mathcal{C}(R)^{op}$ to $\mathcal{M}_{gr}(S)$. For every degreewise split exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}(R)$ the induced sequence $0 \to G(M') \to G(M) \to G(M'') \to 0$ is degreewise split exact in $\mathcal{C}(S)$.

**EXERCISES**

E 2.1.1 Let $L$ be a graded $R$-module and let $E = \{e_u\}_{u \in U}$ be a subset of $L$ consisting of homogeneous elements. Show that $L$ is graded-free with graded basis $E$ if and only if it has...
2.2 Homology

SYNOPSIS. Shift; homology; connecting morphism; homotopy.

SHIFT

2.2.1 Definition. Let $M$ be an $R$-complex and let $s$ be an integer. The \textit{s-fold shift} of $M$ is the complex $\Sigma^s M$ given by
\[
(S^s M)_v = M_{v-s} \quad \text{and} \quad \partial^s_v = (-1)^s \partial^M_{v-s}
\]
for all \( v \in \mathbb{Z} \). For a homomorphism \( \alpha = (\alpha_v)_{v \in \mathbb{Z}} : M \to N \) of \( R \)-complexes, the \( s \)-fold shift \( \Sigma^s \alpha : \Sigma^s M \to \Sigma^s N \) is the homomorphism with \( (\Sigma^s \alpha)_v = \alpha_{v-s} \) for all \( v \in \mathbb{Z} \).

**Remark.** Other words for shift are suspension and translation.

**2.2.2.** For an \( R \)-complex \( M \) one has \( \partial \Sigma^s M = (-1)^s \Sigma^s \partial M \). For a homomorphism of \( R \)-complexes \( \alpha \) one has \( |\Sigma^s \alpha| = |\alpha| \), and for composable homomorphisms \( \alpha \) and \( \beta \) one has \( (\Sigma^s \beta)(\Sigma^s \alpha) = \Sigma^s (\beta \alpha) \). It follows that if \( \alpha : M \to N \) is a chain map of \( R \)-complexes then so is \( \Sigma^s \alpha \); indeed, there are equalities

\[
\partial \Sigma^s N (\Sigma^s \alpha) = (-1)^s (\Sigma^s \partial N) (\Sigma^s \alpha) = (-1)^s \Sigma^s (\partial N \alpha)
\]

\[
= (-1)^s \Sigma^s (\partial N \alpha)
\]

\[
= (-1)^{|\alpha|} (-1)^s \Sigma^s (\alpha \partial M)
\]

\[
= (-1)^{|\Sigma^s \alpha|} (\Sigma^s \alpha) \partial \Sigma^s M.
\]

In particular, \( \Sigma^s \) takes morphisms to morphisms, and it follows that \( \Sigma^s \) is an exact \( k \)-linear autofunctor on \( \mathcal{C}(R) \) with inverse \( \Sigma^{-s} \). Evidently, \( \Sigma^s \) is the \( s \)-fold composite of the functor \( \Sigma = \Sigma^1 \).

**2.2.3.** Let \( M \) be an \( R \)-complex and \( s \) be an integer. There is a canonical chain map \( \varsigma_{-s}^M : M \to \Sigma^{-s} M \) of degree \( s \) that maps a homogeneous element \( m \) in \( M \) to the corresponding element of degree \( |m| + s \) in \( \Sigma^{-s} M \). The map is invertible, and \( (\varsigma_{-s}^M)^{-1} \) is the chain map \( \varsigma_{s}^{\Sigma^{-s} M} : \Sigma^{s} M \to M \). Notice that for every homomorphism \( \alpha : M \to N \) of complexes the following diagram is commutative,

\[
M \xrightarrow{\varsigma_{-s}^M} \Sigma^{-s} M \xrightarrow{\varsigma_{s}^{\Sigma^{-s} M}} M
\]

\[
N \xrightarrow{\varsigma_{-s}^N} \Sigma^{-s} N \xrightarrow{\varsigma_{s}^{\Sigma^{-s} N}} N.
\]

These degree shifting maps are occasionally suppressed. In particular, a homomorphism (chain map) \( \beta : \Sigma^{-s} M \to N \) is identified with the homomorphism (chain map) \( \beta \varsigma_{-s}^M : M \to N \) of degree \( |\beta| + s \), and a homomorphism (chain map) \( \gamma : L \to \Sigma^s M \) is identified with the homomorphism (chain map) \( \varsigma_{s}^{\Sigma^s M} \gamma : L \to M \) of degree \( |\gamma| - s \).

**Homology**

**2.2.4.** Let \( M \) be an \( R \)-complex. The differential \( \partial M \) is a chain map, so it follows from 2.1.31 that \( \text{Ker} \partial M \) and \( \text{Im} \partial M \) are subcomplexes of \( M \). As \( \partial M \partial M = 0 \) holds, there is an inclusion \( \text{Im} \partial M \subseteq \text{Ker} \partial M \), and the induced differential on either subcomplex is 0. Likewise, the differential on the quotient complex \( \text{Coker} \partial M \) is 0.

**2.2.5 Definition.** Let \( M \) be an \( R \)-complex \( M \); set

\[
\text{Homology}
\]

\[
24-\text{Feb-2012}
\]

Draft, not for circulation
\[ Z(M) = \ker \partial^M, \quad B(M) = \operatorname{Im} \partial^M, \quad \text{and} \quad C(M) = \operatorname{Coker} \partial^M. \]

Notice that \( C(M) \) is the quotient complex \( M/B(M) \). Elements in the subcomplex \( Z(M) \) are called cycles, and elements in \( B(M) \) are called boundaries. The quotient
\[ H(M) = Z(M) / B(M) \]
is called the homology complex of \( M \). If one has \( H(M) = 0 \), then \( M \) is called acyclic.

The module of homogeneous cycles in \( M \) of degree \( v \), that is, the module in degree \( v \) in the complex \( Z(M) \), is written \( Z_v(M) \). Similarly the modules in the complexes \( B(M) \), \( C(M) \), and \( H(M) \) are written \( B_v(M) \), \( C_v(M) \), and \( H_v(M) \).

Notice that a complex is acyclic (see 2.2.5) as an object in \( \mathcal{M}(R) \) if and only if it is exact (see 1.1.1) when regarded as a sequence in \( \mathcal{M}(R) \).

**Remark.** Other words for acyclic are exact and homologically trivial. For a complex written in cohomological notation \( M = \cdots \rightarrow M^{v-1} \rightarrow M^v \rightarrow M^{v+1} \rightarrow \cdots \) it is standard to write the homology complex \( H(M) \) in cohomological notation as well. The module in cohomological degree \( v \) of \( H(M) \) is denoted \( H^v(M) \) and called the \( v \)-th cohomology module of \( M \).

The terms "boundary" and "cycle" are borrowed from specific homology theories, where this nomenclature comes natural. Briefly: paths \( \pi \) and \( \pi' \in \mathbb{R}^2 \) are deemed equivalent if they differ by a boundary; that is, the concatenation of \( \pi \) and \( -\pi' \) is the boundary of a region. The reason is that by Stokes’ theorem the path integral of an exact differential form is then the same along \( \pi \) and \( \pi' \). In a topological space that has a structure of a simplicial complex—a higher triangulation; cf. 2.1.22—it is compelling to use the name cycle for a potential boundary. For a rigid, but still brief, explanation see the first section of Rotman’s book [45]. Weibel’s historical survey [53] has plenty of illuminating details.

**2.2.6 Example.** The Dold complex in 2.1.20 is acyclic.

**2.2.7 Example.** The Koszul homology of a sequence \( x_1, \ldots, x_n \) in \( k \) is the homology of the Koszul complex \( K^k(x_1, \ldots, x_n) \) from 2.1.21. It is elementary to verify that the homology in degree 0 is \( H_0(K^k(x_1, \ldots, x_n)) = k/(x_1, \ldots, x_n) \).

For a single element \( x \) the Koszul complex \( K^k(x) \) has the form
\[ 0 \rightarrow k\langle e \rangle \xrightarrow{\partial} k \rightarrow 0, \]
where \( \partial(1) = x \). Hence one has \( H_1(K^k(x)) \cong (0 :_k x) \). In particular, this homology module vanishes if and only if \( x \) is not a zero-divisor in \( k \).

The Koszul complex \( K = K^k(x_1, x_2) \) has the form
\[ 0 \rightarrow k\langle e_1 \wedge e_2 \rangle \xrightarrow{\partial^1} k\langle e_1, e_2 \rangle \xrightarrow{\partial^2} k \rightarrow 0, \]
where \( \partial^1 K(k_1 e_1 + k_2 e_2) = k_1 x_1 + k_2 x_2 \) and \( \partial^2 K(e_1 \wedge e_2) = x_1 e_2 - x_2 e_1 \). The cycles in degree 2 are elements \( ke_1 \wedge e_2 \) with \( k x_1 = 0 = k x_2 \); hence there is an isomorphism \( H_2(K) \cong (0 :_k x_1) \cap (0 :_k x_2) \). The cycles in degree 1 embody the relations between \( x_1 \)
and $x_2$ while the boundaries embody the simple relation $x_1x_2 - x_2x_1 = 0$. Notice that $H_2(K)$ vanishes if $x_1$ or $x_2$ is not a zero-divisor in $k$. The homology module $H_1(K)$ vanishes if and only if $[x_2]_{x_1}$ is not a zero-divisor in the ring $k[x_1]$ and $[x_1]_{x_2}$ is not a zero-divisor in $k/x_1x_2$.

The complex $K = K^d(x_1, \ldots, x_n)$ is concentrated in degrees $0, \ldots, n$. The homology module $H_n(K)$ is isomorphic to the ideal of common annihilators of the elements $x_1, \ldots, x_n$. The module $H_i(K)$ is the module generated by all relations between the generators modulo the one generated by the simple ones $x_ix_j - x_jx_i = 0$. In intermediate degrees $n > i > 1$, the module $H_i(K)$ is generated by “higher” relations between the generators, and the boundaries that are factored out are simple “higher” relations usually referred to as Koszul relations.

**2.2.8 Example.** The singular homology $H_*(X; A)$ of a topological space $X$ with coefficients in an Abelian group $A$ is the homology of the singular chain complex $S(X) \otimes_Z A$ from 2.1.22. Singular homology with coefficients in $\mathbb{Z}$ is written $H_*(X)$. The singular homology group $H_n(X)$ detects “$n$-dimensional holes” in $X$.

For every space $X$, the abelian group $H_0(X)$ is free, and its rank (possibly an infinite cardinal) is the number of path components of $X$. The singular chain complex of a one-point space $pt$ is

$$S(pt) = \cdots \to \mathbb{Z} \langle \sigma_2 \rangle \xrightarrow{\partial^2_n} \mathbb{Z} \langle \sigma_1 \rangle \xrightarrow{\partial^1_n} \mathbb{Z} \langle \sigma_0 \rangle \to 0,$$

where $\sigma_n$ for each $n > 0$ denotes the unique (continuous) map $\Delta_n \to pt$ from the standard $n$-simplex to a point. The definition 2.1.22 yields

$$\partial^2_n = 0 \quad \text{for } n \text{ odd} \quad \text{and} \quad \partial^1_n(\sigma_n) = \sigma_{n-1} \quad \text{for } n \text{ even},$$

and thus one has $H_0(pt) \cong \mathbb{Z}$ and $H_n(pt) = 0$ for all $n \neq 0$. Every contractible space has the same singular homology as $pt$; see also 2.2.28. For $m > 0$ the singular homology of the $m$-sphere $S^m$ is $H_0(S^m) \cong \mathbb{Z} \cong H_m(S^m)$ and $H_n(S^m) = 0$ for all $n \notin \{0, m\}$.

**2.2.9 Example.** The de Rham cohomology $H^d_{\text{dR}}(M)$ of a smooth real manifold $M$ is the cohomology of the de Rham complex $\Omega(M)$ from 2.1.23; it detects closed differential forms on $M$ that are not exact. A closed 0-form is constant on every connected component, so the rank of the real vector space $H^0_{\text{dR}}(M)$ is the number of connected components of $M$. The cohomology group $H^1_{\text{dR}}(M)$ vanishes if and only if every closed differential 1-form is the differential of a smooth function $f: M \to \mathbb{R}$.

The de Rham complex $\Omega(S^1)$ of the 1-sphere is non-zero only in degrees 0 and 1; in particular, the cohomology modules $H^n_{\text{dR}}(S^1)$ vanish for $n \notin \{0, 1\}$. As $S^1$ is connected, one has $H^0_{\text{dR}}(M) \cong \mathbb{R}$. Every 1-form is closed ($d^1 = 0$). Despite the appearance, $d\theta$, the differential of the polar coordinate “function” $\theta$ is not exact, so $H^1_{\text{dR}}(M)$ is non-vanishing and, in fact, isomorphic to $\mathbb{R}$.

The de Rham cohomology $H^n_{\text{dR}}(B^n)$ of the open unit ball in $\mathbb{R}^n$ vanishes for $n \neq 0$.

**2.2.10.** Let $M$ be an $R$-complex. The subcomplexes and (sub-)quotient complexes introduced in 2.2.5 are linked by the following exact sequences in $\mathcal{C}(R)$. 

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For the proof of the second claim, let $M$ be a morphism of $R$-complexes. It follows from the equality $\alpha \partial^M = \partial^N \alpha$ that $\alpha$ restricts to a morphism of cycle complexes $Z(M) \to Z(N)$ and further to a morphism of boundary complexes $B(M) \to B(N)$. In particular, $\alpha$ induces a morphism of cokernel complexes, $\bar{\alpha}: C(M) \to C(N)$. It is straightforward to verify that $Z(-)$, $B(-)$, and $C(-)$ are endofunctors on $\mathcal{C}(R)$. For brevity, we continue to write $\bar{\alpha}$ for $\bar{\alpha}(\alpha)$ and $\bar{\alpha} for $Z(\alpha) = \alpha |_{Z(M)}$ and $B(\alpha) = \alpha |_{B(M)}$.

2.2.11. Let $\alpha: M \to N$ be a morphism of $R$-complexes. It follows from 2.2.11 that $\alpha$ induces morphism of $R$-complexes

$$H(\alpha): H(M) \to H(N),$$

which is given by the assignment $[z]_{B(M)} \mapsto [\alpha(z)]_{B(N)}$ for $z \in Z(M)$.

The equivalence class $[z]_{B(M)}$ is called the homology class of $z$. Hereafter we drop the subscript on homology classes and write $[z]$ for the homology class of a cycle $z$.

By the definition of $H(\alpha)$ there is a commutative diagram,

$$\begin{array}{c}
0 \to B(M) \to Z(M) \to H(M) \to 0 \\
\downarrow \alpha \downarrow \alpha \downarrow \alpha \\
0 \to B(N) \to Z(N) \to H(N) \to 0.
\end{array}$$

2.2.12. Let $\alpha: M \to N$ be a morphism of $R$-complexes. It follows from 2.2.11 that $\alpha$ induces morphism of $R$-complexes

$$H(\alpha): H(M) \to H(N),$$

which is given by the assignment $[z]_{B(M)} \mapsto [\alpha(z)]_{B(N)}$ for $z \in Z(M)$.

The equivalence class $[z]_{B(M)}$ is called the homology class of $z$. Hereafter we drop the subscript on homology classes and write $[z]$ for the homology class of a cycle $z$.

By the definition of $H(\alpha)$ there is a commutative diagram,

$$\begin{array}{c}
0 \to B(M) \to Z(M) \to H(M) \to 0 \\
\downarrow \alpha \downarrow \alpha \downarrow \alpha \\
0 \to B(N) \to Z(N) \to H(N) \to 0.
\end{array}$$

2.2.13 Proposition. Homology $H$ is a $\mathbb{Z}$-linear endofunctor on $\mathcal{C}(R)$. Moreover, there is an equality $H \Sigma = \Sigma H$ of endofunctors on $\mathcal{C}(R)$.

Proof. It is straightforward to verify that $H: \mathcal{C}(R) \to \mathcal{C}(R)$ is a $\mathbb{Z}$-linear functor. For the proof of the second claim, let $M$ be an $R$-complex. The equality $\partial^\Sigma M = -\Sigma \partial^M$ and exactness of the shift functor yields

$$H(\Sigma M) = \text{Ker} \partial^\Sigma M/\text{Im} \partial^\Sigma M$$

$$= \text{Ker} \Sigma \partial^M/\text{Im} \Sigma \partial^M$$

$$= (\Sigma \text{Ker} \partial^M)/(\Sigma \text{Im} \partial^M)$$

$$= \Sigma (\text{Ker} \partial^M/\text{Im} \partial^M)$$

$$= \Sigma H(M).$$

2.2.14. For a complex $Z$ with zero differential one has $H(Z) = Z$. In particular, $H(H(M)) = H(M)$ holds for every complex.

2.2.15. A functor $F: \mathcal{M}(R) \to \mathcal{M}(S)$ extends to a functor $\mathcal{C}(R) \to \mathcal{C}(S)$ that acts as $F$ in each degree. Notice that the extended functor, which is also denoted $F$, commutes
with shift. If the functor $F: \mathcal{M}(R) \to \mathcal{M}(S)$ is exact, then so is the extended functor $F: \mathcal{C}(R) \to \mathcal{C}(S)$, and it commutes with homology. Indeed, in view of 2.2.10 one has
\[ B(F(M)) = F(B(M)), \quad Z(F(M)) = F(Z(M)), \quad \text{and} \quad H(F(M)) = F(H(M)), \]
and similarly $H(F(\alpha)) = F(H(\alpha))$ for every morphism $\alpha$ in $\mathcal{C}(R)$.

Similarly, a functor $G: \mathcal{M}(R)^{\text{op}} \to \mathcal{M}(S)$ extends to a functor $G: \mathcal{C}(R)^{\text{op}} \to \mathcal{C}(S)$. If the original functor is exact, then the extended functor is exact and commutes with homology.

2.2.16. Let $M$ be an $R$-complex. It follows from 2.2.15 that for an injective $R$-module $E$ there are inequalities
\[ (2.2.16.1) \quad \inf \text{Hom}_R(M, E) \geq -\sup M \quad \text{and} \quad \sup \text{Hom}_R(M, E) \leq -\inf M, \]
where equalities hold if $E$ is faithfully injective. Similarly, if $F$ is a flat $R^{\text{op}}$-module, then there are inequalities
\[ (2.2.16.2) \quad \inf (F \otimes_R M) \geq \inf M \quad \text{and} \quad \sup (F \otimes_R M) \leq \sup M, \]
where equalities hold if $F$ is faithfully flat.

CONNECTING MORPHISM IN HOMOLOGY

2.2.17 Construction. Consider a short exact sequence of $R$-complexes,
\[ 0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0. \]
It induces a commutative diagram in $\mathcal{C}(R)$ with exact rows,
\[
\begin{array}{c}
\xymatrix{ 
C(M') \ar[r]^{\alpha'} \ar[d]^{\partial M'} & C(M) \ar[r]^{\alpha} \ar[d]^{\partial M} & C(M'') \ar[d]^{\partial M''} \\
0 \ar[r] & \Sigma Z(M') \ar[r]^{\Sigma \alpha'} & \Sigma Z(M) \ar[r]^{\Sigma \alpha} & \Sigma Z(M'') 
}
\end{array}
\]
and the Snake Lemma 2.1.39 yields an exact sequence
\[ (2.2.17.1) \quad H(M') \xrightarrow{H(\alpha')} H(M) \xrightarrow{H(\alpha)} H(M'') \xrightarrow{\delta} \Sigma H(M') \xrightarrow{\Sigma H(\alpha')} \Sigma H(M), \]
where the connecting morphism in homology, $\delta$, maps a homology class $[z']$ with $z' = \alpha(m)$ to $[z']$ with $(\Sigma \alpha')(z') = \partial M(\alpha(m))$; cf. 2.1.38 and 1.1.3.

The complexes in the exact sequence (2.2.17.1) have zero differentials, and it is often written as an exact sequence in $\mathcal{M}(R)$:
\[ \cdots \longrightarrow H_0(M') \xrightarrow{H_0(\alpha')} H_0(M) \xrightarrow{H_0(\alpha)} H_0(M'') \xrightarrow{\delta} H_0(M^{-1}) \longrightarrow \cdots. \]

**2.2.18.** It follows from (2.2.17.1) that if two of the complexes \( M', M, \) and \( M'' \) in a short exact sequence \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) are acyclic, then so is the third.

The connecting morphism is natural in the following sense.

**2.2.19 Proposition.** For every commutative diagram of \( R \)-complexes

\[
\begin{array}{c}
0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0 \\
| \quad | \quad | \\
0 \longrightarrow N' \xrightarrow{\beta} N \xrightarrow{\beta'} N'' \longrightarrow 0
\end{array}
\]

with exact rows, there is a commutative diagram

\[
\begin{array}{c}
H(M') \xrightarrow{H(\alpha')} H(M) \xrightarrow{H(\alpha)} H(M'') \xrightarrow{\delta} \Sigma H(M') \xrightarrow{\Sigma H(\alpha')} \Sigma H(M) \\
| \quad | \quad | \quad | \quad | \quad | \quad | \\
| \quad | \quad | \quad | \quad | \quad | \quad | \\
H(N') \xrightarrow{H(\beta')} H(N) \xrightarrow{H(\beta)} H(N'') \xrightarrow{\delta} \Sigma H(N') \xrightarrow{\Sigma H(\beta')} \Sigma H(N')
\end{array}
\]

with exact rows. Here \( \delta \) and \( \delta \) are the connecting morphisms from 2.2.17.

**PROOF.** In view of 2.2.13 and 2.2.17 it remains to prove that the square

\[
\begin{array}{c}
H(M'') \xrightarrow{\delta} \Sigma H(M') \\
| \quad | \\
H(N'') \xrightarrow{\delta} \Sigma H(N')
\end{array}
\]

is commutative. To this end, let \([\bar{z}]\) be an element in \( H(M'') = \text{Ker} \partial_{M''} \). By the definition of the connecting morphism \( \delta \), see 2.2.17, one has \( \delta([\bar{z}]) = [\bar{z}'] \), where \( \bar{z}' \in \Sigma Z(M) \) satisfies \( (\Sigma \alpha')(\bar{z}') = \bar{z}' \). Thus, one has \( (\Sigma H(\alpha')(\bar{z}')) = ([\Sigma(\alpha')])(\bar{z}') \). The cycle \( (\Sigma \varphi')(\bar{z}') \) satisfies

\[
(\Sigma \beta')(\Sigma \varphi')(\bar{z}') = (\Sigma \varphi')(\Sigma \alpha')(\bar{z}') = (\Sigma \varphi)\Sigma \delta M([x]_{B(M)}) = \Sigma \delta N([\varphi(x)]_{B(N)}),
\]

where the last equality follows from (2.2.31). The element \([\varphi(x)]_{B(N)} \) satisfies

\[
\bar{\beta}([\varphi(x)]_{B(N)}) = [\varphi(x)]_{B(M')} = [\varphi' \alpha(x)]_{B(N')} = H(\varphi')([\alpha(x)]) = H(\varphi')([\bar{z}']) .
\]

By the definition of \( \delta \) one now has \( (\delta H(\varphi'))([\bar{z}']) = [(\Sigma \varphi')(\bar{z}')] \). That is, the square \((*)\) is commutative. \( \square \)
Like many other notions in homological algebra, homotopy originates in topology. Homotopy works behind the scenes: it is preserved by additive functors, but it has zero homological footprint.

2.2.20 Definition. A chain map of \( R \)-complexes \( \alpha : M \to N \) is called null-homotopic if there exists a graded homomorphism \( \sigma : M^\bullet \to N^\bullet \) of degree \(|\alpha| + 1\) such that the equality \( \alpha = \partial N \sigma + (-1)^{|\alpha|} \sigma \partial M \) holds.

Two chain maps of \( R \)-complexes \( \alpha, \alpha' : M \to N \) of the same degree are called homotopic, in symbols \( \alpha \sim \alpha' \), if \( \alpha - \alpha' \) is null-homotopic. A degree 1 homomorphism \( \sigma \) with \( \alpha - \alpha' = \partial N \sigma + (-1)^{|\alpha|} \sigma \partial M \) is called a homotopy from \( \alpha \) to \( \alpha' \).

Example. Consider an \( R \)-complex

\[
M = 0 \to M_2 \to M_1 \to M_0 \to 0.
\]

One has \( H(1^M) = 0 \) if and only if \( M \) is exact as a sequence in \( M(R) \), while \( 1^M \) is null-homotopic if and only if \( M \) is split exact as a sequence in \( M(R) \).

2.2.25 Definition. A morphism of \( R \)-complexes \( \alpha : M \to N \) is called a homotopy equivalence if there exists a morphism \( \beta : N \to M \) such that \( 1^M - \beta \alpha \) and \( 1^N - \alpha \beta \) are null-homotopic; such a morphism \( \beta \) is called a homotopy inverse of \( \alpha \). If there exists a homotopy equivalence \( \alpha : M \to N \), then \( M \) and \( N \) are called homotopy equivalent.

2.2.26 Example. Let \( M \) be an \( R \)-module. The unique morphism from the complex \( 0 \to M \to 0 \) to the zero complex is a homotopy equivalence; see 2.2.21.

2.2.27. Let \( \alpha : M \to N \) be a morphism of \( R \)-complexes. If \( \alpha \) is an isomorphism, then \( \alpha^{-1} \) is a homotopy inverse of \( \alpha \). If \( \alpha \) is a homotopy equivalence with homotopy inverse \( \beta \), then 2.2.23 implies that \( H(\alpha) \) is an isomorphism with inverse \( H(\beta) \).
2.2.28 Example. Let $X$ be a contractible topological space. That is, there is a point $x_0$ in $X$ such that the identity map $1^X: X \to X$ and the constant map $c: X \to X$ with value $x_0$ are homotopic, meaning that there is a continuous map $H: X \times [0,1] \to X$ with $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$. The continuous map $\pi: X \to \{x_0\}$ induces a morphism of singular chain complexes, see 2.1.22 and 2.2.8.

$S(X) = \cdots \xrightarrow{\partial_2^X} \mathbb{Z}(C(\Delta_2,X)) \xrightarrow{\pi_0} \mathbb{Z}(C(\Delta_1,X)) \xrightarrow{\partial_1^X} \mathbb{Z}(C(\Delta_0,X)) \xrightarrow{\pi_0} 0$

$S(\{x_0\}) = \cdots \xrightarrow{\partial_2^{\{x_0\}}} \mathbb{Z}(\sigma_2) \xrightarrow{\pi_0} \mathbb{Z}(\sigma_1) \xrightarrow{\partial_1^{\{x_0\}}} \mathbb{Z}(\sigma_0) \xrightarrow{\pi_0} 0.$

The morphism $\pi_*$ is a homotopy equivalence with homotopy inverse $\iota$, induced by the embedding $\iota: \{x_0\} \to X$. In particular, the spaces $X$ and $\{x_0\}$ have isomorphic singular homology by 2.2.27. Moreover, it follows from 2.2.8 and 2.2.26 that the complex $S(\{x_0\})$, and hence also $S(X)$, is homotopy equivalent to the complex with $\mathbb{Z}$ in degree 0 and zero elsewhere.

2.2.29. A diagram in $\mathbb{C}(R)$,

$$
\begin{array}{ccc}
M & \xrightarrow{\alpha} & M' \\
\varphi \downarrow & & \varphi' \\
N & \xrightarrow{\beta} & N',
\end{array}
$$

is called commutative up to homotopy if one has $\varphi'\alpha \sim \beta\varphi$. Notice that, in view of 2.2.23, the induced diagram in homology is then commutative.

EXERCISES

E 2.2.1 (Cf. 2.2.13) Show that homology $H: \mathbb{C}(R) \to \mathbb{C}(R)$ is a $k$-linear functor.

E 2.2.2 Assume that $R$ is semi-simple. Show that for every $R$-complex $M$ there are morphisms $\alpha: H(M) \to M$ and $\beta: M \to H(M)$ such that $H(\alpha)$ and $H(\beta)$ are isomorphisms.

E 2.2.3 For $n \geq 0$ show that singular homology $H_n(S(\{\cdot\}))$ is a functor from the category $\mathcal{T}$ of topological spaces to $\mathcal{M}(\mathbb{Z})$. Hint: E 2.1.3 and E 2.1.7.

E 2.2.4 Let $f$ and $g$ be elements in the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients, and let $K$ denote the Koszul complex $K^\mathbb{Z}(f,g)$. (a) Show that if $H_0(K)$ vanishes, then $K$ is acyclic. (b) Show that if $H_1(K)$ vanishes, then $H_2(K)$ vanishes as well.

E 2.2.5 Let $f, g: X \to Y$ be continuous maps between topological spaces, and assume that $f$ and $g$ are homotopic in the topological sense; that is, there exists a continuous map $H: X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Show that the induced morphisms $S(f), S(g): S(X) \to S(Y)$ in $\mathbb{C}(\mathbb{Z})$ are homotopic in the sense of 2.2.20. Is the converse true?

E 2.2.6 Let $M$ be an $R$-complex. Show that $\partial^M$ is a null-homotopic chain map.

E 2.2.7 (Cf. 2.2.22) Let $M$ and $N$ be $R$-complexes and let $n$ be an integer. Show that ’$\sim$’ is an equivalence relation on the set of chain maps $M \to N$ of degree $n$. 

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E 2.2.8 Show that a homotopy between homotopic chain maps may not be unique.

E 2.2.9 Let \( \alpha, \beta : M \to N \) be null-homotopic chain maps of \( R \)-complexes. Show that \( \alpha + \beta \) is null-homotopic and that \( \alpha x \) is null-homotopic for every \( x \in \mathbb{k} \).

E 2.2.10 Let \( \gamma : K \to L \) and \( \beta, \beta' : L \to M \) and \( \alpha : M \to N \) be chain maps of \( R \)-complexes. Show that if \( \beta \) and \( \beta' \) are homotopic, \( \beta \sim \beta' \), then one has \( \alpha \beta \sim \alpha \beta' \) and \( \beta \gamma \sim \beta' \gamma \).

E 2.2.11 Let \( \alpha : M \to N \) be a morphism of \( R \)-complexes. (a) Show that a homotopy inverse of \( \alpha \) is unique up to homotopy. (b) Show that \( \alpha \) is a homotopy equivalence if it has both a left and a right homotopy inverse; that is, if there exist morphisms \( \beta, \beta' : N \to M \) such that \( \alpha \beta \sim 1_N \) and \( \beta \alpha \sim 1_M \) hold.

E 2.2.12 Show that a morphism that is homotopic to a homotopy equivalence is a homotopy equivalence.

E 2.2.13 Show that the identity morphism on the Koszul complex \( K^{2,3} \) is null-homotopic.

E 2.2.14 Show that the inclusion functor \( \mathcal{M}_{gr}(R) \to \mathcal{C}(R) \) has a left adjoint \( C : \mathcal{C}(R) \to \mathcal{M}_{gr}(R) \).

E 2.2.15 Show that the inclusion functor \( \mathcal{M}_{gr}(R) \to \mathcal{C}(R) \) has a right adjoint \( Z : \mathcal{C}(R) \to \mathcal{M}_{gr}(R) \).

2.3 Homomorphisms

SYNOPSIS. Hom complex; chain map; homotopy; Hom functor.

Complexes \( M \) and \( N \) are graded modules with differentials, and those differentials induce a differential on the graded module \( \text{Hom}_R(M, N) \), which then becomes the Hom complex.

2.3.1 Definition. A homomorphism of complexes is a graded homomorphism of the underlying graded modules; see 2.1.6. For \( R \)-complexes \( M \) and \( N \), the complex \( \text{Hom}_R(M, N) \) is the \( \mathbb{k} \)-complex with underlying graded module

\[
\text{Hom}_R(M, N)^{\mathbb{k}} = \text{Hom}_R(M^{\mathbb{k}}, N^{\mathbb{k}})
\]

and differential given by

\[
\partial^{\text{Hom}_R(M,N)}(\alpha) = \partial^N \alpha - (-1)^{|\alpha|} \alpha \partial^M
\]

for a homogeneous element \( \alpha \).

2.3.2. It is elementary to verify that \( \partial^{\text{Hom}_R(M,N)} \) is square zero. Indeed, one has

\[
\partial^{\text{Hom}_R(M,N)} \partial^{\text{Hom}_R(M,N)}(\alpha) = \partial^{\text{Hom}_R(M,N)}(\partial^N \alpha - (-1)^{|\alpha|} \alpha \partial^M)
= \partial^N(\partial^N \alpha - (-1)^{|\alpha|} \alpha \partial^M)
= \partial^N(\partial^N \alpha - (-1)^{|\alpha|} \alpha \partial^M)
= (-1)^{|\alpha|-1}(\partial^N \alpha - (-1)^{|\alpha|} \alpha \partial^M) \partial^M
= 0 .
\]

2.3.3 Proposition. Let \( \alpha : M \to N \) be a homomorphism of \( R \)-complexes.

(a) The homomorphism \( \alpha \) is a cycle in the complex \( \text{Hom}_R(M, N) \) if and only if it is a chain map.
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(b) The homomorphism $\alpha$ is a boundary in the complex $\text{Hom}_R(M,N)$ if and only if it is a null-homotopic chain map.

**Proof.** The assertions are immediate from the definition of the differential on the Hom complex; see 2.3.1.  

2.3.4. Let $L \xrightarrow{\beta} M \xrightarrow{\alpha} N$ be homomorphisms of $R$-complexes. The differentials on the Hom complexes interact with the composition rule from 2.1.9 as follows,

\[
\partial^{\text{Hom}_{(L,N)}}(\alpha \beta) = \partial^N \alpha \beta - (-1)^{[\alpha][\beta]} \alpha \delta^L \\
= (\partial^N \alpha - (-1)^{[\alpha]} \alpha \delta^M) \beta + (-1)^{[\alpha]} \alpha (\partial^M \beta - (-1)^{[\beta]} \beta \delta^L) \\
= \partial^{\text{Hom}_{(M,N)}}(\alpha) \beta + (-1)^{[\alpha]} \alpha \partial^{\text{Hom}_{(L,M)}}(\beta).
\]

Assume that $\alpha$ and $\beta$ are chain maps. It follows from the identity above and 2.3.3 that $\alpha \beta$ is a chain map; this recovers 2.1.27.

The gist of the next proposition is that homotopy is a congruence relation: an equivalence relation that is compatible with composition. From part (c) it follows, in particular, that a homotopy inverse is unique up to homotopy.

2.3.5 **Proposition.** Let $\beta, \beta' : N \to M$ and $\alpha, \alpha' : M \to N$ be morphisms in $\mathbb{C}(R)$.

(a) If $\alpha$ or $\beta$ is null-homotopic, then $\alpha \beta$ is null-homotopic.

(b) If one has $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha \beta$ and $\alpha' \beta'$ are homotopic: $\alpha \beta \sim \alpha' \beta'$.

(c) If $\gamma, \gamma' : N \to M$ are morphisms in $\mathbb{C}(R)$ such that $1^M \sim \gamma \alpha$ and $1^N \sim \alpha' \gamma'$ hold, then $\gamma$ and $\gamma'$ are both homotopy inverses of $\alpha$ and they are homotopic: $\gamma \sim \gamma'$.

**Proof.** (a): If $\alpha$ is null-homotopic, then there is a homomorphism $\vartheta : M \to N$ with $\partial^{\text{Hom}_{(M,N)}}(\vartheta) = \alpha$. As $\beta$ is a cycle in Hom$_R(L,M)$, it follows from 2.3.4 that one has $\partial^{\text{Hom}_{(L,N)}}(\vartheta \beta) = \alpha \beta$. Similarly, 2.3.4 yields $\partial^{\text{Hom}_{(L,N)}}(\alpha \vartheta) = \alpha \beta$ if there is a homomorphism $\varphi$ with $\partial^{\text{Hom}_{(L,M)}}(\varphi) = \beta$.

(b): It follows from (a) that the morphism $\alpha \beta - \alpha' \beta' = \alpha (\beta - \beta') + (\alpha - \alpha') \beta'$ is null-homotopic.

(c): By part (b) one has $\gamma = \gamma' 1^N \sim \gamma \alpha \gamma' \sim 1^M \gamma' = \gamma'$. Another application of (b) yields $1^N \sim \alpha' \gamma' \sim \alpha' \gamma$, so $\gamma$ is a homotopy inverse of $\alpha$; a similar computation shows that $\gamma'$ is a homotopy inverse of $\alpha$.

**Remark.** A differential graded (for short, DG) $\mathbb{k}$-algebra is a $\mathbb{k}$-complex $A$ endowed with a graded $\mathbb{k}$-algebra structure, such that the Leibniz rule $\partial^A(ab) = \partial^A(a)b + (-1)^{|a|}a \partial^A(b)$ holds for all homogeneous elements $a, b \in A$. In other words: the differential $\partial^A$ is a derivation on $A$. For an $R$-complex $M$, it follows from 2.1.9 and 2.3.4 that the complex $\text{Hom}_R(M,M)$ is a DG $\mathbb{k}$-algebra. Moreover, the Koszul complex from 2.1.21 is a DG $\mathbb{k}$-algebra; so is the de Rham complex under the wedge product, and the singular cochain complex $\text{Hom}_{\mathbb{Z}}(S(X), \mathbb{k})$ from 2.1.22 endowed with the so-called cup product is a DG algebra as well. An algebra structure on a complex $A$ induces an algebra structure, notably a product, in homology; this yields another tool for investigating $H(A)$.
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2.3.6 Definition. Let $\alpha : M' \to M$ and $\beta : N \to N'$ be homomorphisms of $R$-complexes. Denote by $\text{Hom}_R(\alpha, \beta)$ the degree $|\alpha| + |\beta|$ homomorphism of $\mathbb{k}$-complexes

$$\text{Hom}_R(M, N) \to \text{Hom}_R(M', N')$$

given by $\theta \mapsto (-1)^{|\alpha||\beta|} \beta \theta \alpha$.

Furthermore, set $\text{Hom}_R(\alpha, N) = \text{Hom}_R(\alpha, 1^N)$ and $\text{Hom}_R(M, \beta) = \text{Hom}_R(1^M, \beta)$.

2.3.7. It is straightforward to verify that the assignment $(\alpha, \beta) \mapsto \text{Hom}_R(\alpha, \beta)$, for homomorphisms $\alpha$ and $\beta$ of $R$-complexes, is $\mathbb{k}$-bilinear. It is also immediate from the definition that one has

$$\text{Hom}_R(1^M, 1^N) = 1^{\text{Hom}_R(M, N)}.$$  

For homomorphisms $M'' \to M' \to M'$ and $N'' \to N \to N'$ of $R$-complexes there is an equality

$$\text{Hom}_R(\alpha', \beta') = (-1)^{|\alpha'||\beta'|} \text{Hom}_R(\alpha', \beta') \text{Hom}_R(\alpha, \beta).$$

Indeed, for every homomorphism $\theta : M' \to N'$ one has

$$\text{Hom}_R(\alpha', \beta') (\theta) = (-1)^{|\alpha'\alpha||\beta'\beta|} \beta' \theta (\alpha' \alpha)$$

$$= (-1)^{|\alpha'||\beta'| (|\theta|)} (-1)^{|\alpha||\beta| (|\theta|)} \text{Hom}_R(\alpha, \beta) (\beta' \theta \alpha')$$

$$= (-1)^{|\alpha'||\beta'| (|\theta|) + |\alpha||\beta| (|\theta|)} \text{Hom}_R(\alpha, \beta) (\beta' \theta \alpha')$$

$$= (-1)^{|\alpha'||\beta'| (|\theta|)} \text{Hom}_R(\alpha', \beta') (\theta).$$

In particular, there are equalities,

$$\text{Hom}_R(\alpha', N) = (-1)^{|\alpha'||\alpha|} \text{Hom}_R(\alpha, N) \text{Hom}_R(\alpha', N),$$

$$\text{Hom}_R(M, \beta') = \text{Hom}_R(M, \beta) \text{Hom}_R(M, \beta'),$$

and, furthermore, a commutative diagram of homomorphisms of $\mathbb{k}$-complexes,

$$\begin{array}{ccc}
\text{Hom}_R(M, N) & \xrightarrow{\text{Hom}(M, \beta)} & \text{Hom}_R(M, N') \\
(-1)^{|\alpha||\beta|} \text{Hom}(\alpha, N) & \xrightarrow{\text{Hom}(\alpha, \beta)} & \text{Hom}(\alpha, N') \\
\text{Hom}_R(M'', N) & \xrightarrow{\text{Hom}(M'', \beta)} & \text{Hom}_R(M'', N').
\end{array}$$

2.3.8 Lemma. Let $\alpha : M' \to M$ and $\beta : N \to N'$ be homomorphisms of $R$-complexes. With $H = \text{Hom}_R(\text{Hom}_R(M', N), \text{Hom}_R(M, N'))$ there is an equality

$$\partial^H(\text{Hom}_R(\alpha, \beta)) = \text{Hom}_R(\partial^\text{Hom}(M', M)(\alpha), \beta) + (-1)^{|\alpha|} \text{Hom}_R(\alpha, \partial^\text{Hom}(N, N')(\beta)).$$
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PROOF. Let \( \theta : M \rightarrow N \) be a homomorphism. The definitions yield

\[
(\partial^H(\text{Hom}_R(\alpha, \beta)))(\theta) = (\partial^{\text{Hom}_R(M', N')} \text{Hom}_R(\alpha, \beta))(\theta) - (-1)^{|\alpha|+|\beta|} (\text{Hom}_R(\alpha, \beta) \partial^{\text{Hom}_R(M, N)})(\theta)
\]

Further, one has

\[
(\text{Hom}_R(\partial^{\text{Hom}_R(M', M)}(\alpha), \beta) + (-1)^{|\alpha|} \text{Hom}_R(\alpha, \partial^{\text{Hom}_R(N', N)}(\beta)))(\theta)
\]

Compare the two expressions above to see that they are identical.

Now we focus on chain maps.

2.3.9 Proposition. Let \( \alpha \) and \( \beta \) be chain maps of \( R \)-complexes. The homomorphism \( \text{Hom}_R(\alpha, \beta) \) is a chain map of degree \(|\alpha| + |\beta|\), and if \( \alpha \) or \( \beta \) is null-homotopic, then \( \text{Hom}_R(\alpha, \beta) \) is null-homotopic.

PROOF. Let \( \alpha : M' \rightarrow M \) and \( \beta : N \rightarrow N' \) be chain maps; by 2.3.3 they are cycles in the complexes \( \text{Hom}_R(M', M) \) and \( \text{Hom}_R(N, N') \). That is, one has \( \partial^{\text{Hom}_R(M', M)}(\alpha) = 0 \) and \( \partial^{\text{Hom}_R(N, N')}(\beta) = 0 \). With \( H = \text{Hom}_R(M, N), \text{Hom}_R(M', N') \), 2.3.8 yields \( \partial^H(\text{Hom}_R(\alpha, \beta)) = 0 \), so \( \text{Hom}_R(\alpha, \beta) \) is a chain map by 2.3.3.

If \( \alpha \) is null-homotopic, then one has \( \alpha = \partial^{\text{Hom}_R(M', M)}(\theta) \) for a \( \theta \) in \( \text{Hom}_R(M', M) \); see 2.3.3. Now 2.3.8 yields \( \text{Hom}_R(\alpha, \beta) = \partial^H(\text{Hom}_R(\theta, \beta)) \), whence \( \text{Hom}_R(\alpha, \beta) \) is null-homotopic. Similarly, if one has \( \beta = \partial^{\text{Hom}_R(N', N)}(\theta) \) for some \( \theta \) in \( \text{Hom}_R(N, N') \), then 2.3.8 yields \( \text{Hom}_R(\alpha, \beta) = \partial^H(\text{Hom}_R(\alpha, \beta)) \).

2.3.10 Corollary. Let \( M \) be an \( R \)-complex.

(a) If \( \beta \) and \( \beta' \) are homotopic chain maps between \( R \)-complexes, then \( \text{Hom}_R(\beta, \beta) \) and \( \text{Hom}_R(M, \beta') \) are homotopic chain maps.

(b) If \( \beta \) is a homotopy equivalence between \( R \)-complexes, then \( \text{Hom}_R(M, \beta) \) is a homotopy equivalence.
PROOF. (a): By 2.3.9 the homomorphisms $\text{Hom}_R(M, \beta)$ and $\text{Hom}_R(M, \beta')$ are chain maps. By assumption $\beta - \beta'$ is null-homotopic, so the chain map $\text{Hom}_R(M, \beta - \beta')$ is null-homotopic by 2.3.9, and by 2.3.7 one has $\text{Hom}_R(M, \beta - \beta') = \text{Hom}_R(M, \beta) - \text{Hom}_R(M, \beta')$. Thus, $\text{Hom}_R(M, \beta)$ and $\text{Hom}_R(M, \beta')$ are homotopic.

(b): Assume that $\beta : N \to N'$ is a homotopy equivalence. By definition there exists a morphism $\alpha : N' \to N$, such that $1^N - \alpha \beta$ and $1^{N'} - \beta \alpha$ are null-homotopic. Note from 2.3.9 that $\text{Hom}_R(M, \alpha)$ and $\text{Hom}_R(M, \beta)$ are morphisms. It follows from 2.3.7 and 2.3.9 that the morphisms

\[
\text{Hom}_R(M, 1^N - \alpha \beta) = 1^{\text{Hom}_R(M, N)} - \text{Hom}_R(M, \alpha) \text{Hom}_R(M, \beta)
\]

and

\[
\text{Hom}_R(M, 1^{N'} - \beta \alpha) = 1^{\text{Hom}_R(M, N')} - \text{Hom}_R(M, \beta) \text{Hom}_R(M, \alpha)
\]

are null-homotopic. Thus, $\text{Hom}_R(M, \beta)$ is a homotopy equivalence. \hfill $\square$

2.3.11 Corollary. Let $N$ be an $R$-complex.

(a) If $\alpha$ and $\alpha'$ are homotopic chain maps between $R$-complexes, then $\text{Hom}_R(\alpha, N)$ and $\text{Hom}_R(\alpha', N)$ are homotopic chain maps.

(b) If $\alpha$ is a homotopy equivalence between $R$-complexes, then $\text{Hom}_R(\alpha, N)$ is a homotopy equivalence.

PROOF. Parallel to the proof of 2.3.10. \hfill $\square$

The functor described in the first part of the next theorem is called the $\text{Hom}$ functor. The theorem’s second part requires a few preparatory remarks. In the $\kappa$-linear category $\mathcal{C}(R)$, each hom-set $\mathcal{C}(R)(M, N)$ is a $\kappa$-module. Homotopy '$\sim$' is by 2.2.22 an equivalence relation on $\mathcal{C}(R)(M, N)$, and it is straightforward to verify that the set $\mathcal{C}(R)(M, N)/\sim$ of equivalence classes is a $\kappa$-module under induced addition, $[\alpha] + [\beta] = [\alpha + \beta]$., and $\kappa$-multiplication $x[\alpha] = [x\alpha]$.

2.3.12 Theorem. The functions

\[
\text{Hom}_R(-, -) : \mathcal{C}(R)^{\text{op}} \times \mathcal{C}(R) \to \mathcal{C}(\kappa),
\]

defined on objects in 2.3.1 and on morphisms in 2.3.6, constitute a $\kappa$-bilinear and left exact functor.

For $R$-complexes $M$ and $N$, the complex $\text{Hom}_R(M, N)$ and the module $\mathcal{C}(R)(M, N)$ are related by the following identities of $\kappa$-modules,

\[
Z_0(\text{Hom}_R(M, N)) = \mathcal{C}(R)(M, N) \quad \text{and} \quad H_0(\text{Hom}_R(M, N)) = \mathcal{C}(R)(M, N)/\sim.
\]

PROOF. It follows from 2.3.1 and 2.3.9 that $\text{Hom}_R(-, -)$ takes objects and morphisms in $\mathcal{C}(R)^{\text{op}} \times \mathcal{C}(R)$ to objects and morphisms in $\mathcal{C}(\kappa)$. The functoriality and the $\kappa$-bilinearity are established in 2.3.7. To prove left exactness, let

\[
0 \to (M', N') \xrightarrow{(\alpha', \beta')} (M, N) \xrightarrow{(\alpha, \beta)} (M'', N'') \to 0.
\]
be an exact sequence in \( \mathcal{C}(R)^{op} \times \mathcal{C}(R) \). It is sufficient to verify that the sequence

\[
\begin{align*}
(*) \quad 0 & \longrightarrow \text{Hom}_R(M', N')_v \\
 & \longrightarrow \text{Hom}_R(M, N)_v \\
 & \longrightarrow \text{Hom}_R(M'', N'')_v
\end{align*}
\]

in \( \mathcal{M}(\mathcal{X}) \) is exact for every \( v \in \mathbb{Z} \). By 2.1.6 and 2.3.6 such a sequence is a product of exact sequences; indeed \( \text{Hom}_R(a', b')_v \) is the product

\[
\prod_{i \in \mathbb{Z}} \text{Hom}_R(a_i', b_i')_v : \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i', N_i')_v \longrightarrow \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i, N_{i+v})_v,
\]

and \( \text{Hom}_R(\alpha, \beta)_v \) has a similar form. It follows that \((*)\) is exact; cf. 1.1.14.

Finally, the equalities of \( \mathcal{X} \)-modules follow from 2.3.3.

**Remark.** The proof above uses the fact, immediate from 1.1.14, that a product of exact sequences in \( \mathcal{M}(R) \) is exact. This property does not hold in every Abelian category; a counterexample is the category of sheaves of Abelian groups on a suitable topological space; see [22].

2.3.13 Addendum. If \( M \) is in \( \mathcal{C}(R-Q^o) \) and \( N \) is in \( \mathcal{C}(R-S^o) \), then it follows from 2.1.8 that \( \text{Hom}_R(M, N)^{op} \) is a graded \( Q-S^{op} \)-bimodule, and it is elementary to verify that \( \partial \text{Hom}_R(M, N) \) is \( Q \)- and \( S^{op} \)-linear. That is, \( \text{Hom}_R(M, N) \) is an object in \( \mathcal{C}(Q-S^{op}) \). For morphisms \( \alpha \) in \( \mathcal{C}(R-Q^o) \) and \( \beta \) in \( \mathcal{C}(R-S^o) \) it is straightforward to verify that \( \text{Hom}_R(\alpha, \beta) \) is a morphism in \( \mathcal{C}(Q-S^{op}) \). Thus, there is an induced \( \mathcal{X} \)-bilinear functor,

\[
\text{Hom}_R(-,-) : \mathcal{C}(R-Q^o)^{op} \times \mathcal{C}(R-S^o) \longrightarrow \mathcal{C}(Q-S^{op}).
\]

2.3.14. Let \( M \) be an \( R \)-complex. It follows from 2.3.1 and 2.3.6 that one has \( \text{Hom}_R(M, N)^{op} = \text{Hom}_R(M, N^o)^{op} \) for every \( R \)-complex \( N \). Thus, for every degree-wise split exact sequence \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) in \( \mathcal{C}(R) \) the induced sequence of \( \mathcal{X} \)-complexes, \( 0 \rightarrow \text{Hom}_R(M, N') \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N'') \rightarrow 0 \), is degree-wise split exact by 2.1.43.

Similarly, for every \( R \)-complex \( N \) the functor \( \text{Hom}_R(-, N) \) preserves degree-wise split exact sequences; see 2.1.44.

**Hom and Shift**

Recall from 2.2.3 that for every complex \( X \) and every integer \( s \) there is an invertible chain map \( \zeta^X_s : X \rightarrow \Sigma^s X \). If one suppresses these maps, then the isomorphism in 2.3.15 below is given by the assignment \( \psi \mapsto (-1)^{s|\psi|} \psi \).

2.3.15 Proposition. Let \( M \) and \( N \) be \( R \)-complexes, and let \( s \) be an integer. The composite \( \psi \in \text{Hom}(M, N) \), \( \text{Hom}(s^{\mathcal{X}}_s, M, N) \) is an isomorphism of \( \mathcal{X} \)-complexes,

\[
\text{Hom}_R(\Sigma^{-s} M, N) \xrightarrow{\sim} \Sigma^s \text{Hom}_R(M, N),
\]

and it is natural in \( M \) and \( N \).
As recalled above, $\varsigma^s_{\mathrm{Hom}(M,N)}$ is an invertible chain map of degree $s$, and $\varsigma^s_M$ is a degree $-s$ invertible chain map with inverse $\varsigma^{-s}_M$; see 2.2.3. It follows from 2.3.7 and 2.3.9 that $\mathrm{Hom}_R(\varsigma^s_{\Sigma^{-s}M}, N)$ is a degree $-s$ invertible chain map with inverse $(-1)^s \mathrm{Hom}_R(\varsigma^{\Sigma^{-s}M}, N)$. Thus, the composite $\varsigma^s_{\mathrm{Hom}(M,N)} \circ \mathrm{Hom}_R(\varsigma^{-s}_M, N)$ is an invertible chain map of degree 0, i.e. an isomorphism in the category $\mathcal{C}(\mathbb{k})$.

To prove that this isomorphism is natural in $M$ and $N$, let $\alpha: M' \to M$ and $\beta: N \to N'$ be morphisms of $R$-complexes. It suffices to show that the following diagram of homomorphisms of $\mathbb{k}$-complexes is commutative,

$$
\begin{array}{ccc}
\mathrm{Hom}_R(\Sigma^{-s}M, N) & \xrightarrow{\mathrm{Hom}(\varsigma^{M}_{\Sigma^{-s}N})} & \mathrm{Hom}_R(M, N) \\
\mathrm{Hom}(\Sigma^{-s}\alpha, \beta) & \searrow \downarrow & \mathrm{Hom}(\alpha, \beta) \\
\mathrm{Hom}_R(\Sigma^{-s}M', N') & \xrightarrow{\mathrm{Hom}(\varsigma^{M'}_{\Sigma^{-s}N'})} & \mathrm{Hom}_R(M', N')
\end{array}
$$

The equality $\varsigma^s_{\Sigma^{-s}M, \alpha} = (\Sigma^{-s} \alpha) \varsigma^s_{M'}$ from (2.2.3.1) conspires with 2.3.7 to give

$$
\mathrm{Hom}_R(\alpha, \beta) \mathrm{Hom}_R(\varsigma^{M'}_{\Sigma^{-s}N'}) = \mathrm{Hom}_R(\varsigma^{M}_{\Sigma^{-s}N}, \alpha, \beta)
= \mathrm{Hom}_R((\Sigma^{-s} \alpha) \varsigma^{M'}_{\Sigma^{-s}N'}, \beta)
= \mathrm{Hom}_R(\varsigma^{M'}_{\Sigma^{-s}N'}, \beta) \mathrm{Hom}_R(\Sigma^{-s} \alpha, \beta).
$$

This shows that the left-hand square is commutative. The right-hand square is commutative by (2.2.3.1).

If one suppresses the degree changing chain maps $\varsigma$, defined in 2.2.3, then the isomorphism in 2.3.16 below is an equality.

**2.3.16 Proposition.** Let $M$ and $N$ be $R$-complexes, and let $s$ be an integer. The composite $\varsigma^s_{\mathrm{Hom}(M,N)} \circ \mathrm{Hom}_R(\varsigma^{-s}_M, N)$ is an isomorphism of $\mathbb{k}$-complexes,

$$
\mathrm{Hom}_R(M, \Sigma^s N) \xrightarrow{\varsigma^s} \Sigma^s \mathrm{Hom}_R(M, N),
$$

and it is natural in $M$ and $N$.

**Proof.** Parallel to the proof of 2.3.15.

**Exercises**

**E 2.3.1** Generalize the identity in 2.3.4 as follows. If $M^1 \xrightarrow{a_1} M^2 \xrightarrow{a_2} \cdots \xrightarrow{a_n} M^n \xrightarrow{a_{n+1}} M^{n+1}$ are homomorphisms of $R$-complexes, then one has
2.4 Tensor Products

SYNOPSIS. Tensor product complex; chain map, homotopy; tensor product functor.

The graded tensor product of two complexes can be endowed with a differential constructed from the differentials of the factors.

2.4.1 Definition. Let $M$ be an $R^o$-complex and let $N$ be an $R$-complex. The tensor product complex $M \otimes_R N$ is the $k$-complex with underlying graded module

$$(M \otimes_R N)^\cdot = M^\cdot \otimes_R N^\cdot$$

and differential given by

$$\partial^{M \otimes_R N}(m \otimes n) = \partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n)$$

for homogeneous elements $m$ and $n$.

2.4.2. It is elementary to verify that $\partial^{M \otimes_R N}$ is square zero. Indeed, one has

$$\partial^{M \otimes_R N} \partial^{M \otimes_R N}(m \otimes n) = \partial^{M \otimes_R N}(\partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n))$$

$$= \partial^M \partial^M(m) \otimes n + (-1)^{|m|} \partial^M(m) \otimes \partial^N(n)$$

$$+ (-1)^{|m|} (\partial^M(m) \otimes \partial^N(n) + (-1)^{|m|} m \otimes \partial^N \partial^N(n))$$

$$= 0.$$
2.4.3 Definition. Let \( \alpha : M \to M' \) be a homomorphism of \( R^\circ \)-complexes and let \( \beta : N \to N' \) a homomorphism of \( R \)-complexes. Denote by \( \alpha \otimes_R \beta \) the degree \(|\alpha| + |\beta|\) homomorphism of \( \kappa \)-complexes

\[
M \otimes_R N \to M' \otimes_R N'
\]
given by \( m \otimes n \mapsto (-1)^{|\beta||m|} \alpha(m) \otimes \beta(n) \).

Furthermore, set \( \alpha \otimes_R N = \alpha \otimes_R 1^N \) and \( M \otimes_R \beta = 1^M \otimes_R \beta \).

2.4.4. It is straightforward to verify that the assignment \( (\alpha, \beta) \mapsto \alpha \otimes_R \beta \), for homomorphisms \( \alpha \) and \( \beta \) of complexes, is \( \kappa \)-bilinear. It is also immediate from the definition that one has

\[
1^M \otimes_R 1^N = 1^{M \otimes_R N}.
\]

For homomorphisms \( M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \) of \( R^\circ \)-complexes and homomorphisms \( N' \xrightarrow{\beta'} N \xrightarrow{\beta} N'' \) of \( R \)-complexes there is an equality

\[
(\alpha \alpha') \otimes_R (\beta \beta') = (-1)^{|\alpha'| |\beta|} (\alpha \otimes_R \beta)(\alpha' \otimes_R \beta')
\]

Indeed, for homogeneous elements \( m' \) in \( M' \) and \( n' \) in \( N' \) one has

\[
((\alpha \alpha') \otimes_R (\beta \beta'))(m' \otimes n') = (-1)^{|\beta'| |m'|}((\alpha \alpha')(m') \otimes (\beta \beta')(n'))
\]

\[
= (-1)^{|\beta'| |m'|}(-1)^{|\alpha'| |m'|}((\alpha \otimes_R \beta)(\alpha'(m') \otimes (\beta'(n'))
\]

\[
= (-1)^{|\alpha'| |\beta|}((-1)^{|\beta'| |m'|}((\alpha \otimes_R \beta)(\alpha'(m') \otimes (\beta'(n')
\]

\[
= (-1)^{|\alpha'| |\beta|}((\alpha \otimes_R \beta)((\alpha' \otimes_R \beta')(m \otimes n))
\]

In particular, there are equalities,

\[
(\alpha \alpha') \otimes_R N = (\alpha \otimes_R N)(\alpha' \otimes_R N)
\]

\[
M \otimes_R (\beta \beta') = (M \otimes_R \beta)(M \otimes_R \beta')
\]

and, furthermore, a commutative diagram of homomorphisms of \( \kappa \)-complexes,

2.4.5 Lemma. Let \( \alpha : M \to M' \) be a homomorphism of \( R^\circ \)-complexes and \( \beta : N \to N' \) be a homomorphism of \( R \)-complexes. With \( H = \text{Hom}_\kappa(M \otimes_R N, M' \otimes_R N') \) one has

\[
\partial^H(\alpha \otimes_R \beta) = \partial^{\text{Hom}_{\kappa}(M')}(\alpha) \otimes_R \beta + (-1)^{|\alpha|} \alpha \otimes_R \partial^\text{Hom}_\kappa(N')(\beta).
\]
PROOF. Let $m$ and $n$ be homogeneous elements in $M$ and $N$. There are equalities.

\[ (\partial^H (\alpha \otimes_R \beta))(m \otimes n) = (\partial^{M \otimes N}(\alpha \otimes_R \beta) - (-1)^{|\alpha|+|\beta|}(\alpha \otimes R \beta)\partial^{M \otimes R N})(m \otimes n) \]

\[ = (-1)^{|\beta|m}(\partial^{M')(\alpha(m) \otimes \beta(n))) \]

\[ - (-1)^{|\alpha|+|\beta|}(\alpha \otimes_R \beta)(\partial^{M}(m) \otimes n + (-1)^{|m|m}m \otimes \partial^N(n)) \]

\[ = (-1)^{|\beta|m}(\partial^{M'}\alpha(m) \otimes \beta(n) + (-1)^{|\alpha|+|m|m}(-1)^{|\alpha|}\alpha \otimes \partial^N) \]

\[ + (-1)^{|m|m}(-1)^{|\beta|m}\alpha(m) \otimes \beta^N(n)) \].

Further, one has

\[ (\partial^{\text{Hom}_R(M,M')}(\alpha) \otimes_R \beta + (-1)^{|\alpha|}\alpha \otimes_R \partial^{\text{Hom}_R(N,N')}) \]

\[ (m \otimes n) = (-1)^{|\beta|m}(\partial^{\text{Hom}_R(M,M')}(\alpha))(m) \otimes \beta(n) \]

\[ + (-1)^{|\alpha|\alpha(m) \otimes (\partial^{\text{Hom}_R(N,N')})(n) \]

\[ = (-1)^{|\beta|m}(\partial^{M'}\alpha - (-1)^{|\beta|}\alpha \partial^M)(m) \otimes \beta(n) \]

\[ + (-1)^{|\alpha|\alpha(m) \otimes (\partial^N \beta - (-1)^{|\beta|} \beta^N)(n)) \].

Compare the two expressions above to see that they are identical. 

\[ \square \]

2.4.6 Proposition. Let $\alpha : M \to M'$ be a chain map of $R^n$-complexes and $\beta : N \to N'$ be a chain map of $R$-complexes. The homomorphism $\alpha \otimes_R \beta$ is a chain map of degree $|\alpha| + |\beta|$, and if $\alpha$ or $\beta$ is null-homotopic, then $\alpha \otimes_R \beta$ is null-homotopic.

PROOF. By 2.3.3 the chain maps $\alpha$ and $\beta$ are cycles in the complexes $\text{Hom}_R(M,M')$ and $\text{Hom}_R(N,N')$. That is, one has $\partial^{\text{Hom}_R(M,M')}(\alpha) = 0$ and $\partial^{\text{Hom}_R(N,N')}(\beta) = 0$. Denote by $H$ the complex $\text{Hom}_R(M \otimes_R N, M' \otimes_R N')$, then 2.4.5 yields $\partial^H(\alpha \otimes_R \beta) = 0$, so $\alpha \otimes_R \beta$ is a chain map by 2.3.3.

If $\alpha$ is null-homotopic, then $\alpha = \partial^{\text{Hom}_R(M,M')}(\theta)$ for some $\theta \in \text{Hom}_R(M,M')$; see 2.3.3. Now 2.4.5 yields $\alpha \otimes_R \beta = \partial^H(\theta \otimes_R \beta)$, whence $\alpha \otimes_R \beta$ is null-homotopic. Similarly, $\beta = \partial^{\text{Hom}_R(N,N')}(\theta)$ implies $\alpha \otimes_R \beta = \partial^H((-1)^{|\alpha|} \alpha \otimes_R \theta)$. 

\[ \square \]

2.4.7 Corollary. Let $M$ be an $R^n$-complex.

(a) If $\beta$ and $\beta'$ are homotopic chain maps between $R$-complexes, then $M \otimes_R \beta$ and $M \otimes_R \beta'$ are homotopic chain maps.

(b) If $\beta$ is a homotopy equivalence between $R$-complexes, then $M \otimes_R \beta$ is a homotopy equivalence.

PROOF. (a): By 2.4.6 the homomorphisms $M \otimes_R \beta$ and $M \otimes_R \beta'$ are chain maps. By assumption $\beta - \beta'$ is null-homotopic, so it follows from 2.4.6 that $M \otimes_R (\beta - \beta')$ is a null-homotopic chain map, and by 2.4.4 one has $M \otimes_R (\beta - \beta') = M \otimes_R \beta - M \otimes_R \beta'$.
(b): Assume that $\beta: N \to N'$ is a homotopy equivalence. By definition there exists a morphism $\alpha: N' \to N$ such that $1^N - \alpha \beta$ and $1^{N'} - \beta \alpha$ are null-homotopic. Note from 2.4.6 that $M \otimes_R \alpha$ and $M \otimes_R \beta$ are morphisms. It follows from 2.4.4 and 2.4.6 that the morphisms

\[ M \otimes_R (1^N - \alpha \beta) = 1^{M \otimes_R N} - (M \otimes_R \alpha)(M \otimes_R \beta) \quad \text{and} \quad M \otimes_R (1^{N'} - \beta \alpha) = 1^{M \otimes_R N'} - (M \otimes_R \beta)(M \otimes_R \alpha) \]

are null-homotopic. Thus, $M \otimes_R \beta$ is a homotopy equivalence.

2.4.8 Corollary. Let $N$ be an $R$-complex.

(a) If $\alpha$ and $\alpha'$ are homotopic chain maps between $R^o$-complexes, then $\alpha \otimes_R N$ and $\alpha' \otimes_R N$ are homotopic chain maps.

(b) If $\alpha$ is a homotopy equivalence between $R^o$-complexes, then $\alpha \otimes_R N$ is a homotopy equivalence.

PROOF. Parallel to the proof of 2.4.7.

The functor described in the next theorem is called the tensor product functor.

2.4.9 Theorem. The functions

\[ \ominus: \mathcal{C}(R^o) \times \mathcal{C}(R) \to \mathcal{C}(k), \]

defined on objects in 2.4.1 and on morphisms in 2.4.3, constitute a $k$-bilinear and right exact functor.

PROOF. It follows from 2.4.1 and 2.4.6 that $\ominus$ takes objects and morphisms in $\mathcal{C}(R^o) \times \mathcal{C}(R)$ to objects and morphisms in $\mathcal{C}(k)$. The functoriality and the $k$-bilinearity are established in 2.4.4. To prove right exactness, let

\[ 0 \to (M', N') \xrightarrow{(\alpha', \beta')} (M, N) \xrightarrow{(\alpha \otimes \beta)} (M'', N'') \to 0 \]

be an exact sequence in $\mathcal{C}(R^o) \times \mathcal{C}(R)$. It is sufficient to verify that the sequence

\[ (\star) \]

\[ (M' \otimes_R N')_v \xrightarrow{(\alpha \otimes \beta)_v} (M \otimes_R N)_v \xrightarrow{(\alpha \otimes \beta)_v} (M'' \otimes_R N'')_v \to 0 \]

in $\mathcal{M}(k)$ is exact for every $v \in \mathbb{Z}$. By 2.1.11 and 2.4.3 such a sequence is a coproduct of exact sequences; indeed $(\alpha \otimes \beta)_v$ is the coproduct

\[ \bigsqcup_{i \in \mathbb{Z}} \alpha'_{i} \otimes_R \beta'_{v-i} : \bigsqcup_{i \in \mathbb{Z}} M'_{i} \otimes_R N'_{v-i} \to \bigsqcup_{i \in \mathbb{Z}} M_{i} \otimes_R N_{v-i}, \]

and $(\alpha \otimes \beta)_v$ has a similar form. It follows that $(\star)$ is exact; cf. 1.1.15.

2.4.10 Addendum. If $M$ is in $\mathcal{C}(Q^o)$ and $N$ is in $\mathcal{C}(R^o)$, then it follows from 2.1.12 that $(M \otimes_R N)^3$ is a graded $Q^o$-$S^o$-bimodule, and it is elementary to verify...
that $\partial^{M \otimes R^N}$ is $Q$- and $S^0$-linear. That is, $M \otimes_R N$ is an object in $\mathcal{C}(Q-S^0)$. For morphisms $\alpha$ in $\mathcal{C}(R-Q^0)$ and $\beta$ in $\mathcal{C}(R-S^0)$ it is straightforward to verify that $\alpha \otimes_R \beta$ is a morphism in $\mathcal{C}(Q-S^0)$. Thus, there is an induced $k$-bilinear functor

$$- \otimes_R : \mathcal{C}(Q-R^0) \times \mathcal{C}(R-S^0) \to \mathcal{C}(Q-S^0).$$

2.4.11. Let $M$ be an $R^0$-complex. It follows from 2.4.1 and 2.4.3 that one has $(M \otimes_R N)^\Sigma = (M \otimes_R N^\Sigma)$ for every $R$-complex $N$. Thus, for every degreewise split exact sequence $0 \to N' \to N \to N'' \to 0$ in $\mathcal{C}(R)$ the induced sequence in $\mathcal{C}(k)$, $0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$, is degreewise split exact by 2.1.43.

Similarly, for every $R$-complex $N$ the functor $- \otimes_R N$ preserves degreewise split exact sequences.

**Tensor Product and Shift**

Recall from 2.2.3 that for every complex $X$ and every integer $s$ there is an invertible chain map $\varsigma^s_X : X \to \Sigma^s X$. If one suppresses these maps, then the isomorphism in 2.4.12 below is given by the assignment $m \otimes n \mapsto (-1)^{\deg m} m \otimes n$.

2.4.12 Proposition. Let $M$ be an $R^0$-complex and let $N$ be an $R$-complex. For every $s \in \mathbb{Z}$ the composite $\varsigma^s_M \otimes N \circ (M \otimes_R \varsigma^s_N)$ is an isomorphism of $\mathcal{C}$-complexes,

$$M \otimes_R \Sigma^s N \cong \Sigma^s (M \otimes_R N),$$

and it is natural in $M$ and $N$.

**Proof.** As recalled above, $\varsigma^s_M \otimes N$ is an invertible chain map of degree $s$, and $\varsigma^s_{-N}$ is a degree $-s$ invertible chain map with inverse $\varsigma^{-s}_{-N}$; see 2.2.3. It follows from 2.4.4 and 2.4.6 that $M \otimes_R \varsigma^s_{-N}$ is a degree $-s$ invertible chain map with inverse $M \otimes_R \varsigma^{-s}_{-N}$. Thus, the composite $\varsigma^s_M \otimes N \circ (M \otimes_R \varsigma^s_{-N})$ is an invertible chain map of degree 0, i.e. an isomorphism in the category $\mathcal{C}(k)$.

To prove that this isomorphism is natural in $M$ and $N$, let $\alpha : M \to M'$ and $\beta : N \to N'$ be morphisms of complexes. It suffices to show that the following diagram of homomorphisms of $k$-complexes is commutative,

$$\begin{array}{cccc}
M \otimes_R \Sigma^s N & \xrightarrow{M \otimes \varsigma^s_{-N}} & M \otimes_R N & \xrightarrow{\varsigma^s_M \otimes N} & \Sigma^s (M \otimes_R N) \\
\alpha \otimes \varsigma^{-s}_{-N} & & \alpha \otimes \beta & & \Sigma^s (\alpha \otimes \beta) \\
M' \otimes_R \Sigma^s N' & \xrightarrow{M' \otimes \varsigma^s_{-N'}} & M' \otimes_R N' & \xrightarrow{\varsigma^s_{-N'}} & \Sigma^s (M' \otimes_R N').
\end{array}$$

The equality $\beta \varsigma^s_{-N} = \varsigma^s_{-N'} \Sigma^s \beta$, which follows from (2.2.3.1), conspires with 2.4.4 to give
\[(\alpha \otimes_R \beta)(M \otimes_R \varsigma^{\Sigma'}_s N) = \alpha \otimes_R (\beta \varsigma^{\Sigma'}_s) = \alpha \otimes_R (\varsigma^{\Sigma'}_s \Sigma \beta) = (M' \otimes_R \varsigma^{\Sigma'}_s N)(\alpha \otimes_R \Sigma \beta) \,.
\]

This shows that the left-hand square is commutative. The right-hand square is commutative by (2.2.3.1).

If one suppresses the degree changing chain maps \(\varsigma\) defined in 2.2.3, then the isomorphism in 2.4.13 below is an equality.

**2.4.13 Proposition.** Let \(M\) be an \(R^s\)-complex and let \(N\) be an \(R\)-complex. For every \(s \in \mathbb{Z}\) the composite \(\varsigma_s^M \otimes \varsigma_s^N \circ (\varsigma_{s-1}^M \otimes_R N)\) is an isomorphism of \(k\)-complexes,

\[\Sigma' M \otimes_R N \xrightarrow{\sim} \Sigma' (M \otimes_R N),\]

and it is natural in \(M\) and \(N\).

**Proof.** Parallel to the proof of 2.4.12.

---

**Exercises**

**E 2.4.1** Let \(M\) be an \(R^s\)-complex and \(N\) be an \(R\)-complex. Consider the degree \(-1\) homomorphisms \(\partial^M \otimes_R N\), \(\partial^M \otimes_R N\), and \(M \otimes_R \partial^N\) from \(M \otimes_R N\) to itself. Verify the identity

\[\partial^M \otimes_R N = \partial^M \otimes_R N + M \otimes_R \partial^N.\]

**E 2.4.2** Give a proof of 2.4.8.

**E 2.4.3** (Cf. 2.4.10) Let \(\alpha\) be a homomorphism of complexes of \(Q\)-\(R^s\)-bimodules and let \(\beta\) be a homomorphism of complexes of \(R\)-\(S^s\)-bimodules. Show that \(\alpha \otimes_R \beta\) is a homomorphism of complexes of \(Q\)-\(S^s\)-bimodules.

---

**2.5 Boundedness and Finiteness**

**Synopsis.** Degreewise finite generation/presentation; boundedness (above/below); supremum; infimum; amplitude; boundedness of Hom and tensor product complexes; hard/soft truncation.

**2.5.1 Definition.** An \(R\)-complex \(M\) is called **degreewise finitely generated** if the \(R\)-module \(M_v\) is finitely generated for every \(v \in \mathbb{Z}\). Similarly, \(M\) is called **degreewise finitely presented** if the \(R\)-module \(M_v\) is finitely presented for every \(v \in \mathbb{Z}\).

**Remark.** A complex may be degreewise finitely generated without the underlying graded module being finitely generated; see E 2.5.1.
2.5 Boundedness and Finiteness

2.5.2 Definition. Let $M$ be a complex, and let $w \leq u$ be integers. The complex $M$ is said to be \textit{concentrated in degrees} $w, \ldots, u$ if one has $M_v = 0$ for all $v \notin \{w, \ldots, u\}$; it is then visualized like this:

$$0 \to M_u \to M_{u-1} \to \cdots \to M_{w+1} \to M_w \to 0.$$ 

The complex $M$ is called \textit{bounded above} if $M_v = 0$ holds for $v \gg 0$, \textit{bounded below} if $M_v = 0$ holds for $v \ll 0$, and \textit{bounded} if it is bounded above and below.

\textbf{Remark.} Other words for bounded above/below are \textit{bounded on the left/right}.

2.5.3. Suppressing the full embeddings of categories $\mathcal{M}(R) \to \mathcal{M}_{gr}(R) \to \mathcal{C}(R)$, see 2.1.18 and 2.1.31, an $R$-module $M$ is considered as an $R$-complex $0 \to M \to 0$ concentrated in degree 0. Conversely, an $R$-complex $M$ that is concentrated in degree 0 is identified with the module $M_0$; this amounts to suppressing the forgetful functors $\mathcal{C}(R) \to \mathcal{M}_{gr}(R) \to \mathcal{M}(R)$.

More generally, a diagram in $\mathcal{M}(R)$,

$$M^0 \xrightarrow{\alpha^0} \cdots \xrightarrow{\alpha^{p-1}} M^p,$$

such that $\alpha^n \alpha^{n-1} = 0$ holds for all $n$ in $\{1, \ldots, p-1\}$, can be considered, for every $u \in \mathbb{Z}$, as a complex concentrated in degrees $u, \ldots, p+u$.

The supremum and infimum of a complex, which will be defined next, capture its homological position; the amplitude captures its homological size.

2.5.4 Definition. Let $M$ be a complex. The \textit{supremum} and \textit{infimum} of $M$ are defined as follows,

$$\sup M = \sup \{ v \in \mathbb{Z} \mid H_v(M) \neq 0 \} \quad \text{and} \quad \inf M = \inf \{ v \in \mathbb{Z} \mid H_v(M) \neq 0 \},$$

with the convention that one sets $\sup M = -\infty$ and $\inf M = \infty$ if $M$ is acyclic. The \textit{amplitude} of $M$ is the difference

$$\amp M = \sup M - \inf M,$$

with the convention that one sets $\amp M = -\infty$ if $M$ is acyclic.

2.5.5. For every complex $M$ one has $H(M^2) = M^2$ and, therefore,

$$\sup M^2 = \sup \{ v \in \mathbb{Z} \mid H_v(M) \neq 0 \} \quad \text{and} \quad \inf M^2 = \inf \{ v \in \mathbb{Z} \mid H_v(M) \neq 0 \}.$$ 

Thus, a complex $M \neq 0$ is bounded above if and only if $\sup M^2$ is not $\infty$; and $M$ is bounded below if and only if $\inf M^2$ is not $-\infty$.

2.5.6. Let $M$ be a graded $R$-module. It follows from 1.3.11 that there is a surjective morphism $L \to M$ of graded $R$-modules, where $L$ is graded-free. If $M$ is degreewise finitely generated, then $L$ can be chosen degreewise finitely generated.
It is elementary to verify that if $M$ is (bounded and) degreewise finitely presented, then $M$ has a degreewise finite presentation: an exact sequence $L' \to L \to M \to 0$, where $L$ and $L'$ are (bounded) graded-free and degreewise finitely generated.

**HOM COMPLEXES**

**2.5.7 Lemma.** Let $M$ and $N$ be $R$-complexes with $M_v = 0$ for all $v < 0$ and $N_v = 0$ for all $v > 0$. There are isomorphisms of $k$-modules,

$$H_0(\text{Hom}_R(M,N)) \cong Z_0(\text{Hom}_R(M,N)) \cong \text{Hom}_R(H_0(M),H_0(N)).$$

**Proof.** It follows by the assumptions that the only degree 1 homomorphism from $M$ to $N$ is the zero map. In particular, the only null-homotopic morphism $M \to N$ is the zero map. In view of this fact and 2.3.12 one sees that the canonical map

$$Z_0(\text{Hom}_R(M,N)) = C(R)(M,N) \to C(R)(M,N)/\sim = H_0(\text{Hom}_R(M,N))$$

is an isomorphism. This proves the first isomorphism in 2.5.7.

To prove the second isomorphism, note that the assumptions on $M$ and $N$ yield equalities $H_0(M) = C_0(M)$ and $H_0(N) = Z_0(N)$, so that one has

$$\text{Hom}_R(H_0(M),H_0(N)) = \text{Hom}_R(C_0(M),Z_0(N)).$$

Let $\pi_0: M_0 \to C_0(M)$ be the canonical map, and let $\iota_0: Z_0(N) \to N_0$ be the embedding. There is an isomorphism of $k$-modules,

$$\varphi: \text{Hom}_R(C_0(M),Z_0(N)) \cong C(R)(M,N),$$

defined by assigning to a homomorphism $\alpha: C_0(M) \to Z_0(N)$ of modules the morphism of complexes given by:

$$
\begin{array}{c}
\cdots \to M_1 \xrightarrow{\partial^M_1} M_0 \xrightarrow{\partial^M_0} 0 \xrightarrow{\partial^M_0} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \to 0 \xrightarrow{\partial^N_0} N_0 \xrightarrow{\partial^N_0} N^{-1} \xrightarrow{\partial^N_{-1}} \cdots
\end{array}
$$

The inverse of $\varphi$ is defined as follows. Let $\beta = (\beta_v)_{v \in \mathbb{Z}}: M \to N$ be a morphism. The homomorphism $\beta_0: M_0 \to N_0$ satisfies $\beta_0 \partial^M_0 = 0$ and $\partial^N_0 \beta_0 = 0$. Hence there is a unique homomorphism $\tilde{\beta}: C_0(M) \to Z_0(N)$ that makes the diagram
is finitely generated for every $v < u$. Now, set $\varphi^{-1}(\beta) = \tilde{\beta}$.

**2.5.8.** Let $M$ and $N$ be $R$-complexes. Suppose there exist integers $w$ and $u$ such that one has $M_v = 0$ for all $v < u$ and $N_v = 0$ for all $v > w$. For each $v \in \mathbb{Z}$ the module $\text{Hom}_R(M,N)_v$ is then a direct sum

$$
\text{Hom}_R(M,N)_v = \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i,N_{i+v}) = \bigoplus_{i=u}^{w-v} \text{Hom}_R(M_i,N_{i+v}).
$$

If one has $w - v < u$, then $\text{Hom}_R(M,N)_v = 0$ holds.

**2.5.9 Proposition.** Let $M$ and $N$ be $R$-complexes. If $M$ is bounded below and $N$ is bounded above, then the complex $\text{Hom}_R(M,N)$ is a bounded above. More precisely, if one has $M_v = 0$ for all $v < u$ and $N_v = 0$ for all $v > w$, then the next assertions hold.

(a) $\text{Hom}_R(M,N)_v = 0$ for $v > w - u$.

(b) $\text{Hom}_R(M,N)_{w-u} = \text{Hom}_R(M_u,N_w)$.

(c) $H_{w-u}(\text{Hom}_R(M,N)) \cong \text{Hom}_R(H_0(M),H_w(N))$.

**PROOF.** Parts (a) and (b) are immediate from 2.5.8.

(c): By 2.2.13, 2.3.15, and 2.3.16 there is an isomorphism

$$
H_{w-u}(\text{Hom}_R(M,N)) = H_0(\Sigma^{w-u}\text{Hom}_R(M,N)) \cong H_0(\text{Hom}_R(\Sigma^{-u}M,\Sigma^{-w}N)).
$$

The complexes $\Sigma^{-u}M$ and $\Sigma^{-w}N$ are concentrated in non-negative and non-positive degrees, respectively, so by 2.5.7 there is an isomorphism

$$
H_0(\text{Hom}_R(\Sigma^{-u}M,\Sigma^{-w}N)) \cong \text{Hom}_R(H_0(\Sigma^{-u}M),H_0(\Sigma^{-w}N)) = \text{Hom}_R(H_u(M),H_w(N)).
$$

**2.5.10 Proposition.** Assume that $k$ is Noetherian and that $R$ is finitely generated as a $k$-module. Let $M$ be a bounded below $R$-complex and let $N$ be a bounded above $R$-complex. If $M$ and $N$ are degreewise finitely generated, then the $k$-complex $\text{Hom}_R(M,N)$ is bounded above and degreewise finitely generated.

**PROOF.** For every $v \in \mathbb{Z}$ and $i \in \mathbb{Z}$ the $k$-module $\text{Hom}_R(M_i,N_{i+v})$ is finitely generated; see 1.3.14. By assumption there exist integers $w$ and $u$ such that $M_v = 0$ for all $v < u$ and $N_v = 0$ for all $v > w$. It follows from 2.5.8 that the module $\text{Hom}_R(M,N)_v$ is finitely generated for every $v$, and by 2.5.9 it vanishes for $v > w - u$. $\square$
TENSOR PRODUCT COMPLEXES

2.5.11 Lemma. Let $M$ be an $R^\circ$-complex and let $N$ be an $R$-complex with $M_v = 0$ and $N_v = 0$ for all $v < 0$. There are isomorphisms of $\mathcal{K}$-modules

$$H_0(M \otimes_R N) \cong C_0(M \otimes_R N) \cong H_0(M) \otimes_R H_0(N).$$

PROOF. By definition of the tensor product complex, see 2.4.1, and by the assumptions on $M$ and $N$, it follows that the sequence

$$(M \otimes_R N)_1 \xrightarrow{\partial_1^M \otimes_R N} (M \otimes_R N)_0 \xrightarrow{\partial_0^M \otimes_R N} (M \otimes_R N)_{-1}$$

equals

$$(M_1 \otimes_R N_0) \oplus (\partial_1^M \otimes_R N_0, M_0 \otimes_R N_0) \to M_0 \otimes_R N_0 \to 0.$$  \hfill (*)

The first isomorphism in 2.5.11 follows as $\partial_0^M \otimes_R N$ is 0.

To prove the second isomorphism, notice that one has $H_0(M) = C_0(M)$ and $H_0(N) = C_0(N)$, as $\partial_0^M = 0$ and $\partial_0^N = 0$. Thus, we must establish an isomorphism of $\mathcal{K}$-modules, $C_0(M \otimes_R N) \cong C_0(M) \otimes_R C_0(N)$. Consider the homomorphism,

$$\varphi : M_0 \otimes_R N_0 \to C_0(M) \otimes_R C_0(N),$$

defined by $m \otimes n \mapsto [m]_{B_0(M)} \otimes [n]_{B_0(N)}$. From (*) one gets $B_0(M \otimes_R N) \subset Ker \varphi$, and hence $\varphi$ induces a homomorphism,

$$\bar{\varphi} : C_0(M \otimes_R N) \to C_0(M) \otimes_R C_0(N),$$

given by $[m \otimes n]_{B_0(M \otimes_R N)} \mapsto [m]_{B_0(M)} \otimes [n]_{B_0(N)}$. To see that $\bar{\varphi}$ is an isomorphism, note that there is a well-defined map,

$$\psi : C_0(M) \times C_0(N) \to C_0(M_0 \otimes_R N_0),$$

given by $([m]_{B_0(M)}, [n]_{B_0(N)}) \mapsto [m] \otimes [n]_{B_0(M \otimes_R N)}$. Indeed, if $m - m'$ is in $B_0(M)$ and $n - n'$ is in $B_0(N)$, then one has $m - m' = \partial_1^M(m_0)$ and $n - n' = \partial_1^N(n_0)$ for suitable $m_0 \in M_0$ and $n_0 \in N_0$, and consequently,

$$m \otimes n - m' \otimes n' = (\partial_1^M \otimes_R N_0)(m_0 \otimes n) + (M_0 \otimes_R \partial_1^N)(m' \otimes n_0) \in B_0(M \otimes_R N).$$

It is straightforward to verify that $\psi$ is $\mathcal{K}$-bilinear and middle $R$-linear, so by the universal property of tensor products it induces a homomorphism of $\mathcal{K}$-modules,

$$\bar{\psi} : C_0(M) \otimes_R C_0(N) \to C_0(M_0 \otimes_R N_0).$$
given by \([m]_{B_0(M)} \otimes [n]_{B_0(N)} \mapsto [m \otimes n]_{B_0(M \otimes_R N)}\). Clearly, \(\tilde{\psi}\) is an inverse of \(\bar{\psi}\). □

2.5.12. Let \(M\) be an \(R^0\)-complex and let \(N\) be an \(R\)-complex. Suppose there exist integers \(u\) and \(w\) such that \(M_v = 0\) for \(v < u\) and \(N_v = 0\) for \(v < w\). For each \(v \in \mathbb{Z}\) the module \((M \otimes_R N)_v\) is then a direct sum

\[
(M \otimes_R N)_v = \prod_{i \in \mathbb{Z}} M_i \otimes_R N_{v-i} = \bigoplus_{i=u}^{v-w} M_i \otimes_R N_{v-i}.
\]

If one has \(v - w < u\), then \((M \otimes_R N)_v = 0\) holds.

2.5.13 Proposition. Let \(M\) be an \(R^0\)-complex and let \(N\) be an \(R\)-complex. If \(M\) and \(N\) are bounded below, then the complex \(M \otimes_R N\) is a bounded below. More precisely, if one has \(M_v = 0\) for \(v < u\) and \(N_v = 0\) for \(v < w\), then the next assertions hold.

(a) \((M \otimes_R N)_v = 0\) for \(v < u + w\).
(b) \((M \otimes_R N)_{u+w} = M_u \otimes_R N_w\).
(c) \(H_{u+w}(M \otimes_R N) \cong H_u(M) \otimes_R H_w(N)\).

Proof. Parts (a) and (b) are immediate from 2.5.12.

(c): By 2.2.13, 2.4.12, and 2.4.13 there is an isomorphism

\[
H_{u+w}(M \otimes_R N) = H_0(\Sigma^{-u-w} (M \otimes_R N)) \cong H_0((\Sigma^{-u} M) \otimes_R (\Sigma^{-w} N)).
\]

The complexes \(\Sigma^{-u} M\) and \(\Sigma^{-w} N\) are concentrated in non-negative degrees, so by 2.5.11 there is an isomorphism

\[
H_0((\Sigma^{-u} M) \otimes_R (\Sigma^{-w} N)) \cong H_0(\Sigma^{-u} M) \otimes_R H_0(\Sigma^{-w} N) = H_u(M) \otimes_R H_w(N).
\]

2.5.14 Proposition. Assume that \(R\) is finitely generated as a \(k\)-module. Let \(M\) be a bounded below \(R^0\)-complex and let \(N\) be a bounded below \(R\)-complex. If \(M\) and \(N\) are degreewise finitely generated, then the \(k\)-complex \(M \otimes_R N\) is bounded below and degreewise finitely generated.

Proof. For every \(v \in \mathbb{Z}\) and \(i \in \mathbb{Z}\) the \(k\)-module \(M_i \otimes_R N_{v-i}\) is finitely generated; see 1.3.15. By assumption there exist integers \(u\) and \(w\) such that \(M_v = 0\) for \(v < u\) and \(N_v = 0\) for \(v < w\). It follows from 2.5.12 that the module \((M \otimes_R N)_v\) is finitely generated for every \(v\), and by 2.5.13 it vanishes for \(v < u + w\). □

Truncations

To handle unbounded complexes it is at times convenient to cut them into bounded pieces. The instruments for such procedures are known as truncations.
2.5.15 Definition. Let $M$ be an $R$-complex and $n$ be an integer. The hard truncation above of $M$ at $n$ is the complex $M_{\leq n}$ defined by $(M_{\leq n})_v = 0$ for $v > n$ and $\partial_v^{M_{\leq n}} = \partial_v^M$ for $v \leq n$. It can be visualized as follows,

$$M_{\leq n} = 0 \rightarrow M_n \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} M_{n-2} \rightarrow \cdots .$$

Similarly, the hard truncation below of $M$ at $n$ is the complex $M_{\geq n}$ defined by $(M_{\geq n})_v = 0$ for $v < n$ and $\partial_v^{M_{\geq n}} = \partial_v^M$ for $v > n$. It can be visualized as follows,

$$M_{\geq n} = \cdots \rightarrow M_{n+2} \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} M_n \rightarrow 0 .$$

2.5.16. Let $M$ be an $R$-complex. For every $n \in \mathbb{Z}$, the truncation $M_{\leq n}$ is a subcomplex of $M$, and the quotient complex $M/M_{\leq n}$ is the truncation $M_{\geq n+1}$. In particular, there is a degreewise split exact sequence of $R$-complexes

$$0 \rightarrow M_{\leq n} \rightarrow M \rightarrow M_{\geq n+1} \rightarrow 0 .$$

2.5.17 Definition. Let $M$ be an $R$-complex and $n$ be an integer. The soft truncation above of $M$ at $n$ is the complex $M_{< n}$ defined by

$$(M_{< n})_v = \begin{cases} 0 & \text{for } v > n \\ C_n(M) & \text{for } v = n \end{cases} \quad \text{and} \quad \partial_v^{M_{< n}} = \begin{cases} \partial_v^M & \text{for } v = n \\ \partial_v^M & \text{for } v < n \end{cases}$$

where $\partial_v^M : C_n(M) \rightarrow M_{n-1}$ is the homomorphism induced by $\partial_v^M$. The complex can be visualized as follows,

$$M_{< n} = 0 \rightarrow C_n(M) \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} M_{n-2} \rightarrow \cdots .$$

Similarly, the soft truncation below of $M$ at $n$ is the complex $M_{> n}$ defined by:

$$(M_{> n})_v = \begin{cases} Z_n(M) & \text{for } v = n \\ 0 & \text{for } v < n \end{cases} \quad \text{and} \quad \partial_v^{M_{> n}} = \partial_v^M \quad \text{for } v > n .$$

The complex can be visualized as follows,

$$M_{> n} = \cdots \rightarrow M_{n+2} \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} Z_n(M) \rightarrow 0 .$$

2.5.18. Let $M$ be an $R$-complex. The canonical map

$$(2.5.18.1) \quad \tau_{< n}^M : M \rightarrow M_{< n}$$

induces an isomorphism in homology in degree $n$ and below. That is, $H_v(\tau_{< n}^M)$ is an isomorphism for $v \leq n$. Similarly, the embedding
(2.5.18.2) \[ \tau_{\geq n}^M : M_{\geq n} \rightarrow M \]

induces an isomorphism in homology in degree \( n \) and above. That is, \( H_v(\tau_{\geq n}^M) \) is an isomorphism for \( v \geq n \).

**Exercises**

E 2.5.1 Let \( M \) be an \( R \)-complex. Show that if \( M^\bullet \) is finitely generated in \( \mathcal{M}_{gr}(R) \), then \( M \) is degreewise finitely generated. Show that the converse is not true.

E 2.5.2 Let \( M \) be a bounded above \( R \)-complex and \( N \) be a bounded below \( R \)-complex. Establish a result about \( \text{Hom}_R(M, N) \) akin to 2.5.9.

E 2.5.3 Let \( M \) be a bounded above \( R \)-complex and \( N \) be a bounded above \( R \)-complex. Establish a result about \( \text{Hom}_R(M, N) \) akin to 2.5.13.

E 2.5.4 Let \( M \) be an \( R \)-complex and \( n \) be an integer. Show that the canonical maps \( \tau_{\leq n}^M : M_{\leq n} \rightarrow M \) and \( \tau_{\geq n}^M : M_{\leq n} \rightarrow M \) are morphisms of \( R \)-complexes.

E 2.5.5 Show that hard truncations yield exact endofunctors \((-)_{\leq n}\) and \((-)_{\geq n}\) on \( \mathcal{C}(R) \).

E 2.5.6 For \( n \in \mathbb{Z} \) define full subcategories of \( \mathcal{C}(R) \) by specifying their objects as follows,

\[
\mathcal{C}_{\leq n}(R) = \{ M \in \mathcal{C}(R) \mid M_v = 0 \text{ for all } v > n \} \quad \text{and} \quad \mathcal{C}_{\geq n}(R) = \{ M \in \mathcal{C}(R) \mid M_v = 0 \text{ for all } v < n \}.
\]

Show that the functors \((-)_{\leq n}\) and \((-)_{\geq n}\) from E 2.5.6 are right and left adjoints to the inclusion functors \( \mathcal{C}_{\leq n}(R) \rightarrow \mathcal{C}(R) \) and \( \mathcal{C}_{\geq n}(R) \rightarrow \mathcal{C}(R) \), respectively.

E 2.5.7 Show that soft truncations yield right exact endofunctors \((-)_{\leq n}\) on \( \mathcal{C}(R) \).

E 2.5.8 Show that soft truncations below yield left exact endofunctors \((-)_{\geq n}\) on \( \mathcal{C}(R) \).
Chapter 3
Categorical Constructions

In the first section of this chapter it is established that the category \( \mathcal{C}(R) \) has indexed products and coproducts. As \( \mathcal{C}(R) \) is Abelian, it will then follow from general principles that it has small limits and colimits. Our detailed treatment in Sect. 3.2 and 3.3 is, however, restricted to (co)limits over preordered sets.

3.1 Products and Coproducts

SYNOPSIS. Coproduct; product; universal properties; direct sum; (co)product preserving functors.

COPRODUCTS

3.1.1 Construction. Let \( \{M^u\}_{u \in U} \) be a family of \( R \)-complexes. One defines a complex \( \coprod_{u \in U} M^u \) by setting

\[
(\coprod_{u \in U} M^u)_v = \coprod_{u \in U} M^u_v \quad \text{and} \quad \partial_v \coprod_{u \in U} M^u = \coprod_{u \in U} \partial_v M^u,
\]

where the right-hand side of either equality is given by the coproduct in \( \mathcal{M}(R) \). For every \( u \in U \) the embeddings \( \iota^u_v : M^u_v \to \coprod_{u \in U} M^u_v \) in \( \mathcal{M}(R) \) yield an embedding

\[
(3.1.1.1) \quad \iota^u : M^u \to \coprod_{u \in U} M^u
\]

of \( R \)-complexes. It is straightforward to verify that every element in \( \coprod_{u \in U} M^u \) has the form \( \sum_{u \in U} \iota^u(m^u) \) with \( m^u = 0 \) in \( M^u \) for all but finitely many \( u \in U \). We often use the notation \( (m^u)_{u \in U} \) for the element \( \sum_{u \in U} \iota^u(m^u) \).
The next definition is justified by 3.1.3, which shows that the complex \( \coprod_{u \in U} M^u \) and the morphisms \( i^u \) have the universal property that defines a coproduct. In any category this property determines the coproduct uniquely up to isomorphism.

**3.1.2 Definition.** For a family of \( R \)-complexes \( \{ M^u \}_{u \in U} \), the complex \( \coprod_{u \in U} M^u \) together with the family of embeddings \( \{ i^u \}_{u \in U} \), constructed in 3.1.1, is called the **coproduct** of \( \{ M^u \}_{u \in U} \) in \( \mathcal{C}(R) \).

**Remark.** Other names for the coproduct defined above are **categorical sum** and **direct sum**; we reserve the latter for the (iterated) biproduct; see 3.1.29.

**3.1.3 Lemma.** For every family \( \{ \alpha^u : M^u \to N \}_{u \in U} \) of morphisms in \( \mathcal{C}(R) \), there is a unique morphism \( \alpha \) that makes the next diagram commutative for every \( u \in U \),

\[
\begin{array}{ccc}
M^u & \longrightarrow & \coprod_{u \in U} M^u \\
\downarrow \alpha^u & & \downarrow \alpha \\
N^u & \longrightarrow & \coprod_{u \in U} N^u \\
\end{array}
\]

The morphism \( \alpha \) is given by the assignment \( \sum_{u \in U} i^u(m^u) \mapsto \sum_{u \in U} \alpha^u(m^u) \).

**Proof.** The assignment defines a morphism of graded \( R \)-modules with \( \alpha^u = \alpha i^u \) for all \( u \in U \); it is straightforward to verify that it is a morphism of \( R \)-complexes. For any morphism \( \alpha' : \coprod_{u \in U} M^u \to N \) in \( \mathcal{C}(R) \) with \( \alpha^u = \alpha'i^u \) for all \( u \in U \), one has \( \alpha'((\sum_{u \in U} i^u(m^u))) = \sum_{u \in U} \alpha'i^u(m^u) = \sum_{u \in U} \alpha^u(m^u) \), hence \( \alpha' = \alpha \).

**3.1.4.** It follows readily from 3.1.1 and 3.1.3 that the full subcategory \( \mathcal{M}_{gr}(R) \) of \( \mathcal{C}(R) \) is closed under coproducts.

**3.1.5 Definition.** Let \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) be a family of morphisms in \( \mathcal{C}(R) \). By the universal property of coproducts, the map given by \( (m^u)_{u \in U} \mapsto (\alpha^u(m^u))_{u \in U} \) is the unique morphism that makes the next diagram commutative for every \( u \in U \),

\[
\begin{array}{ccc}
M^u & \longrightarrow & \coprod_{u \in U} M^u \\
\downarrow \alpha^u & & \downarrow \\
N^u & \longrightarrow & \coprod_{u \in U} N^u \\
\end{array}
\]

This morphism is called the **coproduct** of the family \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) in \( \mathcal{C}(R) \), and it is denoted \( \coprod_{u \in U} \alpha^u \).

One readily sees that the coproduct is exact.

**3.1.6.** Let \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) and \( \{ \beta^u : N^u \to X^u \}_{u \in U} \) be families of morphisms in \( \mathcal{C}(R) \). The sequence
3.1 Products and Coproducts

\[
\prod_{u \in U} M^u \overset{\prod_{u \in U} \alpha^u}{\longrightarrow} \prod_{u \in U} N^u \overset{\prod_{u \in U} \beta^u}{\longrightarrow} \prod_{u \in U} X^u
\]

is exact if and only if the sequence \( M^u \overset{\alpha^u}{\longrightarrow} N^u \overset{\beta^u}{\longrightarrow} X^u \) is exact for every \( u \in U \).

Coproduct Preserving Functors

3.1.7. Let \( \{M^u\}_{u \in U} \) be a family of \( R \)-complexes and let \( s \) be an integer. There is an equality of complexes \( \coprod_{u \in U} \Sigma^s M^u = \Sigma^s \coprod_{u \in U} M^u \). Moreover, if \( \{\alpha^u : M^u \to N^u\}_{u \in U} \) is a family of morphisms in \( \mathcal{C}(R) \), then one has \( \coprod_{u \in U} \Sigma^s \alpha^u = \Sigma^s \coprod_{u \in U} \alpha^u \).

3.1.8. Let \( \{M^u\}_{u \in U} \) be a family of \( R \)-complexes. The differential \( \partial \coprod_{u \in U} M^u \), considered as a morphism \( \coprod_{u \in U} M^u \to \Sigma \coprod_{u \in U} M^u \), is the coproduct of the family of morphisms \( \{\partial M^u : M^u \to \Sigma M^u\}_{u \in U} \). Hence, there are equalities

\[
\text{Z}(\coprod_{u \in U} M^u) = \coprod_{u \in U} \text{Z}(M^u) \quad \text{and} \quad \text{B}(\coprod_{u \in U} M^u) = \coprod_{u \in U} \text{B}(M^u)
\]

of subcomplexes of \( \coprod_{u \in U} M^u \); cf. (2.2.10.2) and 3.1.6.

3.1.9. Let \( F : \mathcal{C}(R) \to \mathcal{C}(S) \) be a functor and \( \{M^u\}_{u \in U} \) be a family of \( R \)-complexes. The embedding (3.1.1.1) induces a morphism \( F(\iota^u) : F(M^u) \to F(\coprod_{u \in U} M^u) \) for every \( u \in U \). By the universal property of coproducts, the map given by the assignment \( (x^u)_{u \in U} \mapsto \sum_{u \in U} F(\iota^u)(x^u) \) is the unique morphism that makes the following diagram in \( \mathcal{C}(S) \) commutative for every \( u \in U \).

\[
\begin{array}{ccc}
F(M^u) & \longrightarrow & \prod_{u \in U} F(M^u) \\
\rotatebox{90}{$\longleftarrow$} & \ & \downarrow F(\iota^u) \\
F(\coprod_{u \in U} M^u). & \ & \\
\end{array}
\]

If the canonical morphism in 3.1.9 is an isomorphism, then one says that \( F \) preserves coproducts. While this is a condition on objects, it carries over to morphisms.

3.1.10. Let \( F : \mathcal{C}(R) \to \mathcal{C}(S) \) be a functor and let \( \{\alpha^u : M^u \to N^u\}_{u \in U} \) be a family of morphisms in \( \mathcal{C}(R) \). It is elementary to verify that there is a commutative diagram,

\[
\begin{array}{ccc}
\prod_{u \in U} F(M^u) & \longrightarrow & \prod_{u \in U} F(N^u) \\
\rotatebox{90}{$\longleftarrow$} & \ & \downarrow F(\iota^u) \\
F(\coprod_{u \in U} M^u) & \ & \coprod_{u \in U} F(N^u). \\
\end{array}
\]
where the vertical maps are the canonical morphisms from 3.1.9.

Homology, as a functor \( \mathcal{H}: \mathcal{C}(R) \to \mathcal{C}(\mathbb{k}) \), preserves coproducts.

**3.1.11 Proposition.** Let \( \{M^u\}_{u \in U} \) be a family of \( R \)-complexes. The canonical map

\[
\prod_{u \in U} \mathcal{H}(M^u) \longrightarrow \mathcal{H}(\prod_{u \in U} M^u),
\]

given by \((h^u)_{u \in U} \mapsto \sum_{u \in U} \mathcal{H}(i^u)(h^u)\), is an isomorphism of \( R \)-complexes.

**Proof.** The map is a morphism of \( R \)-complexes by 3.1.9. A homology class in \( \mathcal{H}(\sqcup_{u \in U} M^u) \) has by (3.1.8.1) the form

\[
[(z^u)_{u \in U}] = [\sum_{u \in U} i^u(z^u)] = \sum_{u \in U} \mathcal{H}(i^u)([z^u]),
\]

for cycles \( z^u \in Z(M^u) \). Thus, the assignment \((z^u)_{u \in U} \mapsto ([z^u])_{u \in U}\) defines an inverse to the canonical morphism (3.1.11.1). \( \square \)

The tensor product, as a functor from \( \mathcal{C}(R) \) to \( \mathcal{C}(\mathbb{k}) \), preserves coproducts.

**3.1.12 Proposition.** Let \( \{M^u\}_{u \in U} \) be a family of \( R^o \)-complexes and let \( N \) be an \( R \)-complex. The canonical map

\[
\prod_{u \in U} (M^u \otimes_R N) \longrightarrow (\prod_{u \in U} M^u) \otimes_R N,
\]

given by \((t^u)_{u \in U} \mapsto \sum_{u \in U} (i^u \otimes_R N)(t^u)\), is an isomorphism of \( \mathbb{k} \)-complexes.

**Proof.** The map is a morphism of \( \mathbb{k} \)-complexes by 3.1.9. To show that it is an isomorphism, it is sufficient to construct an inverse at the level of graded modules. One readily verifies that the map \( \sqcup_{u \in \mathbb{Z}} (\sqcup_{u \in U} M^u)_i \times N_{-i} \to \sqcup_{u \in U} (M^u \otimes_R N)_i \) defined by \((\{m^u\}_{u \in U}, n) \mapsto (m^u \otimes n)_{u \in U}\) is \( \mathbb{k} \)-bilinear and middle \( R \)-linear. By the universal property of graded tensor products 2.1.13 there is a morphism of graded \( \mathbb{k} \)-modules \( (\sqcup_{u \in U} M^u) \otimes_R N) \to \prod_{u \in U} (M^u \otimes_R N) \) that maps \( (m^u)_{u \in U} \otimes n \) to \( (m^u \otimes n)_{u \in U}\). It is straightforward to verify that this is an inverse of (3.1.12.1). \( \square \)

**3.1.13 Proposition.** Let \( M \) be an \( R^o \)-complex and let \( \{N^u\}_{u \in U} \) be a family of \( R \)-complexes. The canonical map

\[
\prod_{u \in U} (M \otimes_R N^u) \longrightarrow M \otimes_R \prod_{u \in U} N^u,
\]

given by \((r^u)_{u \in U} \mapsto \sum_{u \in U} (M \otimes_r t^u)(r^u)\), is an isomorphism of \( \mathbb{k} \)-complexes.

**Proof.** Similar to the proof of 3.1.12. \( \square \)

**Remark.** For an \( R^o \)-complex \( M \), the functor \( M \otimes_R -: \mathcal{C}(R) \to \mathcal{C}(\mathbb{k}) \) has a right adjoint, namely the functor \( \text{Hom}_R(M, -) \); see E 4.3.3. From this fact alone, it follows that \( M \otimes_R - \) preserves coproducts; see E 3.1.5.
PRODUCTS

3.1.14 Construction. Let \( \{ N^u \}_{u \in U} \) be a family of \( R \)-complexes. One defines a complex \( \prod_{u \in U} N^u \) by setting

\[
( \prod_{u \in U} N^u )_v = \prod_{u \in U} N^u_v \quad \text{and} \quad \partial_v \prod_{u \in U} N^u = \prod_{u \in U} \partial_v N^u_v,
\]

where the right-hand side of either equality is given by the product in \( M(R) \). For every \( u \in U \) the projections \( \pi^u_v : \prod_{u \in U} N^u \to N^u_v \) yield a projection

(3.1.14.1) \[ \pi^u : \prod_{u \in U} N^u \to N^u \]
of \( R \)-complexes. It is straightforward to verify that an element \( n \) in \( \prod_{u \in U} N^u \) has the form \( n = (n^u)_{u \in U} = (\pi^u(n))_{u \in U} \), where \( n^u = \pi^u(n) \) belongs to \( N^u \).

The next definition is justified by 3.1.16, which shows that the complex \( \prod_{u \in U} N^u \) and the morphisms \( \pi^u \) have the universal property that defines a product. In any category this property determines the product uniquely up to isomorphism.

3.1.15 Definition. For a family of \( R \)-complexes \( \{ N^u \}_{u \in U} \), the complex \( \prod_{u \in U} N^u \) together with the family of projections \( \{ \pi^u \}_{u \in U} \), constructed in 3.1.14, is called the product of \( \{ N^u \}_{u \in U} \) in \( C(R) \).

REMARK. Another name for the product defined above is direct product.

3.1.16 Lemma. For every family \( \{ \alpha^u : M \to N^u \}_{u \in U} \) of morphisms in \( C(R) \), there is a unique morphism \( \alpha \) that makes the next diagram commutative for every \( u \in U \),

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha^u} & N^u \\
\vspace{0.5cm} \alpha \downarrow & & \downarrow \\
\prod_{u \in U} N^u & \xrightarrow{\sum} & N^u
\end{array}
\]

The morphism \( \alpha \) is given by the assignment \( m \mapsto (\alpha^u(m))_{u \in U} \).

PROOF. The assignment defines a morphism of graded \( R \)-modules with \( \alpha^u = \pi^u \alpha \) for all \( u \in U \), and it is straightforward to verify that it is a morphism of \( R \)-complexes. For any morphism \( \alpha' : M \to \prod_{u \in U} N^u \) in \( C(R) \) with \( \alpha^u = \pi^u \alpha' \) for all \( u \in U \), one has \( \alpha'(m) = (\pi^u \alpha'(m))_{u \in U} = (\alpha^u(m))_{u \in U} = \alpha(m) \), so \( \alpha \) is unique. \( \square \)

3.1.17. It follows readily from 3.1.14 and 3.1.16 that the full subcategory \( \mathcal{N}_{gr}(R) \) of \( C(R) \) is closed under products.

3.1.18 Definition. Let \( \{ \beta^u : N^u \to M^u \}_{u \in U} \) be a family of morphisms in \( C(R) \). By the universal property of products, the map given by \( (n^u)_{u \in U} \mapsto \sum_{u \in U} (\beta^u(n^u))_{u \in U} \) is the unique morphism that makes the next diagram commutative for every \( u \in U \),

\[
\begin{array}{ccc}
N^u & \xrightarrow{\beta^u} & M^u \\
\vspace{0.5cm} \sum \downarrow & & \downarrow \\
\prod_{u \in U} N^u & \xrightarrow{\beta} & \prod_{u \in U} M^u
\end{array}
\]
This morphism is called the product of the family \( \{ \beta^u : N^u \rightarrow M^u \} \) in \( \mathcal{C}(R) \), and it is denoted \( \prod_{u \in U} \beta^u \).

The product is exact.

**3.1.19.** Let \( \{ \alpha^u : X^u \rightarrow N^u \} \) and \( \{ \beta^u : N^u \rightarrow M^u \} \) be families of morphisms in \( \mathcal{C}(R) \). The sequence

\[
\prod_{u \in U} X^u \xrightarrow{\prod_{u \in U} \alpha^u} \prod_{u \in U} N^u \xrightarrow{\prod_{u \in U} \beta^u} \prod_{u \in U} M^u
\]

in \( \mathcal{C}(R) \) is exact if and only if each sequence \( X^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} M^u \) is exact.

**Product Preserving Functors**

**3.1.20.** Let \( \{ N^u \} \) be a family of \( R \)-complexes and let \( s \) be an integer. There is an equality of complexes \( \sum \prod_{u \in U} N^u = \prod_{u \in U} \Sigma N^u \). Moreover, if \( \{ \beta^u : N^u \rightarrow M^u \} \) is a family of morphisms in \( \mathcal{C}(R) \), then one has \( \Sigma \prod_{u \in U} \beta^u = \prod_{u \in U} \Sigma \beta^u \).

**3.1.21.** Let \( \{ N^u \} \) be a family of \( R \)-complexes. The differential \( \partial \prod_{u \in U} N^u \), considered as a morphism \( \prod_{u \in U} N^u \rightarrow \Sigma \prod_{u \in U} N^u \), is the product of the family of morphisms \( \{ \partial N^u : N^u \rightarrow \Sigma N^u \} \). It follows that there are equalities

\[
(3.1.21.1) \quad Z(\prod_{u \in U} N^u) = \prod_{u \in U} Z(N^u) \quad \text{and} \quad B(\prod_{u \in U} N^u) = \prod_{u \in U} B(N^u)
\]

of subcomplexes of \( \prod_{u \in U} N^u \); cf. (2.2.10.2) and 3.1.19.

**3.1.22.** Let \( F : \mathcal{C}(R) \rightarrow \mathcal{C}(S) \) be a functor and \( \{ N^u \} \) be a family of \( R \)-complexes. The projection (3.1.14.1) induces a morphism \( F(\pi^u) : F(\prod_{u \in U} N^u) \rightarrow F(N^u) \) for every \( u \in U \). By the universal property of products, the map given by the assignment \( x \mapsto (F(\pi^u)(x))_{u \in U} \) is the unique morphism that makes the following diagram in \( \mathcal{C}(S) \) commutative for every \( u \in U \),

\[
\begin{array}{ccc}
\prod_{u \in U} F(N^u) & \xrightarrow{F(\pi^u)} & F(\prod_{u \in U} N^u) \\
\downarrow & & \downarrow \\
F(\prod_{u \in U} N^u) & \xrightarrow{F(\pi^u)} & F(N^u)
\end{array}
\]
If the canonical morphism in 3.1.22 is an isomorphism, then one says that $F$ preserves products. While this is a condition on objects, it carries over to morphisms.

3.1.23. Let $F: \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor and let $\{\beta^u: N^u \to M^u\}_{u \in U}$ be a family of morphisms in $\mathcal{C}(R)$. It is elementary to verify that there is a commutative diagram,

$$
\begin{array}{c}
F(\prod_{u \in U} N^u) \xrightarrow{F(\prod_{u \in U} \beta^u)} F(\prod_{u \in U} M^u) \\
\downarrow \downarrow \\
\prod_{u \in U} F(N^u) \xrightarrow{\prod_{u \in U} F(\beta^u)} \prod_{u \in U} F(M^u),
\end{array}
$$

where the vertical maps are the canonical morphisms from 3.1.22.

Homology, as a functor $H: \mathcal{C}(R) \to \mathcal{C}(R)$, preserves products.

3.1.24 Proposition. Let $\{N^u\}_{u \in U}$ be a family of $R$-complexes. The canonical map

$$(3.1.24.1) \quad H(\prod_{u \in U} N^u) \longrightarrow \prod_{u \in U} H(N^u),$$

given by $h \mapsto (H(\pi^u)(h))_{u \in U}$, is an isomorphism of $R$-complexes.

PROOF. The map is a morphism of $R$-complexes by 3.1.22. An element $h$ in $H(\prod_{u \in U} N^u)$ is by 3.1.21 a class $[(z^u)_{u \in U}]$, where each $z^u$ belongs to $Z(N^u)$, and it gets mapped to $[(z^u)_{u \in U}]$. The assignment $[(z^u)_{u \in U}] \mapsto [(z^u)_{u \in U}]$, therefore, defines an inverse to (3.1.24.1). \qed

The functor $\text{Hom}_R(M, -)$, from $\mathcal{C}(R)$ to $\mathcal{C}(\mathbb{k})$, preserves products.

3.1.25 Proposition. Let $M$ be an $R$-complex and let $\{N^u\}_{u \in U}$ be a family of $R$-complexes. The canonical map

$$(3.1.25.1) \quad \text{Hom}_R(M, \prod_{u \in U} N^u) \longrightarrow \prod_{u \in U} \text{Hom}_R(M, N^u),$$

given by $\theta \mapsto (\text{Hom}_R(M, \pi^u)(\theta))_{u \in U} = (\pi^u \theta)_{u \in U}$, is an isomorphism of $\mathbb{k}$-complexes.

PROOF. The map is a morphism of $\mathbb{k}$-complexes by 3.1.22. Assign to an element $(\theta^u)_{u \in U}$ in $\prod_{u \in U} \text{Hom}_R(M, N^u)$ the homomorphism $\theta$ in $\text{Hom}_R(M, \prod_{u \in U} N^u)$ that maps an element $m \in M$ to $(\theta^u(m))_{u \in U}$ in $\prod_{u \in U} N^u$. This assignment defines an inverse to (3.1.25.1). \qed

REMARK. For an $R$-complex $M$, the functor $\text{Hom}_R(M, -): \mathcal{C}(R) \to \mathcal{C}(\mathbb{k})$ has a left adjoint, namely the functor $M \otimes_{\mathbb{k}} -$; see E 4.3.4. From this fact alone, it follows that $\text{Hom}_R(M, -)$ preserves products; see E 3.1.11.
3.1.26. Let $F: \mathcal{C}(R)^{\text{op}} \to \mathcal{C}(S)$ be a functor; let $\{M^u\}_{u \in U}$ be a family of $R$-complexes. The embedding (3.1.1.1) induces a morphism $F(\iota^u) : F(\coprod_{u \in U} M^u) \to F(M^u)$ for every $u \in U$. By the universal property of products, the map given by the assignment $x \mapsto (F(\iota^u(x))_{u \in U}$ is the unique morphism that makes the following diagram in $\mathcal{C}(S)$ commutative for every $u \in U$.

\[
\begin{array}{ccc}
\prod_{u \in U} F(M^u) \\
\downarrow \\
F(\coprod_{u \in U} M^u) & \xrightarrow{\text{universal property}} & F(M^u).
\end{array}
\]

3.1.27. Let $F: \mathcal{C}(R)^{\text{op}} \to \mathcal{C}(S)$ be a functor and let $\{\alpha^u : M^u \to N^u\}_{u \in U}$ be a family of morphisms in $\mathcal{C}(R)$. It is elementary to verify that there is a commutative diagram,

\[
\begin{array}{ccc}
F(\coprod_{u \in U} N^u) & \xrightarrow{\text{universal property}} & F(\coprod_{u \in U} M^u) \\
\downarrow & & \downarrow \\
\prod_{u \in U} F(N^u) & \xrightarrow{\text{universal property}} & \prod_{u \in U} F(M^u),
\end{array}
\]

where the vertical maps are the canonical morphisms from 3.1.26.

Products in a category correspond to coproducts in the opposite category and vice versa. In particular, products in $\mathcal{C}(R)^{\text{op}}$ correspond to coproducts in $\mathcal{C}(R)$, so $\mathcal{C}(R)^{\text{op}}$ has products; that is the content of the next proposition, and together with 3.1.25 it sums up as: Hom preserves products.

3.1.28 Proposition. Let $N$ be an $R$-complex and let $\{M^u\}_{u \in U}$ be a family of $R$-complexes. The canonical map

\[(3.1.28.1) \quad \text{Hom}_R(\prod_{u \in U} M^u, N) \to \prod_{u \in U} \text{Hom}_R(M^u, N), \]

given by $\theta \mapsto (\text{Hom}_R(\iota^u, N)(\theta))_{u \in U} = (\theta^u)_{u \in U}$, is an isomorphism of $\mathcal{C}$-complexes.

PROOF. The map is a morphism of $\mathcal{C}$-complexes by 3.1.26. Assign to an element $(\theta^u)_{u \in U}$ in $\prod_{u \in U} \text{Hom}_R(M^u, N)$ the homomorphism in $\text{Hom}_R(\prod_{u \in U} M^u, N)$ given by $\sum_{u \in U} \iota^u(m^u) \mapsto \sum_{u \in U} \theta^u(m^u)$. This defines an inverse to (3.1.28.1). \[\square\]
3.1 Products and Coproducts

**Boundedness and Finiteness**

3.1.29. For every family \( \{M^u\}_{u \in U} \) of \( R \)-complexes, the coproduct \( \prod_{u \in U} M^u \) is a subcomplex of the product \( \prod_{u \in U} M^u \). If \( U \) is a finite set, then the product and coproduct coincide. Per 1.1.10 the biproduct notation \( \bigoplus_{u \in U} M^u \) is used for this complex, which is called the direct sum of the family \( \{M^u\}_{u \in U} \) in \( \mathcal{C}(R) \); each complex \( M^u \) is called a direct summand.

Under suitable finiteness conditions on \( M \) the functor \( M \otimes_R - \) preserves products.

3.1.30 Proposition. Let \( M \) be an \( R^0 \)-complex and let \( \{N^u\}_{u \in U} \) be a family of \( R \)-complexes. The canonical map

\[
(3.1.30.1) \quad M \otimes_R \prod_{u \in U} N^u \longrightarrow \prod_{u \in U} (M \otimes_R N^u),
\]

given by \( t \mapsto ((M \otimes_R N^u)(t))_{u \in U} \), is a morphism of \( \mathfrak{k} \)-complexes. If \( M \) is bounded and degreewise finitely presented, then (3.1.30.1) is an isomorphism.

**Proof.** The map is a morphism of \( \mathfrak{k} \)-complexes by 3.1.9. Assume that \( M \) is bounded and degreewise finitely presented. It follows from 2.5.6 that the graded module \( M^u \) has a presentation \( L' \to L \to M^u \to 0 \), where \( L \) and \( L' \) are graded-free, bounded, and degreewise finitely generated \( R^0 \)-modules. Consider the following commutative diagram, whose upper row is obtained by application of the functor \(- \otimes_R \prod_{u \in U} N^u\) to the presentation of \( M^u \),

\[
\begin{array}{cccccc}
L' \otimes_R \prod_{u \in U} N^u & \longrightarrow & L \otimes_R \prod_{u \in U} N^u & \longrightarrow & M \otimes_R \prod_{u \in U} N^u & \longrightarrow & 0 \\
\downarrow{\kappa'} & & \downarrow{\kappa} & & \downarrow{\kappa^M} & & \\
\prod_{u \in U} (L' \otimes_R N^u) & \longrightarrow & \prod_{u \in U} (L \otimes_R N^u) & \longrightarrow & \prod_{u \in U} (M \otimes_R N^u) & \longrightarrow & 0.
\end{array}
\]

The rows are exact by right exactness of tensor products, 2.4.9, and exactness of products, 3.1.19. The canonical morphism (3.1.30.1) yields the vertical morphism below. The module \( (L' \otimes_R \prod_{u \in U} N^u)_v \) is a direct sum,

\[
\bigoplus_{i = \inf L'} L'_i \otimes_R \left( \prod_{u \in U} N^u \right)_{v-i} \cong \bigoplus_{i = \inf L'} \left( L'_i \otimes_R N^u \right)_{v-i} \cong \prod_{u \in U} (L'_i \otimes_R N^u)_{v-i},
\]

and this composite isomorphism is \( \kappa^L \) in degree \( v \). Similarly, \( \kappa^L \) is an isomorphism, so it follows from the Five Lemma 2.1.37 that \( \kappa_M \) is an isomorphism. \( \square \)

3.1.31 Proposition. Let \( \{M^u\}_{u \in U} \) be a family of \( R^0 \)-complexes and let \( N \) be an \( R \)-complex. The canonical map
Let \( E \)

\[ \sum_{i \in I} M_i \otimes_R N \longrightarrow \prod_{i \in I} (M_i \otimes_R N) \]

given by \( i \mapsto ((\pi_i \otimes_R N)(i))_{i \in I} \), is a morphism of \( \mathcal{R} \)-complexes. If \( N \) is bounded and degreewise finitely presented, then (3.1.31.1) is an isomorphism.

**Proof.** Similar to the proof of 3.1.30. \( \square \)

Under suitable conditions on \( M \), the functor \( \text{Hom}_R(M, -) \) preserves coproducts.

**3.1.32 Proposition.** Let \( M \) be an \( R \)-complex and let \( \{ N^u \}_{u \in U} \) be a family of \( R \)-complexes. The canonical map

\[ \prod_{u \in U} \text{Hom}_R(M, N^u) \longrightarrow \text{Hom}_R(M, \prod_{u \in U} N^u) \]

given by \( (\theta^u)_{u \in U} \mapsto \sum_{u \in U} \text{Hom}_R(M, i^u)(\theta^u) = \sum_{u \in U} i^u \theta^u \), is a morphism of \( \mathcal{R} \)-complexes. If \( M \) is a bounded and degreewise finitely generated, then (3.1.32.1) is an isomorphism.

**Proof.** The map is a morphism of \( \mathcal{R} \)-complexes by 3.1.9. Assume that \( M \) is bounded and degreewise finitely generated. Then \( M^\ell \) is finitely generated, so a homomorphism \( \theta : M \to \prod_{u \in U} N^u \) factors through a subcomplex \( \bigoplus_{u \in U'} N^u \), where \( U' \) is a finite subset of \( U \). For \( u \in U' \) denote by \( \pi^u : \bigoplus_{u \in U'} N^u \to N^u \) the projection. Assign to a homomorphism \( \theta \) in \( \text{Hom}_R(M, \prod_{u \in U} N^u) \) the element \( (\theta^u)_{u \in U} \) in \( \prod_{u \in U} \text{Hom}_R(M, N^u) \) with \( \theta^u = \pi^u \theta \) for \( u \in U' \) and \( \theta^u = 0 \) for \( u \notin U' \). This defines an inverse to (3.1.32.1). \( \square \)

**Exercises**

**E 3.1.1** Let \( \{ F^u : M^u \to C \}_{u \in U} \) be a family of morphisms in \( \mathcal{C}(R) \) with the property that for every family \( \{ \alpha^u : M^u \to N \}_{u \in U} \) of morphisms in \( \mathcal{C}(R) \) there is a unique morphism \( \alpha : C \to N \) with \( \alpha^u = \alpha \) for all \( u \in U \). Show that there is an isomorphism \( \varphi : \prod_{u \in U} M^u \to C \) with \( \varphi^u = \pi^u \) for all \( u \in U \). Conclude that the universal property determines the coproduct uniquely up to isomorphism.

**E 3.1.2** Let \( \alpha : \prod_{u \in U} M^u \to N \) be the morphism induced by a family \( \{ \alpha^u : M^u \to N \}_{u \in U} \) of morphisms in \( \mathcal{C}(R) \); see 3.1.3. Show that \( \alpha \) is surjective if one has \( \bigcup_{u \in U} \text{Im} \alpha^u = N \).

**E 3.1.3** (Cf. 3.1.4) Show that the coproduct in \( \mathcal{C}(R) \) of a family of graded \( R \)-modules is a graded \( R \)-module. Conclude, in particular, that the category \( \text{Mod}_g(R) \) has coproducts.

**E 3.1.4** Let \( U \) be a set. Show that \( U \)-indexed families of \( R \)-complexes form an Abelian category and that the product and coproduct are exact functors from this category to \( \mathcal{C}(R) \).

**E 3.1.5** Show that every functor \( F : \mathcal{C}(R) \to \mathcal{C}(S) \) that has a right adjoint preserves coproducts.

**E 3.1.6** Show that for every \( R \)-complex \( N \) there are isomorphisms \( \prod_{i \in I} \Sigma^i N_i \cong N \cong \prod_{i \in I} \Sigma^i N_i \).

**E 3.1.7** Let \( \{ M^u \}_{u \in U} \) be a family of \( R \)-complexes. Show that there are equalities
3.2 Colimits

SYNOPSIS. Direct system; colimit; universal property; colimit preserving functors; pushout; filtered colimit; telescope.

3.2.1 Definition. Let \((U, \leq)\) be a preordered set. A \emph{U-direct system} in \(C(R)\) is a family \(\{\mu^u : M^u \to M^v\}_{u \leq v}\) of morphisms in \(C(R)\) with the following properties.

1. \(\mu^{uu} = 1_{M^u}\) for all \(u \in U\).
2. \(\mu^{vw} \mu^{uv} = \mu^{wu}\) for all \(u \leq v \leq w\) in \(U\).

Any mention of a \(U\)-direct system \(\{\mu^u : M^u \to M^v\}_{u \leq v}\) includes the tacit assumption that \((U, \leq)\) is a preordered set. A \(\mathbb{Z}\)-direct system is called a \emph{direct system}.

3.2.2 Construction. Let \(\{\mu^u : M^u \to M^v\}_{u \leq v}\) be a \(U\)-direct system in \(C(R)\). We describe the quotient of the coproduct \(\bigsqcup_{u \in U} M^u\) by the subcomplex generated by the set of elements \(\{v^u(m^u) - v^u \mu^u(m^v) \mid m^u \in M^u, u \leq v\}\) as the cokernel of a morphism between coproducts in \(C(R)\).

Set \(\delta(U) = \{(u, v) \in U \times U \mid u \leq v\}\) and set \(M^{(u,v)} = M^u\) for all \((u, v) \in \delta(U)\). The assignment

\[
(m^{(u,v)})_{(u,v)\in\delta(U)} \mapsto \sum_{(u,v)\in\delta(U)} \iota^v(m^{(u,v)}) - \iota^v \mu^u(m^{(u,v)}),
\]

where \(\iota\) is the embedding (3.1.1.1), defines a morphism of \(R\)-complexes.
\[ A_\mu : \coprod_{(u,v) \in V(U)} M^{(u,v)} \rightarrow \coprod_{u \in U} M^u. \]

The cokernel of \( A_\mu \) is denoted \( \text{colim}_{u \in U} M^u \).

Notice that for every \( u \in U \) the composite of the embedding \( t^u \) with the canonical map onto \( \text{colim}_{u \in U} M^u \) is a morphism of \( R \)-complexes,

\[(3.2.2.1)\]
\[ \mu^u : M^u \rightarrow \text{colim}_{u \in U} M^u, \]

and one has \( \mu^u = \mu^v \mu^v \) for all \( u \leq v \) in \( U \). Every element in \( \text{colim}_{u \in U} M^u \) has the form \( \sum_{u \in U} \mu^u(m^u) \) for some element \( \sum_{u \in U} t^u(m^u) \) in \( \coprod_{u \in U} M^u \), and one has

\[(3.2.2.2)\]
\[ \partial^{\text{colim}_{u \in U} M^u} \left( \sum_{u \in U} \mu^u(m^u) \right) = \sum_{u \in U} \mu^u(\partial^{M^u}(m^u)). \]

**Remark.** Though the complex \( \text{colim}_{u \in U} M^u \) depends on the morphisms \( \mu^u : M^u \rightarrow M^u \), it is standard to use this symbol that suppresses the morphisms.

The next definition is justified by 3.2.4; it shows that the complex \( \text{colim}_{u \in U} M^u \) and the canonical morphisms \( \mu^u \) have the universal property that defines a colimit. In any category, this property determines the colimit uniquely up to isomorphism.

**3.2.3 Definition.** For a \( U \)-direct system \( \{ \mu^u : M^u \rightarrow M^v \}_{u \leq v} \) in \( \mathcal{C}(R) \) the complex \( \text{colim}_{u \in U} M^u \) together with the canonical morphisms \( \{ \mu^u \}_{u \in U} \), constructed in 3.2.2, is called the colimit of \( \{ \mu^u : M^u \rightarrow M^v \}_{u \leq v} \) in \( \mathcal{C}(R) \).

**Remark.** Other names for the colimit defined above are direct limit, inductive limit, and injective limit; other symbols used for this gadget are \( \underline{\text{lim}} \) and \( \text{injlim} \).

**3.2.4 Lemma.** Let \( \{ \mu^u : M^u \rightarrow M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). For every family of morphisms \( \{ \alpha^u : M^u \rightarrow N \}_{u \in U} \) in \( \mathcal{C}(R) \) with \( \alpha^u = \alpha^v \mu^v \) for all \( u \leq v \), there is a unique morphism \( \alpha \) that makes the next diagram commutative for all \( u \leq v \),

\[
\begin{array}{ccc}
M^u & \xrightarrow{\mu^u} & \text{colim}_{u \in U} M^u \\
& \searrow \alpha^u & \downarrow \alpha \\
& \text{colim}_{u \in U} M^u & \xrightarrow{\alpha'} N \\
\end{array}
\]

The morphism \( \alpha \) is given by \( \sum_{u \in U} \alpha^u(m^u) \rightarrow \sum_{u \in U} \alpha^u(m^u) \).

**Proof.** Let \( \{ \alpha^u : M^u \rightarrow N \}_{u \in U} \) be a family of morphisms with \( \alpha^u = \alpha^v \mu^v \) for all \( u \leq v \). The equalities \( \alpha^u = \alpha^v \mu^v \) ensure that the morphism \( \coprod_{u \in U} M^u \rightarrow N \) from 3.1.3 factors through \( \text{colim}_{u \in U} M^u \) to yield a morphism \( \alpha \) with the stipulated definition.

It is evident from the definition that \( \alpha \) satisfies \( \alpha^u = \alpha \mu^u \) for all \( u \in U \). Moreover, for any morphism \( \alpha' : \text{colim}_{u \in U} M^u \rightarrow N \) that satisfies \( \alpha^u = \alpha' \mu^u \) for all \( u \in U \), one has \( \alpha'(\sum_{u \in U} \mu^u(m^u)) = \sum_{u \in U} \alpha'(\mu^u(m^u)) = \sum_{u \in U} \alpha^u(m^u) \), hence \( \alpha' = \alpha \). \qed
3.2.5. For \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) and \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) as in 3.2.4, notice that the morphism \( \alpha : \operatorname{colim}_{u \in U} M^u \to N \) is surjective if one has \( \bigcup_{u \in U} \operatorname{Im} \alpha^u = N \).

3.2.6. It follows readily from 3.2.2 and 3.2.4 that the full subcategories \( \mathcal{M}(R) \) and \( \mathcal{M}_f(R) \) of \( \mathcal{C}(R) \) are closed under colimits.

3.2.7 Example. Let \( \{ M^u \}_{u \in U} \) be a family of \( R \)-complexes. Endowed with the discrete order, \( U \) is a preordered set, and \( \{ \mu^u = 1_{M^u} \}_{u \in U} \) is a \( U \)-direct system with \( \operatorname{colim}_{u \in U} M^u = \coprod_{u \in U} M^u \) and \( \mu^u = \iota^u \) for all \( u \in U \). Thus, a coproduct is a colimit.

3.2.8 Example. Let \( p \) be an integer. The sequence \( \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \cdots \) determines a direct system whose colimit is the \( \mathbb{Z} \)-module \( \mathbb{Z}_p = \{ 1, p, p^2, \ldots \}^{-1} \mathbb{Z} \subseteq \mathbb{Q} \).

3.2.9 Definition. Let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) and \( \{ \nu^u : N^u \to N^v \}_{u \leq v} \) be \( U \)-direct systems in \( \mathcal{C}(R) \). A family of morphisms \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) in \( \mathcal{C}(R) \) that satisfy \( \nu^u \alpha^u = \alpha^v \mu^u \) for all \( u \leq v \) in \( U \) is called a morphism of \( U \)-direct systems. Such a morphism is called injective (surjective) if each map \( \alpha^u \) is injective (surjective).

Given a morphism \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) of \( U \)-direct systems, it follows from the universal property of colimits that the map given by \( \sum_{u \in U} \mu^u(m^u) \mapsto \sum_{u \in U} \nu^u \alpha^u(m^u) \) is the unique morphism that makes the next diagram commutative for all \( u \leq v \),

This morphism is called the colimit of \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) and denoted \( \operatorname{colim}_{u \in U} \alpha^u \).

3.2.10. Let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \) and let \( s \) be an integer. It follows from 3.2.2 and 3.1.7 that \( \{ \Sigma^s \mu^u : \Sigma^s M^u \to \Sigma^s M^v \}_{u \leq v} \) is a \( U \)-direct system with \( \operatorname{colim}_{u \in U} \Sigma^s M^u = \Sigma^s \operatorname{colim}_{u \in U} M^u \). Moreover, if \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) is a morphism of \( U \)-direct systems, then one has \( \operatorname{colim}_{u \in U} \Sigma^s \alpha^u = \Sigma^s \operatorname{colim}_{u \in U} \alpha^u \).

3.2.11 Example. Let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). Because the maps \( \mu^u \) are morphisms in \( \mathcal{C}(R) \), the family \( \{ \partial M^u : M^u \to \Sigma M^u \}_{u \in U} \) is a morphism of \( U \)-direct systems. From the definitions one has \( \operatorname{colim}_{u \in U} \partial M^u = \partial \operatorname{colim}_{u \in U} M^u \).

The next statement sums up as: colimits are right exact. While general colimits are not exact—see 3.2.23 for an example—we shall see in 3.2.27 that colimits over filtered sets are exact.

3.2.12 Lemma. Let \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) and \( \{ \beta^u : N^u \to X^u \}_{u \in U} \) be morphisms of \( U \)-direct systems in \( \mathcal{C}(R) \). If the sequence

\[
M^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} X^u \xrightarrow{} 0
\]

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is exact for every \( u \in U \), then the next sequence is exact,
\[
\colim_{u \in U} M^u \xrightarrow{\colim_{u \in U} \alpha^u} \colim_{u \in U} N^u \xrightarrow{\colim_{u \in U} \beta^u} \colim_{u \in U} X^u \to 0.
\]

**Proof.** Let \( \nabla(U) \) be as in 3.2.2, and for all \( (u, v) \in \nabla(U) \) set \( \alpha^{(u,v)} = \alpha^u \) and \( \beta^{(u,v)} = \beta^u \). By 3.1.6 and 3.2.9 there is a commutative diagram with exact columns and exact upper and middle rows,
\[
\begin{array}{ccc}
\prod_{(u,v) \in \nabla(U)} M^{(u,v)} & \xrightarrow{\prod_{(u,v) \in \nabla(U)} \alpha^{(u,v)}} & \prod_{(u,v) \in \nabla(U)} N^{(u,v)} & \xrightarrow{\prod_{(u,v) \in \nabla(U)} \beta^{(u,v)}} & \prod_{(u,v) \in \nabla(U)} X^{(u,v)} \\
\colim_{u \in U} M^u & \xrightarrow{\colim_{u \in U} \alpha^u} & \colim_{u \in U} N^u & \xrightarrow{\colim_{u \in U} \beta^u} & \colim_{u \in U} X^u \\
\end{array}
\]

where the vertical morphisms \( \Delta \) are defined in 3.2.2 An elementary diagram chase shows that the lower row in the diagram is exact. \( \square \)

**Colimit Preserving Functors**

3.2.13. Let \( F: \mathcal{C}(R) \to \mathcal{C}(S) \) be a functor and let \( \{\mu^u: M^u \to M^v\}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). It is easy to see that the morphisms \( \{F(\mu^u): F(M^u) \to F(M^v)\}_{u \leq v} \) form a \( U \)-direct system in \( \mathcal{C}(S) \). For every \( u \in U \) let \( \lambda^u \) be the canonical morphism \( F(M^u) \to \colim_{u \in U} F(M^u) \). For every \( v \in U \), the canonical morphism (3.2.2.1) induces a morphism \( F(\mu^u): F(M^u) \to F(\colim_{u \in U} M^u) \). These morphisms satisfy \( F(\mu^u) = F(\mu^v)F(\mu^{uv}) \) for all \( u \leq v \) in \( U \), so by the universal property of colimits, the map given by the assignment \( \sum_{u \in U} \lambda^u(x^u) \to \sum_{u \in U} F(\mu^u)(x^u) \) is the unique morphism that makes the following diagram in \( \mathcal{C}(S) \) commutative for all \( u \leq v \),

\[
\begin{array}{ccc}
F(M^v) & \xrightarrow{F(\mu^v)} & F(M^u) \\
\colim_{u \in U} F(M^u) & \xrightarrow{\sum_{u \in U} \lambda^u} & F(\colim_{u \in U} M^u) \\
\end{array}
\]
3.2 Colimits

If the canonical morphism in 3.2.13 is an isomorphism, then one says that the functor $F$ preserves colimits; such functors are also called \textit{cocontinuous}. While this is a condition on objects, it carries over to morphisms.

3.2.14. Let $F: \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor and let $\{ \alpha^u: M^u \to N^u \}_{u \in U}$ be a morphism of $U$-direct systems in $\mathcal{C}(R)$. It is easy to see that there is a commutative diagram,

$$
\begin{array}{ccc}
\text{colim}_{u \in U} F(M^u) & \xrightarrow{\text{colim}_{u \in U} F(\alpha^u)} & \text{colim}_{u \in U} F(N^u) \\
\downarrow & & \downarrow \\
F(\text{colim}_{u \in U} M^u) & \xrightarrow{F(\text{colim}_{u \in U} \alpha^u)} & F(\text{colim}_{u \in U} N^u),
\end{array}
$$

where the vertical maps are the canonical morphisms from 3.2.13.

The tensor product, as a functor from $\mathcal{C}(R)$ to $\mathcal{C}(k)$, preserves colimits.

3.2.15 \textbf{Proposition.} Let $\{ \mu^u: M^u \to M^v \}_{u \leq v}$ be a $U$-direct system in $\mathcal{C}(R^0)$ and let $N$ be an $R$-complex. The canonical map

$$(3.2.15.1)\quad \text{colim}_{u \in U} (M^u \otimes_R N) \to (\text{colim}_{u \in U} M^u) \otimes_R N,$$

given by $\sum_{u \in U} \lambda^u(t^u) \mapsto \sum_{v \in U} (\mu^u \otimes_R N)(t^u)$, cf. 3.2.13, \textit{is an isomorphism} in $\mathcal{C}(k)$.

\textbf{PROOF.} The map is a morphism of $k$-complexes by 3.2.13. There is a commutative diagram in $\mathcal{C}(k)$,

$$
\begin{array}{ccc}
\prod_{(u,v) \in \nabla(U)} (M^{(u,v)} \otimes_R N) & \xrightarrow{\Delta_{u,v} \otimes N} & \prod_{u \in U} (M^u \otimes_R N) \xrightarrow{\text{colim}_{u \in U}} \text{colim}_{u \in U} (M^u \otimes_R N) \xrightarrow{\kappa} 0 \\
\cong & & \cong \downarrow \kappa \\
(\prod_{(u,v) \in \nabla(U)} M^{(u,v)} \otimes_R N) & \xrightarrow{\Delta_{u,v} \otimes N} & (\prod_{u \in U} M^u \otimes_R N) \xrightarrow{\text{colim}_{u \in U}} (\text{colim}_{u \in U} M^u \otimes_R N) \xrightarrow{\kappa} 0,
\end{array}
$$

where $\kappa$ is the canonical morphism (3.2.15.1), and the middle and left-most vertical maps are the isomorphisms from 3.1.12. The rows are exact by the construction of colimits, 3.2.2, and right exactness of tensor products, 2.4.9, and it follows from the Five Lemma 2.1.37 that $\kappa$ is an isomorphism. \hfill $\Box$

3.2.16 \textbf{Proposition.} Let $M$ be an $R^0$-complex and $\{ \nu^u: N^u \to N^v \}_{u \in U}$ be a $U$-direct system in $\mathcal{C}(R)$. The canonical map

$$(3.2.16.1)\quad \text{colim}_{u \in U} (M \otimes_R N^u) \to M \otimes_R \text{colim}_{u \in U} N^u,$$

given by $\sum_{u \in U} \lambda^u(t^u) \mapsto \sum_{v \in U} (M \otimes_R \nu^u)(t^u)$, cf. 3.2.13, \textit{is an isomorphism} in $\mathcal{C}(k)$.

\textbf{PROOF.} Similar to the proof of 3.2.15. \hfill $\Box$
3.2.17 Construction. Let $U = \{u, v, w\}$ be a set, preordered as follows $v > u < w$. Given a diagram $M \leftarrow X \stackrel{\beta}{\rightarrow} N$ in $\mathcal{C}(R)$, set

$$M^v = M, \quad M^u = X, \quad M^w = N,$$

$$\mu^{vv} = 1^M, \quad \mu^{vu} = \alpha, \quad \mu^{uw} = 1^X, \quad \mu^{wu} = \beta, \quad \text{and} \quad \mu^{ww} = 1^N.$$ 

This defines a $U$-direct system in $\mathcal{C}(R)$. It is straightforward to verify that the colimit of this system is the cokernel of the morphism $(-\alpha \beta): X \to M \oplus N$.

3.2.18 Definition. For a diagram $M \leftarrow X \stackrel{\beta}{\rightarrow} N$ in $\mathcal{C}(R)$, the colimit of the $U$-direct system constructed in 3.2.17 is called the pushout of $\langle \alpha, \beta \rangle$ and denoted $M \sqcup X N$.

Remark. As for the colimit, the notation for the pushout supresses the morphisms. Other names for the pushout are fibered coproduct, fibered sum, cocartesian square, and amalgamated product.

3.2.19. Given morphisms $\alpha: X \to M$ and $\beta: X \to N$, the pushouts of $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ are isomorphic via the map induced by the canonical isomorphism $M \oplus N \cong N \oplus M$.

3.2.20 Construction. Given a diagram in $M \leftarrow X \stackrel{\beta}{\rightarrow} N$ in $\mathcal{C}(R)$, let

$$\alpha' : N \longrightarrow M \sqcup X N \quad \text{and} \quad \beta' : M \longrightarrow M \sqcup X N$$

be the canonical morphisms (3.2.2.1); they are given by $n \mapsto [(0, n)]_{\text{Im}(\beta)}$ and $m \mapsto [(m, 0)]_{\text{Im}(-\alpha \beta)}$. There is a commutative diagram with exact rows and columns

$$
\begin{array}{c}
X \xrightarrow{\beta} N \xrightarrow{\text{Coker} \beta} 0 \\
\alpha \downarrow \quad \alpha' \downarrow \quad \bar{\alpha} \\
M \xrightarrow{\beta'} M \sqcup X N \xrightarrow{\text{Coker} \beta'} 0
\end{array}
$$

(3.2.20.1)

where $\bar{\beta}$ is the morphism induced by $\beta'$, it maps $[m]_{\text{Im} \alpha}$ to $[\beta'(m)]_{\text{Im} \alpha'}$, and $\bar{\alpha}$ is the morphism induced by $\alpha'$.

3.2.21. Given a diagram $M \leftarrow Y \xrightarrow{\alpha''} N$ in $\mathcal{C}(R)$ with $\beta' \alpha = \alpha'' \beta$, it follows from 3.2.4 that the assignment

$$[(m, n)]_{\text{Im}(-\alpha \beta)} = \alpha'(n) + \beta'(m) \longrightarrow \alpha''(n) + \beta''(m)$$
defines the unique morphism that makes the next diagram commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{\beta} & N \\
\downarrow{\alpha} & & \downarrow{\alpha'} \\
M & \xrightarrow{\beta'} & M \cup_X N \\
\end{array}
\]

\[\xrightarrow{\beta''} Y.\]

**3.2.22 Proposition.** Let \(M \xleftarrow{\alpha} X \xrightarrow{\beta} N\) be a diagram in \(\mathcal{C}(R)\). The following assertions hold for the morphisms in (3.2.20.1).

(a) If \(\alpha\) is injective, then \(\alpha'\) is injective.

(b) \(\bar{\beta}\) is an isomorphism, whence \(\alpha\) is surjective if and only if \(\alpha'\) is surjective.

(c) If \(\beta\) is injective, then \(\beta'\) is injective.

(d) \(\bar{\alpha}\) is an isomorphism, whence \(\beta\) is surjective if and only if \(\beta'\) is surjective.

**Proof.** By symmetry, see 3.2.19, it is sufficient to prove parts (a) and (b).

(a): If \(n\) is in \(\text{Ker} \alpha'\), then one has \(n = \beta(x)\) for some \(x\) in \(\text{Ker} \alpha\).

(b): If \(\beta([m]_{\text{Im} \alpha})\) is zero, then one has \(\beta'(m) = \alpha'(n)\) for some \(n \in N\). Thus \(\left((m,n)\right)_{\text{Im}(- \alpha \beta)} = \alpha'(n) - \beta'(m)\) is zero in \(M \cup_X N\) and, consequently, \(m\) belongs to \(\text{Im} \alpha\); hence \(\bar{\beta}\) is injective. For every element \(\alpha'(n) + \beta'(m)\) in \(M \cup_X N\) one has \([\alpha'(n) + \beta'(m)]_{\text{Im} \alpha'} = [\beta'(m)]_{\text{Im} \alpha'}\) in \(\text{Coker} \alpha'\), so \(\bar{\beta}\) is surjective.

The following example shows that colimits are not left exact.

**3.2.23 Example.** The dashed morphisms below form an injective morphism of pushouts diagrams; that is, an injective morphism of \(U\)-direct systems as in 3.2.9, where \(U\) is the preordered set described in 3.2.17.

\[
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\mathbb{Z}} & & \downarrow{\text{Id}} \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
\end{array}
\]

The colimit \(\mathbb{Z} \cup \mathbb{Z}/2\mathbb{Z} \rightarrow 0\) of this morphism is \(\mathbb{Z}/2\mathbb{Z} \rightarrow 0\), which is not injective.

**Filtered colimits**

Every complex is a colimit of bounded above complexes indexed by a filtered set.
3.2.24 Example. Let $M$ be an $R$-complex. The inclusions among subcomplexes $M_{\leq u}$ give rise to a direct system $\{\mu^u : M_{\leq u} \to M_{\leq v}\}_{u \leq v}$ in $\mathcal{C}(R)$. The embeddings $\alpha^u : M_{\leq u} \to M$ satisfy $\alpha^u = \alpha^v \mu^u$ for all $u \leq v$, so by the universal property of colimits, there is a morphism $\alpha : \text{colim}_{u \leq v} M_{\leq u} \to M$, given by $\sum_{u \leq v} \mu^u(m^u) = \sum_{u \leq v} m^u$. It is surjective by construction. Let $m = \sum_{u \leq v} \mu^u(m^u)$ be an element in $\text{Ker} \alpha$ and set $v = \max\{u \in \mathbb{Z} | m^u \neq 0\}$, then one has $\sum_{u \leq v} \mu^u(m^u) = \sum_{u \leq v} m^u = 0$. In $\bigcup_{u \leq v} M_{u}$ one then has $\sum_{u \leq v} \mu^u(m^u) = \sum_{u \leq v} (\mu^u(m^u) - \mu^v(m^u))$. This element is in the image of $d_{\mu}$, see 3.2.2. Thus, $\alpha$ is an isomorphism.

3.2.25 Lemma. Let $\{\mu^u : M^u \to M^v\}_{u \leq v}$ be a $U$-direct system in $\mathcal{C}(R)$. If $U$ is filtered, then the following assertions hold.

(a) For every $m$ in $\text{colim}_{u \leq v} M^u$ there is a $v \in U$ and an $m^v \in M^v$ with $\mu^v(m^v) = m$.

(b) If $m^u \in M^u$ has $\mu^v(m^u) = 0$, then there is a $w \in U$ with $v \leq w$ and $\mu^w(m^v) = 0$.

(c) If elements $m^u \in M^u$ and $m^v \in M^v$ satisfy $\mu^w(m^u) = \mu^v(m^v)$, then there is a $w$ in $U$ with $u \leq w$ and $v \leq w$ such that $\mu^w(m^u) = \mu^w(m^v)$ holds.

Proof. (a): Fix an element $m$ in $\text{colim}_{u \leq v} M^u$; it has the form $m = \sum_{u \leq v} \mu^u(m^u)$. As $U$ is filtered, one can choose $v$ with $v \geq u$ for all $u \in U$ with $m^u \neq 0$; then one has

$$m = \sum_{u \leq v} \mu^u(m^u) = \mu^v(\sum_{u \leq v} \mu^u(m^u))$$

(b): If $\mu^v(m^v) = 0$ holds, then one has

$$\mu^v(m^v) = \sum_{(t,u) \in V(U)} \mu^v(m^v(t,u)) = \sum_{(t,u) \in V(U)} \mu^u(m^u(t,u))$$

for some element $(m^v(t,u))_{(t,u) \in V(U)} \in \prod_{(t,u) \in V(U)} M^v(t,u)$; see 3.2.2. Choose a $w$ with $w \geq t$ for all $t \in U$ with $m^t \neq 0$. Now apply the morphism $\prod_{u \leq v} M^u \to M^w$ given by $\sum_{u \leq v} \mu^u(m^u) \to \sum_{w \leq v} \mu^w(m^w)$ to both sides in $(*)$ to get $\mu^w(m^v) = \sum_{(t,u) \in V(U)} \mu^u(m^u(t,u)) = \mu^w(m^t) - \mu^w(m^t) = 0$.

(c): Choose $w' \in U$ with $u, v \leq w'$, then one has $\mu^w(\mu^w(u(m^u)) - \mu^w(v(m^v))) = 0$. By part (b) there is a now a $w$ in $U$ with $w \geq w'$ and $0 = \mu^w(\mu^w(u(m^u)) - \mu^w(v(m^v))) = \mu^w(m^u) - \mu^w(m^v)$. □

A classical application of colimits is to write an arbitrary module as a colimit of finitely generated modules.

3.2.26 Example. Let $M$ be an $R$-module and let $U$ be the set of all finitely generated submodules of $M$. The set $U$ is preordered under inclusion and filtered. For elements $M' \subseteq M''$ in $U$ let $\mu^{M'} : M' \to M''$ be the embedding, then $\{\mu^{M'} : M' \to M''\}_{M' \subseteq M''}$ is a $U$-direct system. The morphisms in the family $\{\alpha^{M'} : M' \to M\}_{M' \subseteq M''}$ of embeddings satisfy $\alpha^{M'} = \alpha^{M''} \mu^{M'}$ for all $M' \subseteq M''$, so by the universal property of colimits there is a morphism $\alpha : \text{colim}_{M' \subseteq U} M' \to M$, given by $\sum_{M' \subseteq U} \mu^{M'}(m') \to \sum_{M' \subseteq U} \alpha^{M'}(m') = \sum_{M' \subseteq U} m'$. An element $m \in M$ is in the image of $\alpha^{R\text{fin}}$, so $\alpha$ is surjective. Assume that $m \in \text{colim}_{M' \subseteq U} M'$ is in the kernel.
of $\alpha$. By 3.2.25 there is an element $M'$ in $U$ and an $m'$ in $M'$ with $\mu^M(m') = m$. In $M$ one now has $m' = \alpha M(m') = \alpha \mu^M(m') = \alpha(m) = 0$. Thus, $\alpha$ is an isomorphism.

A colimit of a $U$-direct system is called *filtered* if the preordered set $(U, \leq)$ is filtered. The next result sums up as: filtered colimits are exact.

### 3.2.27 Proposition

Let $\{\alpha^u : M^u \to N^u\}_{u \in U}$ and $\{\beta^u : N^u \to X^u\}_{u \in U}$ be morphisms of $U$-direct systems in $\mathcal{C}(R)$. If the sequence

$$0 \longrightarrow M^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} X^u \longrightarrow 0$$

is exact for every $u \in U$ and $U$ is filtered, then the next sequence is exact,

$$0 \longrightarrow \operatorname{colim} M^u_{u \in U} \xrightarrow{\operatorname{colim} \alpha^u_{u \in U}} \operatorname{colim} N^u_{u \in U} \xrightarrow{\operatorname{colim} \beta^u_{u \in U}} \operatorname{colim} X^u_{u \in U} \longrightarrow 0.$$

**Proof.** By 3.2.12 it is sufficient to prove that $\alpha = \operatorname{colim} \alpha^u$ is injective. We write $\mu^\nu : M^\nu \to M^\nu$ and $\nu^\nu : N^\nu \to N^\nu$ for the morphisms in the direct systems. Let $m \in \ker \alpha$ and choose by 3.2.25 a $v \in U$ and an element $m^v$ in $M^v$ with $\mu^v(m^v) = m$, then one has $0 = \alpha \mu^\nu(m^v) = \nu^\nu \alpha^v(m^v)$. By 3.2.25 there is a $w \in U$ with $w \geq v$ and $0 = \nu^w \alpha^v(m^v) = \alpha^w \mu^w(m^v)$. Since $\alpha^w$ is injective, one has $\mu^w(m^v) = 0$ and, therefore, $m = \mu^v(m^v) = \mu^w \mu^w(m^v) = 0$. \hfill $\square$

### 3.2.28

Let $\{\mu^u : M^u \to M^v\}_{u \leq v}$ be a $U$-direct system in $\mathcal{C}(R)$. Consider the canonical morphisms

$$\mu_B : \operatorname{colim} B(M^u)_{u \in U} \longrightarrow B(\operatorname{colim} M^u)_{u \in U} \quad \text{and} \quad \mu_Z : \operatorname{colim} Z(M^u)_{u \in U} \longrightarrow Z(\operatorname{colim} M^u)_{u \in U}$$

obtained by applying 3.2.13 to the functors $B(-)$ and $Z(-)$; see 2.2.11. By definition, $\mu_B$ and $\mu_Z$ map the coset of an element $\sum_{u \in U} \nu^u(m^u)$ to $\sum_{u \in U} \mu^u(m^u)$. Note that $\mu_B$ is surjective; cf. (3.2.2.2). By (2.2.10.1) and 3.2.12 there is a commutative diagram with exact rows

$$\begin{array}{ccc}
\operatorname{colim} B(M^u)_{u \in U} & \longrightarrow & \operatorname{colim} M^u_{u \in U} \\
\mu_B & & \mu_C \\
0 & \longrightarrow & \operatorname{colim} M^u_{u \in U} \\
\downarrow & & \downarrow & \\
\operatorname{B}(\operatorname{colim} M^u)_{u \in U} & \longrightarrow & \operatorname{colim} M^u_{u \in U} & \longrightarrow & \operatorname{C}(\operatorname{colim} M^u)_{u \in U} & \longrightarrow & 0 .
\end{array}$$

It follows from the Snake Lemma 2.1.39 that $\mu_C$ is an isomorphism. Observe that if $i : C(M^u) \rightarrow \prod_{u \in U} C(M^u)_{u \in U}$ denotes the embedding, then $\mu_C$ maps the image of $\sum_{u \in U} \mu^u(m^u)$ to $[\sum_{u \in U} \mu^u(m^u)]_{B(\operatorname{colim} M^u)}$. Assume now that $U$ is filtered. By 3.2.27 the morphism $\varphi$ is injective, so $\mu_B$ is an isomorphism; again by the Snake Lemma. Finally, a similar commutative diagram based on (2.2.10.2) shows that $\mu_Z$ is an isomorphism. In particular, one has

$$\begin{align*}
(3.2.28.1) \quad & \operatorname{colim} B(M^u)_{u \in U} \cong B(\operatorname{colim} M^u)_{u \in U} \quad \text{and} \quad \operatorname{colim} Z(M^u)_{u \in U} \cong Z(\operatorname{colim} M^u)_{u \in U} .
\end{align*}$$
Filtered colimits commute with homology.

3.2.29 Theorem. Let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). The canonical map

\[
\text{colim}_{u \in U} \text{H}(M^u) \longrightarrow \text{H}(\text{colim}_{u \in U} M^u),
\]

given by \( \sum_{u \in U} \lambda^u(h^u) \mapsto \sum_{u \in U} \text{H}(\mu^u)(h^u) \), cf. 3.2.13, is a morphism of \( R \)-complexes. If \( U \) is filtered, then (3.2.29.1) is an isomorphism.

Proof. The map is a morphism of \( R \)-complexes by 3.2.13. In the following commutative diagram, the middle and left-hand vertical maps are the isomorphisms established in 3.2.28, and \( \mu_H \) is the canonical morphism (3.2.29.1),

\[
\begin{array}{c}
\text{colim}_{u \in U} B(M^u) \\
\text{colim}_{u \in U} Z(M^u) \\
\text{colim}_{u \in U} \text{H}(M^u)
\end{array} \quad \begin{array}{c}
\mu_B \\
\mu_Z \\
\mu_H
\end{array} \quad \begin{array}{c}
0 \\
0 \\
0
\end{array}
\]

It follows from (2.2.10.4) and 3.2.27 that the rows in the diagram are exact, and then the Five Lemma 2.1.37 implies that \( \mu_H \) is an isomorphism.

3.2.30 Corollary. Let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). If \( U \) is filtered and each complex \( M^u \) is acyclic, then \( \text{colim}_{u \in U} M^u \) is acyclic.

Under finiteness conditions on \( M \) the functor \( \text{Hom}_R(M, -) \) preserves colimits.

3.2.31 Proposition. Let \( M \) be an \( R \)-complex and let \( \{ \nu^u : N^u \to N^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). The canonical map

\[
\text{colim}_{u \in U} \text{Hom}_R(M^u, N^u) \longrightarrow \text{Hom}_R(M, \text{colim}_{u \in U} N^u),
\]

given by \( \sum_{u \in U} \lambda^u(\theta^u) \mapsto \sum_{u \in U} \text{Hom}_R(M, \nu^u)(\theta^u) = \sum_{u \in U} \nu^u \theta^u \), cf. 3.2.13, is a morphism of \( \mathbb{k} \)-complexes. If \( U \) is filtered and \( M \) is bounded and degreewise finitely presented, then (3.2.31.1) is an isomorphism.

Proof. The map is a morphism of \( \mathbb{k} \)-complexes by 3.2.13. Assume that \( M \) is bounded and degreewise finitely presented. It follows from 2.5.6 that the graded module \( M^\circ \) has a presentation \( L' \to L \to M^\circ \to 0 \), where \( L \) and \( L' \) are graded-free, bounded, and degreewise finitely generated \( R \)-modules. Consider the following commutative diagram, whose lower row is obtained by application of the functor \( \text{Hom}_R(-, \text{colim} N^u) \) to the presentation of \( M^\circ \),

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The rows are exact by left exactness of Hom, 2.3.12, and by exactness of filtered colimits 3.2.27. The canonical morphism (3.2.31.1) yields the vertical morphisms. To prove that $\chi^M$ is an isomorphism, it suffices by the Five Lemma 2.1.37 to argue that $\chi^L$ and $\chi^L'$ are isomorphisms. To this end, consider the commutative diagram

$$
\begin{array}{ccc}
\prod_{(u,v)\in \mathcal{V}(U)} \text{Hom}_R(L, N^{(u,v)}) & \rightarrow & \prod_{u\in U} \text{Hom}_R(L, N^u) \\
\downarrow_{\cong} & & \downarrow_{\cong} \\
\text{Hom}_R(L, \prod_{(u,v)\in \mathcal{V}(U)} N^{(u,v)}) & \rightarrow & \text{Hom}_R(L, \prod_{u\in U} N^u)
\end{array}
$$

where the left-hand and middle vertical maps are the isomorphisms from 3.1.32. The upper row is exact by the construction of colimits 3.2.2. Exactness of the lower row is a consequence of the unique extension property, 1.3.5, of the free $R$-modules $L_v$. It follows from the Five Lemma 2.1.37 that $\chi^L$ is an isomorphism. Similarly, $\chi^L'$ is an isomorphism, as desired.

### Telescopes

#### 3.2.32 Construction. Let $\{k^u: M^u \rightarrow M^{u+1}\}_{u\in \mathbb{Z}}$ be a sequence of morphisms in $\mathcal{C}(R)$. It determines a direct system $\{\mu^u: M^u \rightarrow M^v\}_{u \leq v}$ as follows: set

$$
\mu^u = 1_{M^u} \text{ for all } u \in \mathbb{Z} \quad \text{and} \quad \mu^v = k^{v-1} \cdots k^u \text{ for all } u < v \in \mathbb{Z}.
$$

Given additional sequences $\{\lambda^u: N^u \rightarrow N^{u+1}\}_{u\in \mathbb{Z}}$ and $\{\alpha^u: M^u \rightarrow N^u\}_{u\in \mathbb{Z}}$ of morphisms, such that $\alpha^{u+1}k^u = \lambda^u\alpha^u$ holds for all $u \in \mathbb{Z}$, it is elementary to verify that $\{\alpha^u\}_{u\in \mathbb{Z}}$ is a morphism of the direct systems determined by $\{k^u\}_{u\in \mathbb{Z}}$ and $\{\lambda^u\}_{u\in \mathbb{Z}}$.

#### 3.2.33 Definition. A sequence $\{k^u: M^u \rightarrow M^{u+1}\}_{u\in \mathbb{Z}}$ of morphisms in $\mathcal{C}(R)$ with $M^u = 0$ for $u \ll 0$ is called a telescope in $\mathcal{C}(R)$. The colimit $\text{colim}_{u\in \mathbb{Z}} M^u$ of the associated direct system, see 3.2.32, is called the colimit of the telescope in $\mathcal{C}(R)$.

Given telescopes $\{k^u: M^u \rightarrow M^{u+1}\}_{u\in \mathbb{Z}}$ and $\{\lambda^u: N^u \rightarrow N^{u+1}\}_{u\in \mathbb{Z}}$, a sequence of morphisms $\{\alpha^u: M^u \rightarrow N^u\}_{u\in \mathbb{Z}}$ that satisfy $\alpha^{u+1}k^u = \lambda^u\alpha^u$ for all $u \in \mathbb{Z}$ is called a morphism of telescopes. The morphism $\text{colim}_{u\in \mathbb{Z}} \alpha^u: \text{colim}_{u\in \mathbb{Z}} M^u \rightarrow \text{colim}_{u\in \mathbb{Z}} N^u$, see 3.2.32 and 3.2.9, is called the colimit of $\{\alpha^u: M^u \rightarrow N^u\}_{u\in \mathbb{Z}}$.  

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3.2.34. Let \( \{ \kappa^u : M^u \to M^{u+1} \}_{u \in \mathbb{Z}} \) be a telescope and let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) be the associated direct system in \( \mathcal{C}(R) \). Given an \( R \)-complex \( N \) and a sequence of morphisms \( \{ \alpha^u : M^u \to N \}_{u \in \mathbb{Z}} \) that satisfy \( \alpha^u = \alpha^{u+1} \kappa^u \) for all \( u \in \mathbb{Z} \), one has \( \alpha^u = \alpha^v \mu^w \) for all \( u \leq v \). By the universal property of colimits, there is a morphism \( \alpha : \text{colim}_{u \in \mathbb{Z}} M^u \to N \) in \( \mathcal{C}(R) \) with properties as described in 3.2.4.

3.2.35 Lemma. Let \( \{ \kappa^u : M^u \to M^{u+1} \}_{u \in \mathbb{Z}} \) be a telescope and let \( \{ \alpha^u : M^u \to N \}_{u \in \mathbb{Z}} \) be a sequence of morphisms with \( \alpha^u = \alpha^{u+1} \kappa^u \) for all \( u \in \mathbb{Z} \). If \( \alpha^u \) is injective for infinitely many \( u > 0 \), then \( \alpha : \text{colim}_{u \in \mathbb{Z}} M^u \to N \) from 3.2.34 is injective.

Proof. Let \( \{ \mu^u : M^u \to M^v \}_{u \in \mathbb{Z}} \) be the direct system associated to the telescope. By 3.2.25 every element \( m \) in \( \text{colim}_{u \in \mathbb{Z}} M^u \) has the form \( m = \mu^u(m^v) \) for some \( u \in \mathbb{Z} \) and \( m^v \in M^v \). Assume that \( \alpha(m) = \alpha(\mu^v(m^v)) = \alpha^v(m^v) \) is zero in \( N \).

By assumption, there exists \( v > u \) such that \( \alpha^v \) is injective, and thus the identity \( \alpha^v(\mu^v(m^v)) = \alpha^v(m^v) = 0 \) shows that one has \( \mu^u(m^v) = 0 \). It follows that the element \( m = \mu^u(m^v) = \mu^v(\mu^u(m^v)) \) is zero as well. \( \Box \)

3.2.36 Example. Let \( M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots \) be an ascending chain of \( R \)-complexes. The embeddings \( M^u \to M^{u+1} \) define a telescope whose colimit is isomorphic to the complex \( \bigcup_{u \in \mathbb{Z}} M^u \). This is immediate from 3.2.5 and 3.2.35.

3.2.37 Proposition. Let \( \{ \kappa^u : M^u \to M^{u+1} \}_{u \in \mathbb{Z}} \) be a telescope in \( \mathcal{C}(R) \). The following assertions hold.

(a) If \( \kappa^u = 0 \) holds for infinitely many \( u > 0 \), then one has \( \text{colim}_{u \in \mathbb{Z}} M^u = 0 \).

(b) If there exists an integer \( w \) such that \( \kappa^u \) is bijective for all \( u \geq w \), then the canonical map \( M^w \to \text{colim}_{u \in \mathbb{Z}} M^u \) is an isomorphism.

Proof. Let \( \{ \mu^u : M^u \to M^v \}_{u \in \mathbb{Z}} \) be the direct system associated to the telescope.

(a) By 3.2.25 every element \( m \) in \( \text{colim}_{u \in \mathbb{Z}} M^u \) has the form \( m = \mu^u(m^v) \) for some \( u \in \mathbb{Z} \) and \( m^v \in M^v \). It follows from the assumption that the map \( \mu^u \) is zero for some \( v > u \), and consequently one has \( m = \mu^v(m^v) = \mu^v(\mu^u(m^v)) = 0 \).

(b) Define a sequence \( \{ \alpha^u : M^u \to M^v \}_{u \in \mathbb{Z}} \) of morphisms in \( \mathcal{C}(R) \) as follows: set \( \alpha^w = 1_{M^w} \); set \( \alpha^u = \kappa^{w-1} \cdots \kappa^0 \) for \( u < w \); and set \( \alpha^u = (\kappa^{w-1} \cdots \kappa^0)^{-1} \) for \( u > w \). By construction one has \( \alpha^u = \alpha^{w+1} \kappa^u \) for all \( u \in \mathbb{Z} \), so by 3.2.34 there is a morphism \( \alpha : \text{colim}_{u \in \mathbb{Z}} M^u \to M^w \), given by \( \mu^u(m^v) \to \alpha^u(m^v) \). Evidently one has \( \alpha \mu^u = \alpha^w = 1_{M^w} \). It follows from 3.2.35 that \( \alpha \) is injective, and hence it is the inverse of \( \mu^w \). \( \Box \)

Exercises

E 3.2.1 Let \( \{ \mu^u : M^u \to M^v \}_{u \in U} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). Let \( \{ \tilde{\mu}^u : M^u \to C \}_{u \in U} \) be a family of morphisms that satisfy the following conditions. (1) One has \( \tilde{\mu}^u = \tilde{\mu}^v \mu^u \) for all \( u \leq v \). (2) For every family \( \{ \alpha^u : M^u \to N \}_{u \in U} \) of morphisms with \( \alpha^u = \alpha^v \mu^u \) for all \( u \leq v \) there exists a unique morphism \( \alpha : C \to N \) with \( \alpha \mu^u = \alpha^u \) for all \( u \in U \). Show that there is an isomorphism \( \phi : \text{colim}_{u \in U} M^u \to C \) with \( \phi \mu^u = \tilde{\mu}^u \) for every \( u \in U \). Conclude that the universal property determines the colimit uniquely up to isomorphism.
3.3 Limits

SYNOPSIS. Inverse system; limit; universal property; limit preserving functors; pullback; tower; Mittag-Leffler condition.

3.3.1 Definition. Let \((U, \leq)\) be a preordered set. A U-inverse system in \(\mathcal{C}(R)\) is a family \(\{v^{uw}: N^u \to N^w\}_{u \leq v}\) of morphisms in \(\mathcal{C}(R)\) with the following properties.

1. \(v^{uu} = 1_{N^u}\) for all \(u \in U\).
2. \(v^{uw}v^{vw} = v^{uw}\) for all \(u \leq v \leq w\) in \(U\).

E 3.2.2 (Cf. 3.2.6) Show that the colimit in \(\mathcal{C}(R)\) of a direct system of morphisms of graded \(R\)-modules is a graded \(R\)-module. Conclude, in particular, that \(\mathcal{M}_{gr}(R)\) has colimits.

E 3.2.3 (Cf. 3.2.6) Show that the colimit in \(\mathcal{C}(R)\) of a direct system of homomorphisms of \(R\)-modules is an \(R\)-module. Conclude, in particular, that the category \(\mathcal{M}(R)\) has colimits.

E 3.2.4 Fix a preordered set \(U\). Show that \(U\)-direct systems in \(\mathcal{C}(R)\) and their morphisms form an Abelian category and that the colimit is a right exact functor from this category to \(\mathcal{C}(R)\).

E 3.2.5 Generalize the result in 3.1.5 by showing that every functor \(F: \mathcal{C}(R) \to \mathcal{C}(S)\) that has a right adjoint preserves colimits.

E 3.2.6 (Cf. 3.2.17) Verify the isomorphism \(\text{colim}_{\mathcal{U}} M^u \cong \text{Coker}(-\alpha \beta)\) in 3.2.17.

E 3.2.7 Show that the categories \(\mathcal{M}(R)\) and \(\mathcal{M}_{gr}(R)\) have pushouts.

E 3.2.8 Let \(\{\mu^u: M' \to M^u\}_{u \in U}\) be a \(U\)-direct system in \(\mathcal{C}(R)\). Show that the inequalities
\[
\sup_{u \in U} (\text{colim}_v M^u) \leq \sup_{u \in U} \{\sup_{v \in U} M^u\} \quad \text{and} \quad \inf_{v \in U} (\text{colim}_u M^v) \geq \inf_{v \in U} \{\inf_{u \in U} M^v\}
\]
hold if \(U\) is filtered.

E 3.2.9 Let \(\{\alpha^u: M' \rightarrow M^u\}_{u \in U}\) be a \(U\)-direct system in \(\mathcal{C}(R)\), let \(\{\alpha^u: M'^u \to N^u\}_{u \in U}\) be a family of morphisms with \(\alpha^u = \alpha^v\mu^u\) for all \(u \leq v\), and denote by \(\alpha: \text{colim}_u M'^u \to N\) the canonical morphism. Assume that \(U\) is filtered and that for every \(u \in U\) there exists a \(v \geq u\) such that \(u^v\) is injective. Show that \(\alpha\) is injective.

E 3.2.10 Let \(\{\alpha^u: M'^u \to N^u\}_{u \in U}\) and \(\{\beta^u: N'^u \to X^u\}_{u \in U}\) be morphisms of \(U\)-direct systems in \(\mathcal{C}(R)\). Show that if the sequence \(0 \to M'^u \to N'^u \to X^u\) is exact for every \(u \in U\), then the sequence \(0 \to \text{Ker} A_u \to \text{Ker} A_v \to \text{Ker} A \) is exact. Show also that if each morphism \(\beta^u\) is surjective and \(U\) is filtered, then the morphism \(\text{Ker} A_u \to \text{Ker} A_v\) is surjective.

E 3.2.11 Let \(U\) be a preordered filtered set and let \(U'\) be a cofinal subset of \(U\); that is, for every \(u \in U\) there exists an \(u' \in U'\) with \(u' \geq u\). Show that for every \(U\)-direct system \(\{\mu^u: M' \to M^u\}_{u \in U}\) in \(\mathcal{C}(R)\) there is an isomorphism \(\text{colim}_{\mathcal{U}} M^u \cong \text{colim}_{\mathcal{U'}} M^u\).

E 3.2.12 Show that a filtered colimit of flat modules is a flat module.

E 3.2.13 Show that the following conditions on \(R\) are equivalent. (i) \(R\) is von Neumann regular. (ii) \(R/b\) is a flat \(R^b\)-module for every right ideal \(b\) in \(R\). (iii) Every \(R\)-module is flat.

E 3.2.14 Show that a left Noetherian and von Neumann regular ring is semi-simple.

E 3.2.15 As in 3.2.32 let \(\{\xi^u: M'^u \to M^{u+1}\}_{u \in \mathbb{Z}}\) be a sequence (not necessarily a telescope) of morphisms in \(\mathcal{C}(R)\). Show that the colimit of the associated direct system does not depend on \(\xi^u\) for \(u < 0\).

E 3.2.16 Show that every complex is the colimit of a telescope of bounded above complexes.

E 3.2.17 Let \(\{\xi^u: M'^u \to M^{u+1}\}_{u \in \mathbb{Z}}\) be a telescope. Show that \(\text{colim}_{u \in \mathbb{Z}} M^u\) can be realized as the cokernel of an injective endomorphism of \(\bigoplus_{u \in \mathbb{Z}} M^u\).
Any mention of a $U$-inverse system $\{N^u: N^v \to N^u\}_{u \leq v}$ includes the tacit assumption that $(U, \leq)$ is a preordered set. A $\mathbb{Z}$-inverse system is called an inverse system.

### 3.3.2 Construction.

Let $\{N^u: N^v \to N^u\}_{u \leq v}$ be a $U$-inverse system in $C(R)$. We describe the subcomplex of the product $\prod_{u \in U} N^u$ generated by the elements $(n^u)_{u \in U}$ with $n^u = v^{nu}(n^v)$ for all $u \leq v$ as the kernel of a morphism between products in $C(R)$.

Indeed, let $\nabla(U)$ be as in 3.2.2 and set $N^{(u,v)} = N^u$ for all $(u,v) \in \nabla(U)$. The assignment

$$ (n^u)_{u \in U} \mapsto (n^u - v^{nu}(n^v))_{(u,v) \in \nabla(U)} $$

defines a morphism of $R$-complexes

$$ \Delta^v: \prod_{u \in U} N^u \longrightarrow \prod_{(u,v) \in \nabla(U)} N^{(u,v)} $$

The kernel of $\Delta^v$ is denoted $\lim_{u \in U} N^u$.

Note that for every $u \in U$, restriction of the projection (3.1.14.1) yields a morphism of $R$-complexes,

(3.3.2.1) $v^u: \lim_{u \in U} N^u \longrightarrow N^u$,

and one has $v^u = v^{nu} v^v$ for all $u \leq v$.

**Remark.** As for colimits, it is standard to use notation that suppresses the morphisms $v^{nu}$, though the complex $\lim_{u \in U} N^u$ depends on them.

The next definition is justified by 3.3.4, which shows that the complex $\lim_{u \in U} M^u$ and the projections $v^u$ have the universal property that defines a limit. In any category, this property determines the limit uniquely up to isomorphism.

### 3.3.3 Definition.

For a $U$-inverse system $\{N^u: N^v \to N^u\}_{u \leq v}$ in $C(R)$ the complex $\lim_{u \in U} N^u$ together with the canonical morphisms $v^u$ constructed in 3.3.2, is called the limit of $\{N^u: N^v \to N^u\}_{u \leq v}$ in $C(R)$.

**Remark.** Other names for the limit defined above are inverse limit and projective limit; other symbols used for this gadget are $\lim_n$ and proj lim.

### 3.3.4 Lemma.

Let $\{N^u: N^v \to N^u\}_{u \leq v}$ be a $U$-inverse system in $C(R)$. For every family of morphisms $\{\alpha^u: M \to N^u\}_{u \in U}$ in $C(R)$ with $\alpha^u = v^{nu} \alpha^v$ for all $u \leq v$, there is a unique morphism $\alpha$ that makes the next diagram commutative for all $u \leq v$.

The morphism $\alpha$ is given by $m \mapsto (\alpha^u(m))_{u \in U}$.
3.3 Limits

**Proof.** Let \( \{a^u: M \to N^u\}_{u \in U} \) be a family of morphisms with \( a^u = v^{vu}a^v \) for all \( u \leq v \). The equalities \( a^u = v^{vu}a^v \) ensure that the morphism \( a: M \to \prod_{u \in U} N^u \) from 3.1.16, given by \( m \mapsto (a^u(m))_{u \in U} \), maps to the subcomplex \( \lim_{u \in U} N^u \).

It is evident from the definition that \( a \) satisfies \( a^u = v^u a \) for all \( u \in U \). Moreover, for any morphism \( a': M \to \lim_{u \in U} N^u \) that satisfies \( a^u = v^u a' \) for all \( u \in U \), one has \( a'(m) = (v^u(a'(m)))_{u \in U} = (a(m))_{u \in U} = a(m) \), so \( a \) is unique. \( \square \)

**3.3.5.** For \( \{v^{uv}: N^u \to N^v\}_{u \leq v} \) and \( \{a^u: M \to N^u\}_{u \in U} \) as in 3.3.4, notice that the morphism \( a: M \to \lim_{u \in U} N^u \) is injective if one has \( \bigcap_{u \in U} \ker a^u = 0 \).

**3.3.6.** It follows readily from 3.3.2 and 3.3.4 that the full subcategories \( \mathcal{M}(R) \) and \( \mathcal{M}_{gr}(R) \) of \( \mathcal{C}(R) \) are closed under limits.

**3.3.7 Example.** Let \( \{N^u\}_{u \in U} \) be a family of \( R \)-complexes. Endowed with the discrete order, \( U \) is a preordered set, and \( \{v^{nu} = 1^N\}_{u \in U} \) is a \( U \)-inverse system with \( \lim_{u \in U} N^u = \prod_{u \in U} N^u \) and \( v^u = n^u \) for all \( u \in U \). Thus, every product is a limit.

Every complex is a limit of bounded below complexes.

**3.3.8 Example.** Let \( N \) be an \( R \)-complex. The canonical surjections among the quotient complexes \( N_{\geq u} \) give rise to an inverse system \( \{v^{uv}: N_{\geq v} \to N_{\geq u}\}_{u \leq v} \) in \( \mathcal{C}(R) \). The canonical maps \( \beta^u: N \to N_{\geq -u} \) satisfy \( \beta^u = v^u \beta^v \) for all \( u \leq v \), so by the universal property of limits, there is a morphism \( \beta: N \to \lim_{u \in U} N_{\geq -u} \), given by \( \beta(n) = (\beta^u(n))_{u \in U} \). It is injective by construction. To see that \( \beta \) is surjective, let \( (n^u)_{u \in U} \) in \( \lim_{u \in U} N_{\geq -u} \) be homogeneous of degree \( -w \), and set \( n = n^w \) considered as an element in the module \( N_{-w} \). Then one has \( \beta^u(n) = 0 = n^u \) for \( u < w \), and for \( u \geq w \) one has \( \beta^u(n) = n^u = n^w \) as the homomorphisms \( v^{nu}_{w}: (N_{\geq -u})_{-w} \to (N_{\geq -w})_{-w} \) are the identity map on \( N_{-w} \) for \( w \leq u \). Thus, \( \beta \) is an isomorphism.

**3.3.9 Definition.** Let \( \{v^{uv}: N^u \to N^v\}_{u \leq v} \) and \( \{\mu^{vu}: M^v \to M^u\}_{u \leq v} \) be \( U \)-inverse systems in \( \mathcal{C}(R) \). A family of morphisms \( \{\beta^u: N^u \to M^u\}_{u \in U} \) in \( \mathcal{C}(R) \) that satisfies \( \beta^u v^{uv} = \mu^u v^{vu} \) for all \( u \leq v \) is called a *morphism of \( U \)-inverse systems*. Such a morphism is called injective (surjective) if each map \( \beta^u \) is injective (surjective).

Given a morphism \( \{\beta^u: N^u \to M^u\}_{u \in U} \) of \( U \)-inverse systems, it follows from the universal property of limits that the map given by \( (n^u)_{u \in U} \mapsto (\beta^u(n^u))_{u \in U} \) is the unique morphism that makes the next diagram commutative for all \( u \leq v \) in \( U \),

![Diagram](attachment:image.png)

This morphism is called the *limit* of \( \{\beta^u: N^u \to M^u\}_{u \in U} \) and denoted \( \lim_{u \in U} \beta^u \).
3.3.10. Let \(\{\nu^u : N^u \to N^u\}_{u \in \mathbb{V}}\) be a \(U\)-inverse system in \(\mathbb{C}(\mathbb{R})\) and let \(s\) be an integer. It follows from 2.2.2 and 3.1.20 that \(\{\Sigma^u \lim_{\partial \in \mathbb{U}} N^\partial = \lim_{\partial \in \mathbb{U}} \Sigma^u N^\partial\}\) is a \(U\)-inverse system with \(\Sigma^u \lim_{\partial \in \mathbb{U}} N^\partial = \lim_{\partial \in \mathbb{U}} \Sigma^u N^\partial\). Moreover, if \(\{\beta^u : N^u \to M^u\}_{u \in \mathbb{U}}\) is a morphism of \(U\)-inverse systems, then one has \(\Sigma^u \lim_{\partial \in \mathbb{U}} \beta^\partial = \lim_{\partial \in \mathbb{U}} \Sigma^u \beta^\partial\).

3.3.11 Example. Let \(\{\nu^u : N^u \to N^u\}_{u \in \mathbb{V}}\) be a \(U\)-inverse system. Because the maps \(\nu^u\) are morphisms in \(\mathbb{C}(\mathbb{R})\), the family \(\{\delta^{u^v} : N^u \to \Sigma^u N^v\}_{u \in \mathbb{U}}\) is a morphism of \(U\)-inverse systems. From the definitions one has \(\lim_{\partial \in \mathbb{U}} \partial M^\partial = \lim_{\partial \in \mathbb{U}} M^\partial\).

3.3.12 Construction. Consider \(U\)-inverse systems in \(\mathbb{C}(\mathbb{R})\),

\[
\{\chi^v : X^v \to X^v\}_{u \in \mathbb{V}}, \quad \{\nu^u : N^u \to N^u\}_{u \in \mathbb{V}}, \quad \text{and} \quad \{\mu^v : M^v \to M^v\}_{v \in \mathbb{V}}.
\]

Let \(\{\alpha^u : X^u \to N^u\}_{u \in \mathbb{U}}\) and \(\{\beta^u : N^u \to M^u\}_{u \in \mathbb{U}}\) be morphisms of \(U\)-inverse systems, such that the sequence \(X^u \to N^u \to M^u\) is exact for every \(u \in \mathbb{U}\). Let \(\nabla(U)\) be as in 3.2.2, and for all \((u,v) \in \nabla(U)\) set \(\alpha^{(u,v)} = \alpha^u\) and \(\beta^{(u,v)} = \beta^u\). In view of 3.1.19 and 3.3.9 there is a commutative diagram with exact columns and exact lower and middle rows

\[
\begin{array}{cccc}
\lim_{\partial \in \mathbb{U}} X^\partial & \xrightarrow{\lim_{\partial \in \mathbb{U}} \chi^\partial} & \lim_{\partial \in \mathbb{U}} N^\partial & \xrightarrow{\lim_{\partial \in \mathbb{U}} \beta^\partial} & \lim_{\partial \in \mathbb{U}} M^\partial \\
\prod_{u \in \mathbb{U}} X^u & \xrightarrow{\prod_{u \in \mathbb{U}} \chi^u} & \prod_{u \in \mathbb{U}} N^u & \xrightarrow{\prod_{u \in \mathbb{U}} \beta^u} & \prod_{u \in \mathbb{U}} M^u \\
\prod_{(u,v) \in \nabla(U)} X^{(u,v)} & \xrightarrow{\prod_{(u,v) \in \nabla(U)} \chi^{(u,v)}} & \prod_{(u,v) \in \nabla(U)} N^{(u,v)} & \xrightarrow{\prod_{(u,v) \in \nabla(U)} \beta^{(u,v)}} & \prod_{(u,v) \in \nabla(U)} M^{(u,v)} \\
\end{array}
\]

(3.3.12.1)

where the vertical morphisms \(\Delta\) are defined in 3.3.2.

The next statement sums up as: limits are left exact. Exactness of limits is a delicate issue, not to say a rare occurrence. A sufficient condition for exactness certain limits is given in 3.3.32, and an example of a non-exact limit is given in 3.3.33.

3.3.13 Lemma. Let \(\{\alpha^u : X^u \to N^u\}_{u \in \mathbb{U}}\) and \(\{\beta^u : N^u \to M^u\}_{u \in \mathbb{U}}\) be morphisms of \(U\)-inverse systems in \(\mathbb{C}(\mathbb{R})\). If the sequence

\[
0 \to X^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} M^u
\]

is exact for every \(u \in \mathbb{U}\), then the next sequence is exact,

\[
0 \to \lim_{\partial \in \mathbb{U}} X^\partial \xrightarrow{\lim_{\partial \in \mathbb{U}} \alpha^\partial} \lim_{\partial \in \mathbb{U}} N^\partial \xrightarrow{\lim_{\partial \in \mathbb{U}} \beta^\partial} \lim_{\partial \in \mathbb{U}} M^\partial.
\]

PROOF. Consider the commutative diagram (3.3.12.1). Assuming that each sequence \(0 \to X^u \to N^u \to M^u\) is exact, it follows from 3.1.19 that the morphisms
The functor \( \text{Hom}_R(M, -) \) from \( \mathcal{C}(R) \) to \( \mathcal{C}(k) \) preserves limits.

**3.3.16 Proposition.** Let \( \{\nu^u: N^u \rightarrow N^v\}_{u \leq v} \) be a \( U \)-inverse system in \( \mathcal{C}(R) \) and let \( M \) be an \( R \)-complex. The canonical map

\[
\text{Hom}_R(M, \lim_{u \in U} N^u) \rightarrow \lim_{u \in U} \text{Hom}_R(M, N^u),
\]

given by \( \vartheta \mapsto (\text{Hom}_R(M, \nu^u(\vartheta))_{u \in U} = (\nu^u \vartheta)_{u \in U} \), is an isomorphism in \( \mathcal{C}(k) \).
PROOF. The map is a morphism of \( \mathcal{C} \)-complexes by 3.3.14. There is a commutative diagram in \( \mathcal{C}(\mathcal{C}) \),

\[
\begin{array}{ccc}
0 \rightarrow \text{Hom}_R(M, \text{lim}_{u \in U} N^u) & \rightarrow & \text{Hom}_R(M, \prod_{u \in U} N^u) \\
& \downarrow \cong & \downarrow \cong \\
0 \rightarrow \text{lim}_{u \in U} \text{Hom}_R(M, N^u) & \rightarrow & \prod_{u \in U} \text{Hom}_R(M, N^u)
\end{array}
\]

where \( \kappa \) is the canonical morphism (3.3.16.1), and the middle and right-hand vertical maps are the isomorphisms from 3.1.25. The rows are exact by left exactness of Hom, 2.3.12, and the construction of limits, 3.3.2; it follows from the Five Lemma 2.1.37 that \( \kappa \) is an isomorphism.

3.3.17. Let \( F: \mathcal{C}(R)^{\text{op}} \rightarrow \mathcal{C}(S) \) be a functor and let \( \{ \mu^u: M^u \rightarrow M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \). It is elementary to verify that \( \{ F(\mu^u): F(M^u) \rightarrow F(M^v) \}_{u \leq v} \) is a \( U \)-inverse system in \( \mathcal{C}(S) \). For every \( u \in U \) let \( \lambda^u \) be the canonical morphism \( \lim_{v \in U} F(M^v) \rightarrow F(M^u) \). The canonical morphism (3.2.2.1) induces a morphism \( F(\lambda^u): F(\text{colim} M^u) \rightarrow F(M^u) \) for every \( u \in U \). By the universal property of limits, the map given by the assignment \( x \mapsto (F(\mu^u)(x))_{u \in U} \) is the unique morphism that makes the following diagram in \( \mathcal{C}(S) \) commutative for all \( u \leq v \),

\[
F(\text{colim}_{u \in U} M^u) \rightarrow \lim_{u \in U} F(M^u)
\]

3.3.18. Let \( F: \mathcal{C}(R)^{\text{op}} \rightarrow \mathcal{C}(S) \) be a functor and let \( \{ \alpha^u: M^u \rightarrow N^u \}_{u \in U} \) be morphisms of \( U \)-direct systems in \( \mathcal{C}(R) \). It is elementary to verify that there is a commutative diagram in \( \mathcal{C}(S) \),

\[
\begin{array}{ccc}
F(\text{colim} N^u) & \rightarrow & F(\text{colim} M^u) \\
\downarrow \lim F(\alpha^u) & & \downarrow \lim F(\alpha^u) \\
\text{lim} F(N^u) & \rightarrow & \text{lim} F(M^u)
\end{array}
\]

where the vertical maps are the canonical morphisms from 3.3.17.

Inverse systems and limits in \( \mathcal{C}(R)^{\text{op}} \) correspond to direct systems and colimits in \( \mathcal{C}(R) \), so \( \mathcal{C}(R)^{\text{op}} \) has limits by 3.2.4. Together with 3.3.16 the next result, therefore, sums up as: the Hom functor preserves limits.
3.3 Limits

3.3.19 Proposition. Let \( \{ \mu^u : M^u \to M^v \}_{u \leq v} \) be a \( U \)-direct system in \( \mathcal{C}(R) \) and let \( N \) be an \( R \)-complex. The canonical map

\[
\text{Hom}_R(\text{colim}_{u \in U} M^u, N) \to \lim_{u \in U} \text{Hom}_R(M^u, N),
\]

given by \( \theta \mapsto (\text{Hom}_R(\mu^u, N)(\theta))_{u \in U} = (\theta \mu^u)_{u \in U} \), is an isomorphism in \( \mathcal{C}(\mathbb{k}) \).

Proof. The map is a morphism of \( k \)-complexes by 3.3.17. There is a commutative diagram in \( \mathcal{C}(\mathbb{k}) \),

\[
0 \to \text{Hom}_R(\text{colim}_{u \in U} M^u, N) \to \text{Hom}_R(\bigoplus_{u \in U} M^u, N) \xrightarrow{\text{Hom}(\mu^u, N)} \text{Hom}_R(\prod_{(u,v) \in \mathcal{V}(U)} M^{(u,v)}, N) \to \lim_{u \in U} \text{Hom}_R(M^u, N) \to \prod_{(u,v) \in \mathcal{V}(U)} \text{Hom}_R(M^{(u,v)}, N),
\]

where \( \kappa \) is the canonical morphism (3.3.19.1), and the middle and right-hand vertical maps the isomorphisms from 3.1.28. The rows are exact by left exactness of \( \text{Hom} \), 2.3.12, and the construction of limits, 3.3.2; it follows from the Five Lemma 2.1.37 that \( \kappa \) is an isomorphism.

Pullbacks

3.3.20 Construction. Let \( U = \{ u, v, w \} \) be a set, preordered as follows \( v > u < w \). Given a diagram \( N \xrightarrow{\beta} Y \xleftarrow{\alpha} M \) in \( \mathcal{C}(R) \), set

\[
N^v = N, \quad N^u = Y, \quad N^w = M, \quad v^v = 1_N, \quad v^u = \beta, \quad v^w = 1_Y, \quad v^w = \alpha, \quad \text{and} \quad v^w = 1_M.
\]

This defines a \( U \)-inverse system in \( \mathcal{C}(R) \). It is straightforward to verify that the limit of this system is the kernel of the morphism \( (\beta - \alpha) : N \oplus M \to Y \).

3.3.21 Definition. For a diagram \( N \xrightarrow{\beta} Y \xleftarrow{\alpha} M \) in \( \mathcal{C}(R) \), the limit of the \( U \)-inverse system constructed in 3.3.20 is called the pullback of \( (\beta, \alpha) \) and denoted \( N \bowtie Y \).

Remark. As for the limit, the notation for the pullback supresses the morphisms. Other names for the pullback are fibered product and cartesian square.

3.3.22. Given morphisms \( \alpha : M \to Y \) and \( \beta : N \to Y \), the pullbacks of \( (\beta, \alpha) \) and \( (\alpha, \beta) \) are isomorphic via the map induced by the canonical isomorphism \( N \oplus M \cong M \oplus N \).

3.3.23 Construction. Given a diagram in \( N \xrightarrow{\beta} Y \xleftarrow{\alpha} M \) in \( \mathcal{C}(R) \), let

\[
\text{Hom}_R(\text{colim}_{u \in U} M^u, N) \to \lim_{u \in U} \text{Hom}_R(M^u, N),
\]

be the canonical morphisms (3.3.2.1); they are given by \((n,m) \mapsto n\) and \((n,m) \mapsto m\).

There is a commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & \downarrow & \\
\text{Ker} \alpha' & \xrightarrow{\bar{\beta}} & \text{Ker} \alpha & \\
\downarrow & & \downarrow & \\
0 & \xrightarrow{\bar{\alpha}} & \text{Ker} \beta' & \xrightarrow{\alpha'} N \oplus Y M \\
\downarrow & & \downarrow & \downarrow & \\
0 & \xrightarrow{0} & \text{Ker} \beta & \xrightarrow{\beta'} & N \oplus Y M \\
\end{array}
\]

(3.3.23.1)

where \(\bar{\alpha}\) and \(\bar{\beta}\) are restrictions of \(\alpha'\) and \(\beta'\).

**3.3.24.** Given a diagram \(N \xleftarrow{\alpha''} X \xrightarrow{\beta''} M\) in \(\mathcal{C}(R)\) with \(\beta\alpha'' = \alpha\beta''\), it follows from 3.3.4 that the assignment

\[x \mapsto (\alpha''(n), \beta''(m))\]

defines the unique morphism that makes the next diagram commutative,

\[
\begin{array}{cccc}
X & \xrightarrow{\beta''} & M \\
\downarrow & & \downarrow & \\
N \oplus Y M & \xrightarrow{\beta'} & M \\
\downarrow & & \downarrow & \\
N & \xrightarrow{\beta} & Y. \\
\end{array}
\]

**3.3.25 Proposition.** Let \(N \xleftarrow{\beta} Y \xrightarrow{\alpha} M\) be a diagram in \(\mathcal{C}(R)\). The following assertions hold for the morphisms in (3.3.23.1).

(a) If \(\alpha\) is surjective, then \(\alpha'\) is surjective.

(b) \(\bar{\beta}\) is an isomorphism, whence \(\alpha\) is injective if and only if \(\alpha'\) is injective.

(c) If \(\beta\) is surjective, then \(\beta'\) is surjective.

(d) \(\bar{\alpha}\) is an isomorphism, whence \(\beta\) is injective if and only if \(\beta'\) is injective.

**Proof.** By symmetry, see 3.3.22, it is sufficient to prove parts (a) and (b).

(a): Assume that \(\alpha\) is surjective. For every \(n\) in \(N\) there is then an \(m \in M\) with \(\alpha(m) = \beta(n)\). The pair \((n,m)\) is, therefore, in \(N \oplus Y M\) and one has \(n = \alpha'(n,m)\).

(b): The kernel of \(\alpha'\) consists of all pairs \((0,m)\) in \(N \oplus M\) with \(\alpha(m) = 0\).  

□
3.3 Limits

Towers

3.3.26 Example. Let \( p \) be a prime. The sequence \( \cdots \to \mathbb{Z}/p^4\mathbb{Z} \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) determines an inverse system whose limit is the \( \mathbb{Z} \)-module \( \mathbb{Z}_p \) of \( p \)-adic integers.

3.3.27 Construction. Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a sequence of morphisms in \( \mathcal{C}(R) \). It determines an inverse system \( \{ \nu^v : N^v \to N^u \}_{u \leq v} \) as follows: set

\[
\nu^u = 1_{N^u} \quad \text{for all } u \in \mathbb{Z} \quad \text{and} \quad \nu^v = \lambda^u \cdots \lambda^v \quad \text{for all } u < v \in \mathbb{Z}.
\]

Given additional sequences \( \{ \kappa^u : M^u \to M^{u-1} \}_{u \in \mathbb{Z}} \) and \( \{ \beta^u : N^u \to M^u \}_{u \in \mathbb{Z}} \) of morphisms, such that \( \beta^u \circ \lambda^u = \kappa^u \beta^u \) holds for all \( u \in \mathbb{Z} \), it is elementary to verify that \( \{ \beta^u \}_{u \in \mathbb{Z}} \) is a morphism of the inverse systems determined by \( \{ \lambda^u \}_{u \in \mathbb{Z}} \) and \( \{ \kappa^u \}_{u \in \mathbb{Z}} \).

3.3.28 Definition. A sequence \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) of morphisms in \( \mathcal{C}(R) \) with \( N^0 = 0 \) for \( u < 0 \) is called a tower in \( \mathcal{C}(R) \). The limit \( \lim_{u \in \mathbb{Z}} N^u \) of the associated inverse system, see 3.3.27, is called the limit of the tower in \( \mathcal{C}(R) \).

Given towers \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) and \( \{ \lambda'^u : M^u \to M^{u-1} \}_{u \in \mathbb{Z}} \) in \( \mathcal{C}(R) \), a sequence of morphisms \( \{ \beta^u : N^u \to M^u \}_{u \in \mathbb{Z}} \) that satisfy \( \beta^u \circ \lambda^u = \lambda'^u \beta^u \) for all \( u \in \mathbb{Z} \) is called a morphism of towers. The morphism \( \lim_{u \in \mathbb{Z}} \beta^u : \lim_{u \in \mathbb{Z}} N^u \to \lim_{u \in \mathbb{Z}} M^u \), see 3.3.27 and 3.3.9, is called the limit of \( \{ \beta^u : N^u \to M^u \}_{u \in \mathbb{Z}} \).

3.3.29. Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower and let \( \{ \nu^v : N^v \to N^u \}_{u \leq v} \) be the associated inverse system in \( \mathcal{C}(R) \). Given an \( R \)-complex \( M \) and a sequence of morphisms \( \{ \alpha^u : M \to N^u \}_{u \in \mathbb{Z}} \) that satisfy \( \alpha^u \circ \lambda^u = \lambda'^u \alpha^u \) for all \( u \in \mathbb{Z} \), one has \( \alpha^u = \nu^v \alpha^u \) for all \( u \leq v \). By the universal property of limits, there is a morphism \( \alpha : M \to \lim_{u \in \mathbb{Z}} N^u \) in \( \mathcal{C}(R) \) with properties as described in 3.3.4.

3.3.30 Example. Let \( N^0 \supseteq N^1 \supseteq N^2 \supseteq \cdots \) be a descending chain of \( R \)-complexes. The embeddings \( \lambda^u : N^u \to N^{u-1} \) define a tower; let \( \{ \nu^v : N^v \to N^u \}_{u \leq v} \) be the associated inverse system; see 3.3.27. Set \( N = \bigcap_{u \in \mathbb{Z}} N^u \), and for every \( u \) let \( \beta^u \) be the embedding \( N \to N^u \). One has \( \beta^u = \lambda^u \beta^{u+1} \) for all \( u \in \mathbb{Z} \), so there is an injective morphism \( \beta : N \to \lim_{u \in \mathbb{Z}} N^u \), given by \( \beta(n) = (\beta^u(n))_{u \in \mathbb{Z}} \); see 3.3.5 and 3.3.29. Let \( (n^u)_{u \in \mathbb{Z}} \) be an element in \( \lim_{u \in \mathbb{Z}} N^u \); one has

\[
0 = n^u - \lambda^{u+1}(n^{u+1}) = n^u - n^{u+1}
\]

for all \( u \in \mathbb{Z} \), so \( (n^u)_{u \in \mathbb{Z}} \) is in the image of \( \beta \). Thus, \( \beta \) is an isomorphism.

3.3.31 Proposition. Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \). The following assertions hold.

(a) If \( \lambda^u = 0 \) holds for infinitely many \( u > 0 \), then one has \( \lim_{u \in \mathbb{Z}} N^u = 0 \).

(b) If there exists an integer \( w \) such that \( \lambda^u \) is bijective for all \( u > w \), then the canonical map \( \lim_{u \in \mathbb{Z}} N^u \to N^w \) is an isomorphism.
PROOF. (a): Let \( n = (n^u)_{u \in \mathbb{Z}} \) be an element in \( \lim_{u \in \mathbb{Z}} N^u \); for \( w > u \) one then has \( n^u = \lambda^u + \cdots + \lambda^w (n^w) \). For every \( u \in \mathbb{Z} \) there is by assumption an integer \( w > u \) with \( \lambda^u = 0 \), whence one has \( n^u = 0 \) and, consequently, \( n = 0 \).

(b): Define a sequence of morphisms \( \{ \alpha^u: N^u \to N^u \}_{u \in \mathbb{Z}} \) in \( \mathcal{C}(R) \) as follows: set \( \alpha^u = 1_{N^u} \), so \( \alpha^u = \lambda^{u+1} \cdots \lambda^w \) for \( u < w \), and set \( \alpha^u = (\lambda^{u+1} \cdots \lambda^w)^{-1} \), for \( u > w \).

By construction one has \( \alpha^{w+1} = \lambda^w \alpha^w \) for all \( u \in \mathbb{Z} \), so by 3.3.29 there is a morphism \( \alpha: N^w \to \lim_{u \in \mathbb{Z}} N^u \) given by \( \alpha(n) = (\alpha^u(n))_{u \in \mathbb{Z}} \). Evidently, there are equalities \( \lambda^u \alpha = \alpha^u = N^u \). For \( n = (n^u)_{u \in \mathbb{Z}} \) in \( \lim_{u \in \mathbb{Z}} N^u \) one has \( \alpha^u(n) = \alpha(n^u) = (\alpha^u(n^u))_{u \in \mathbb{Z}} = (n^u)_{u \in \mathbb{Z}} = n \), where the penultimate equality holds by definition of the maps \( \alpha^u \) and the complex \( \lim_{u \in \mathbb{Z}} N^u \).

\[ \square \]

3.3.32 Theorem. Let \( \{ \alpha^u: X^u \to N^u \}_{u \in \mathbb{Z}} \) and \( \{ \beta^u: N^u \to M^u \}_{u \in \mathbb{Z}} \) be morphisms of towers in \( \mathcal{C}(R) \) and assume that the maps in the tower \( \{ \xi^u: X^u \to X_1^u \}_{u \in \mathbb{Z}} \) are surjective. If the sequence

\[ 0 \longrightarrow X^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} M^u \longrightarrow 0 \]

is exact for every \( u \in U \), then the next sequence is exact,

\[ 0 \longrightarrow \lim_{u \in \mathbb{Z}} X^u \xrightarrow{\lim_{u \in \mathbb{Z}} \alpha^u} \lim_{u \in \mathbb{Z}} N^u \xrightarrow{\lim_{u \in \mathbb{Z}} \beta^u} \lim_{u \in \mathbb{Z}} M^u \longrightarrow 0. \]

PROOF. Let \( \{ \xi^u \}_{u \leq v}, \{ \nu^u \}_{u \leq v}, \{ \mu^u \}_{u \leq v} \) be the inverse systems determined by the towers \( \{ \xi^u \}_{u \in \mathbb{Z}} \), \( \{ \nu^u \}_{u \in \mathbb{Z}} \), \( \{ \mu^u \}_{u \in \mathbb{Z}} \), and \( \{ \kappa^u: M^u \to M_1^u \}_{u \in \mathbb{Z}} \). Consider the associated commutative diagram (3.12.1). Assuming that each sequence \( 0 \to X^u \to N^u \to M^u \to 0 \) is exact, it follows from 3.1.19 and the Snake Lemma that it suffices to prove that the connecting morphism \( \delta: \lim_{u \in \mathbb{Z}} M^u \to \text{Coker} \mathcal{A}^u \) is the zero map. Set \( \tilde{\alpha} = \prod_{(u,v) \in V(U)} \alpha^{(u,v)}; \) given an element \( m \) in \( \lim_{u \in \mathbb{Z}} M^u \), the image \( \tilde{\delta}(m) \) in \( \text{Coker} \mathcal{A}^u \) is the coset of an element \( \tilde{x} = (\xi^{(u,v)})_{(u,v) \in V(U)} \) with \( \tilde{\alpha}(\tilde{x}) = \mathcal{A}^u(n) \) for a preimage \( n = (n^u)_{u \in \mathbb{Z}} \) of \( m \). Notice that for every \( u \in \mathbb{Z} \) one has

\[ \beta^u(n^u - \lambda^u+1(n^u+1)) = m^u - \kappa^u+1(m^u+1) = 0. \]

The goal is to prove that \( \tilde{x} \) is in the image of \( \mathcal{A}^u \). The map \( \tilde{\alpha} \) is injective, so it is sufficient to construct an element \( x \in \prod_{u \in \mathbb{Z}} X^u \) with

\[ \tilde{\alpha}(\mathcal{A}^u(x) - \tilde{x}) = 0. \]

Without loss of generality, assume that one has \( N^u = 0 \) for \( u < 0 \). It follows that \( \mathcal{A}^u(n)_{(u,v)} \) is zero for all \( (u,v) \in V(U) \) with \( u < 0 \). Set \( x^u = 0 \) for \( u < 0 \) and define \( x^u \) for \( u \geq 0 \) recursively as follows. Given \( x^u \) for \( u \geq -1 \) one can, in view of (\( \star \)) and because \( \xi^{u+1} \) is surjective, choose \( x^{u+1} \) such that the next equality holds,

\[ \alpha^u(x^u - \xi^{u+1}(x^{u+1})) = n^u - \lambda^u+1(n^{u+1}) = \mathcal{A}^u(n)_{(u,u+1)}. \]

Set \( x = (x^u)_{u \in \mathbb{Z}} \). The assumption that \( N^u = 0 \) for \( u < 0 \) yields \( (\tilde{\alpha}(\mathcal{A}^u(x) - \tilde{x}))_{(u,v)} = 0 \) for all \( (u,v) \in V(U) \) with \( u < 0 \). For \( u \geq 0 \) one has \( (\tilde{\alpha}(\mathcal{A}^u(x) - \tilde{x}))_{(u,u)} = 0 \) by
the definition of $\mathcal{A}^\nu$, see 3.3.2, and $(\hat{\alpha}(\mathcal{A}^\nu(x) - \chi))_{(a, u, n)} = 0$ in view of (c). An induction argument now finishes the proof. Indeed, fix $n \geq 2$ and assume that $(\hat{\alpha}(\mathcal{A}^\nu(x) - \chi))_{(a, u, n - 1)} = 0$ holds for all $u > 0$. In the computation below, the fourth equality follows from this hypothesis, and the last equality follows from the already established case $n = 2$.

$$
(\hat{\alpha}(\mathcal{A}^\nu(x) - \chi))_{(a, u, n)}
= (\hat{\alpha}(\mathcal{A}^\nu(x)))_{(a, u, n)} - (\mathcal{A}^\nu(n))_{(a, u, n)}
= \alpha^\nu(x^\mu - \chi^\nu(u, n)(x^\mu + n)) - (n^\mu - \nu^\mu(u, n)(n^\mu + n))
= \alpha^\nu(x^\mu - \chi^\nu(u, n - 1)(x^\mu + n - 1) + \chi^\nu(u, n - 1)(x^\mu + n - 1 - \xi^\nu(u, n + 1)(x^\mu + n)))
- (n^\mu - \nu^\mu(u, n)(n^\mu + n))
= \alpha^\nu(\chi^\nu(u, n - 1)(x^\mu + n - 1 - \xi^\nu(u, n)(x^\mu + n))) - (\nu^\mu(u, n - 1)(n^\mu + n - 1) - \nu^\mu(u, n)(n^\mu + n))
= \nu^\mu(u, n - 1)\alpha^\nu((\hat{\alpha}(\mathcal{A}^\nu(x) - \chi))_{(a, u, n - 1, u, n)}) = 0.
$$

**Remark.** One can realize the limit of a tower $\{\chi^\nu : N^u \to N^{u - 1}\}_{u \in \mathbb{Z}}$ as the kernel of a morphism $\prod_{u \in \mathbb{Z}} N^u \to \prod_{u \in \mathbb{Z}} N^u$, and that opens to a simpler proof of 3.3.32; see E 3.3.13.

**3.3.33 Example.** Let $m, n > 1$ be integers and relatively prime. Consider the following commutative diagram of $\mathbb{Z}$-modules

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
| & m & | \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \\
| & m & | \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
| & m & | \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \\
| & m & | \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
| & m & | \\
\downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \\
| & m & | \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
$$

The rows are exact and the vertical maps define towers with limits 0, 0, and $\mathbb{Z}/n\mathbb{Z}$, respectively; cf. 3.3.31(b). Thus the sequence of limits $0 \to 0 \to 0 \to \mathbb{Z}/n\mathbb{Z} \to 0$ is not exact.

**3.3.34.** Let $\{\chi^\nu : N^u \to N^{u - 1}\}_{u \in \mathbb{Z}}$ be a $U$-inverse system in $\mathcal{C}(R)$. By 3.3.14, applied to the functors $B(-)$ and $Z(-)$, see 2.2.11, there are canonical morphisms

$$
\lambda^B : B(\lim_{u \in U} N^u) \longrightarrow \lim_{u \in U} B(N^u) \quad \text{and} \quad \nu^Z : Z(\lim_{u \in U} N^u) \longrightarrow \lim_{u \in U} Z(N^u).
$$

Notice that both maps are injective. By (2.2.10.2) and 3.3.13 there is a commutative diagram with exact rows

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If each morphism \( \lambda \) map given by \( h \), that is, the morphisms in the system \( \{ H_u \} \). It follows from the Snake Lemma 2.1.39 that \( \nu \) where \( \nu \) has \( H_n \) is surjective. Let \( \nu : \text{lim} Z(N^u) \xrightarrow{\nu} \text{lim} Z(N^v) \) be a morphism of ROOF, where \( \nu \) is surjective. The map \( \nu \) is a morphism of ROOF, cf. 3.3.34. It follows from (2.2.10.4) and 3.3.32 that there is a commutative diagram with exact rows,\n
\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & Z(\lim_{u \in U} N^u) & \xrightarrow{} & \lim_{u \in U} N^u & \xrightarrow{} & H(\lim_{u \in U} N^u) & \xrightarrow{} & 0 \\
\quad & & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
(3.3.34.1) & & & & & & & & \\
0 & \xrightarrow{} & \lim_{u \in U} Z(M^u) & \xrightarrow{} & \lim_{u \in U} N^u & \xrightarrow{} & \Sigma \lim_{u \in U} B(N^u) & \xrightarrow{} & 0 \\
\quad & & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

The Snake Lemma 2.1.39 implies that \( \nu \) is an isomorphism. In particular, one has\n
\[
(3.3.34.2) \quad Z(\lim_{u \in U} N^u) \cong \lim_{u \in U} Z(N^u).
\]

**3.35 Theorem.** Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \). The canonical map \( \lambda^u \) is surjective, then the following assertions hold.

(a) The morphism \( \lambda^u \) is surjective.

(b) Let \( n \in \mathbb{Z} \); if each homomorphism \( H_{n+1}(\lambda^u) \) is surjective, then the degree \( n \) component of the canonical map \( \lambda^u \) is an isomorphism; that is, one has \( H_n(\lim_{u \in \mathbb{Z}} N^u) \cong H_n(\lim_{u \in \mathbb{Z}} N^u) \).

**Proof.** The map is a morphism of ROOF by 3.3.34. Let \( \{ \nu^u : \text{lim} Z(N^u) \to \text{lim} Z(N^v) \}_{u \leq v} \) be the inverse system in \( \mathcal{C}(R) \) determined by the tower \( \{ \lambda^u \}_{u \in \mathbb{Z}} \) and assume that each map \( \lambda^u \) is surjective.

(a): The morphisms \( \nu^u \) are surjective, so they are surjective on boundaries; that is, the morphisms in the system \( \{ \nu^u : B(M^u) \to B(M^v) \}_{u \leq v} \) are surjective; cf. 3.3.34. It follows from (2.2.10.4) and 3.3.32 that there is a commutative diagram with exact rows,\n
\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & B(\lim_{u \in \mathbb{Z}} N^u) & \xrightarrow{} & Z(\lim_{u \in \mathbb{Z}} N^u) & \xrightarrow{} & H(\lim_{u \in \mathbb{Z}} N^u) & \xrightarrow{} & 0 \\
\quad & & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
(\ast) & & & & & & & & \\
0 & \xrightarrow{} & \lim_{u \in \mathbb{Z}} B(N^u) & \xrightarrow{\nu} & \lim_{u \in \mathbb{Z}} Z(N^u) & \xrightarrow{\lambda^u} & \lim_{u \in \mathbb{Z}} H(N^u) & \xrightarrow{} & 0 ,
\end{array}
\]

where \( \nu^B \) and \( \nu^Z \) are the canonical morphisms from 3.3.34, and \( \nu^H \) is the morphism \( (3.3.35.1) \). It follows from the Snake Lemma 2.1.39 that \( \nu^H \) is surjective.

(b): By the Snake Lemma the degree \( n \) component of \( \nu^H \) is an isomorphism if (and only if) the degree \( n \) component of \( \nu^B \) is surjective. Let \( (b^u)_{u \in \mathbb{Z}} = (\partial^u_{n+1}(N^u))_{u \in \mathbb{Z}} \) be an element in \( \lim_{u \in \mathbb{Z}} B(N^u) \). It is in the image of \( \nu^B \) if the map \( \partial^u_{n+1} \lim_{u \in \mathbb{Z}} N^u = \lim_{u \in \mathbb{Z}} \partial^u_{n+1} \lim_{u \in \mathbb{Z}} N^u \to \lim_{u \in \mathbb{Z}} B(N^u) \) is surjective. Assume that the homomorphisms \( H_{n+1}(\lambda^u) \) are surjective for all \( u \in \mathbb{Z} \), then all the homomorphisms \( H_n(\lim_{u \in \mathbb{Z}} N^u) \) are surjective. As the homomorphisms \( \nu^u \) are, themselves, surjective, and hence surjective on boundaries, it follows that they are surjective on ker-
Fix a preordered set \( E \). Show that the limit in \( E \) is acyclic for all \( u \in \mathbb{Z} \), that is, the homomorphisms in the system \( \{ v^u_{n+1}: Z_{n+1}(N^u) \to Z_{n+1}(N^v) \}_{u \leq v} \) are surjective. For every \( u \in \mathbb{Z} \) there is an exact sequence

\[
0 \to Z_{n+1}(N^u) \to N^u_{n+1} \xrightarrow{\partial^u_{n+1}} B_n(N^u) \to 0,
\]

and it follows from 3.3.32 that \( \lim_{u \in \mathbb{Z}} \partial^u_{n+1} \) is surjective. \( \square \)

**3.3.36 Corollary.** Let \( \{ \lambda^u: N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \). If \( \lambda^u \) and \( H(\lambda^u) \) are surjective for all \( u \in \mathbb{Z} \), then the canonical map \( H(\lim_{u \in \mathbb{Z}} N^u) \to \lim_{u \in \mathbb{Z}} H(N^u) \) from 3.3.35 is an isomorphism. \( \square \)

**3.3.37 Corollary.** Let \( \{ \lambda^u: N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \). If \( \lambda^u \) is surjective and \( N^u \) is acyclic for all \( u \in \mathbb{Z} \), then the complex \( \lim_{u \in \mathbb{Z}} N^u \) is acyclic. \( \square \)

**Exercises**

**E 3.3.1** Let \( \{ v^u: N^v \to N^u \}_{u \leq v} \) be a \( U \)-inverse system in \( \mathcal{C}(R) \). Let \( \{ \varphi^u: L \to N^u \}_{u \in U} \) be a family of morphisms that satisfy the next conditions. (1) One has \( \varphi^v = v^v \varphi^v \) for all \( u \leq v \). (2) For every family \( \{ \alpha^u: M \to N^u \}_{u \in U} \) of morphisms with \( \alpha^u = v^u \alpha^v \) for all \( u \leq v \) there exists a unique morphism \( \alpha: M \to L \) with \( \varphi^v \alpha = \alpha^v \) for all \( u \in U \). Show that there is an isomorphism \( \varphi: L \to \lim_{u \in U} N^u \) with \( v^u \varphi = \varphi^u \) for every \( u \in U \). Conclude that the universal property determines the limit uniquely up to isomorphism.

**E 3.3.2** (Cf. 3.3.6) Show that the limit in \( \mathcal{C}(R) \) of an inverse system of morphisms of graded \( R \)-modules is a graded \( R \)-module. Conclude, in particular, that \( \mathcal{M}_h(R) \) has limits.

**E 3.3.3** (Cf. 3.3.6) Show that the limit in \( \mathcal{C}(R) \) of a inverse system of homomorphisms of \( R \)-modules is an \( R \)-module. Conclude, in particular, that the category \( \mathcal{M}(R) \) has limits.

**E 3.3.4** Fix a preordered set \( U \). Show that \( U \)-inverse systems in \( \mathcal{C}(R) \) and their morphisms form an Abelian category and that the limit is a left exact functor from this category to \( \mathcal{C}(R) \).

**E 3.3.5** Generalize the result in E 3.1.11 by showing that every functor \( F: \mathcal{C}(R) \to \mathcal{C}(S) \) that has a left adjoint preserves limits.

**E 3.3.6** (a) Show that \( U \)-inverse systems in \( \mathcal{C}(R)^{op} \) correspond to \( U \)-direct systems in \( \mathcal{C}(R) \) and that \( U \)-direct systems in \( \mathcal{C}(R)^{op} \) correspond to \( U \)-inverse systems in \( \mathcal{C}(R) \). (b) Show limits in \( \mathcal{C}(R)^{op} \) correspond to colimits in \( \mathcal{C}(R) \) and that colimits in \( \mathcal{C}(R)^{op} \) correspond to limits in \( \mathcal{C}(R) \).

**E 3.3.7** Let \( U \) be a preordered filtered set and let \( U' \) be a cofinal subset of \( U \); that is, for every \( u \in U \) there exists a \( u' \in U' \) with \( u' \geq u \). Show that for every \( U \)-inverse system \( \{ v^u: N^v \to N^u \}_{u \in U} \) in \( \mathcal{C}(R) \) there is an isomorphism \( \lim_{u \in U} N^u \cong \lim_{u \in U'} N^u \).

**E 3.3.8** (Cf. 3.3.20) Verify the isomorphism \( \lim_{u \in U} N^u \cong \ker(\beta - \alpha) \) in 3.3.20.

**E 3.3.9** Show that the categories \( \mathcal{M}(R) \) and \( \mathcal{M}_h(R) \) have pullbacks.

**E 3.3.10** Consider a diagram of \( R \)-complexes, not \textit{a priori} assumed to be commutative,

\[
\begin{array}{ccc}
X & \xrightarrow{\beta} & M \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
N & \xrightarrow{\delta} & Y
\end{array}
\]

(a) Show that \( Y \) is isomorphic \( M \sqcup_N N \) if and only if the next sequence is exact,
(b) Show that \( X \) is isomorphic to \( N \cap M \) if and only if the next sequence is exact,
\[
0 \to X \xrightarrow{\beta \alpha} M \oplus N \xrightarrow{\gamma - \delta} Y \to 0.
\]

E 3.3.11 As in 3.3.27 let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a sequence (not necessarily a tower) of morphisms in \( \mathcal{C}(R) \). Show that the limit of the associated inverse system does not depend on \( \lambda^u \) for \( u \ll 0 \).

E 3.3.12 Show that every complex is the limit of a tower of bounded below complexes.

E 3.3.13 Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower. Show that \( \lim_{u \in \mathbb{Z}} N^u \) can be realized as the kernel of an endomorphism of \( \prod_{u \in \mathbb{Z}} N^u \). Use this to give an alternative proof of 3.3.32.

E 3.3.14 Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \), let \( \{ \alpha^u : M \to N^u \}_{u \in \mathbb{Z}} \) be a sequence of morphisms with \( \alpha^{u-1} = \lambda^u \alpha^u \) for all \( u \in \mathbb{Z} \), and denote by \( \alpha : M \to \lim_{u \in \mathbb{Z}} N^u \) the canonical morphism. Assume that \( \alpha^u \) is surjective and \( \ker \alpha^u = \ker \alpha^{u-1} \) holds for \( u \gg 0 \). Show that \( \alpha \) is surjective.

E 3.3.15 Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \), let \( \{ \alpha^u : M \to N^u \}_{u \in \mathbb{Z}} \) be a sequence of morphisms with \( \alpha^{u-1} = \lambda^u \alpha^u \) for all \( u \in \mathbb{Z} \), and denote by \( \alpha : M \to \lim_{u \in \mathbb{Z}} N^u \) the canonical morphism. Assume that \( \alpha^u \) is surjective and \( \ker \alpha^u = \ker \alpha^{u-1} \) holds for \( u \gg 0 \). Show that \( \alpha \) is surjective.

E 3.3.16 Let \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be a tower in \( \mathcal{C}(R) \) with \( \lambda^u \) surjective for all \( u \in \mathbb{Z} \). Assume that \( N^u \) is acyclic for infinitely many \( u \gg 0 \). Show that \( \lim_{u \in \mathbb{Z}} N^u \) is acyclic.

E 3.3.17 Show that the conclusion in 3.3.37 may fail if the homomorphisms \( \lambda^u \) are surjective for infinitely many but not all \( u \in \mathbb{Z} \). **Hint:** Set \( N = 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \) and consider the tower given by \( \cdots \xrightarrow{m} N \xrightarrow{m} N \xrightarrow{m} N \xrightarrow{m} N \); cf. 3.3.33.
Chapter 4
Distinguished Morphisms

4.1 The Mapping Cone

SYNOPSIS. Mapping cone; homotopy equivalence; $\Sigma$-functor.

The mapping cone of a continuous map $f : X \to Y$ between topological spaces is a space glued together from $X$ and $Y$ via the map $f$. Here we explore the algebraic version of this construction. In Chap. 6 the mapping cone will be key to the construction of triangulated structures on the homotopy and derived categories.

4.1.1 Definition. Let $\alpha : M \to N$ be a morphism of $R$-complexes. The mapping cone of $\alpha$ is the complex with underlying graded module

$$(\text{Cone } \alpha)^g = \bigoplus \Sigma^g M^i \oplus N^g$$

and differential

$$\partial_{\text{Cone } \alpha} = \begin{pmatrix} \partial^N & \alpha \Sigma M \\ 0 & \partial \Sigma M \end{pmatrix},$$

where $\alpha \Sigma M$ is the degree $-1$ chain map from 2.2.3.

4.1.2. A straightforward computation shows that $\partial_{\text{Cone } \alpha}$ is square zero,

$$\partial_{\text{Cone } \alpha} \partial_{\text{Cone } \alpha} = \begin{pmatrix} \partial^N \partial^N & \partial^N \alpha \Sigma M + \alpha \Sigma M \partial \Sigma M \\ 0 & \partial \Sigma M \partial \Sigma M \end{pmatrix} = 0.$$

4.1.3. For every morphism of $R$-complexes $\alpha : M \to N$ there is a degreewise split exact sequence of $R$-complexes, known as the mapping cone exact sequence,

$$(\text{4.1.3.1}) 0 \to N \xrightarrow{(1_N)} \text{Cone } \alpha \xrightarrow{(0 \ 1 \Sigma M)} \Sigma M \to 0.$$

4.1.4 Theorem. Let $\alpha, \alpha' : M \to N$ be morphisms of $R$-complexes, and consider the following diagram whose rows are the exact sequences (4.1.3.1),
The morphisms $\alpha$ and $\alpha'$ are homotopic if and only if there exists a morphism $\gamma$ that makes the diagram commutative; moreover, any such morphism is an isomorphism.

PROOF. It follows from the Five Lemma that a morphism $\gamma: \text{Cone} \alpha \to \text{Cone} \alpha'$ that makes the diagram commutative is an isomorphism; moreover, it must have the form $\gamma = \left( 1^N \chi \right)$, for some homomorphism $\chi: \Sigma M \to N$ of degree 0. Writing $\gamma$ with this expression for $\chi$ one has

$$\gamma \partial^\text{Cone} \alpha - \partial^\text{Cone} \alpha' = \left( \begin{array}{cc} 0 & 1^N \chi \\ 0 & 1^N \alpha \end{array} \right) \left( \begin{array}{cc} 0 & 1^N \chi \\ 0 & 1^N \alpha \end{array} \right) \left( \begin{array}{cc} 0 & 1^N \chi \\ 0 & 1^N \alpha \end{array} \right) = \left( \begin{array}{cc} 0 & 1^N \chi \\ 0 & 1^N \alpha \end{array} \right) \left( \begin{array}{cc} 0 & 1^N \chi \\ 0 & 1^N \alpha \end{array} \right) = \left( \begin{array}{cc} 0 & 1^N \chi \\ 0 & 1^N \alpha \end{array} \right),$$

so $\gamma$ is a morphism if and only if $\alpha - \alpha' = \partial^N \sigma + \sigma \partial^M$ holds; that is, if and only if $\alpha$ and $\alpha'$ are homotopic. \qed

**Σ-FUNCTORS**

4.1.5 Definition. A functor $F: C(R) \to C(S)$ is called a $Σ$-functor if there exist a natural isomorphism $\varphi: FΣ \to ΣF$ of functors, such that for every morphism $\alpha: M \to N$ in $C(R)$ there exists an isomorphism $\tilde{\alpha}$ that makes the diagram

$$
\begin{array}{ccc}
F(N) & \xrightarrow{F(1^N)} & F(\text{Cone} \alpha) \\
\parallel & \quad & \parallel \\
F(N) & \xrightarrow{\left( \begin{array}{cc} 0 & 1^N \alpha \\ 0 & 1^N \alpha \end{array} \right)} & \text{Cone} F(\alpha) \\
\parallel & \quad & \parallel \\
F(N) & \xrightarrow{\left( \begin{array}{cc} 0 & 1^N \alpha \\ 0 & 1^N \alpha \end{array} \right)} & \Sigma F(M)
\end{array}
\xrightarrow{\sim} \tilde{\alpha} \xrightarrow{\varphi M} \Sigma F(M)
$$

in $C(S)$ commutative.

4.1.6 Definition. A functor $G: C(R)^{op} \to C(S)$ is called a $Σ$-functor if there exist a natural isomorphism $\varphi: GΣ^{-1} \to ΣG$ of functors, such that for every morphism $\alpha: M \to N$ in $C(R)$ there exists an isomorphism $\tilde{\alpha}$ that makes the diagram

$$
\begin{array}{ccc}
C(N) & \xrightarrow{C(1^N)} & C(\text{Cone} \alpha) \\
\parallel & \quad & \parallel \\
C(N) & \xrightarrow{\left( \begin{array}{cc} 0 & 1^N \alpha \\ 0 & 1^N \alpha \end{array} \right)} & \text{Cone} C(\alpha) \\
\parallel & \quad & \parallel \\
C(N) & \xrightarrow{\left( \begin{array}{cc} 0 & 1^N \alpha \\ 0 & 1^N \alpha \end{array} \right)} & \Sigma C(M)
\end{array}
\xrightarrow{\sim} \tilde{\alpha} \xrightarrow{\varphi M} \Sigma C(M)
$$
in \( \mathcal{C}(S) \) commutative.

We prove below that the Hom and tensor product functors are \( \Sigma \)-functors. The homology functor \( H \) commutes with \( \Sigma \) by 2.2.13; however, \( H \) is not a \( \Sigma \)-functor since the complexes \( H(\text{Cone} \alpha) \) and \( \text{Cone} H(\alpha) \) are, in general, not isomorphic. Indeed, for every \( R \)-complex \( M \) one has \( H(\text{Cone} 1^M) = 0 \) by 4.2.18 and 4.2.20, whereas if \( M \) is not acyclic then \( H(1^M) = 1^H(1^M) \) is not the zero morphism, and hence \( \text{Cone} H(1^M) \neq 0 \).

4.1.7 Proposition. Let \( M \) be an \( R \)-complex. The functor \( \text{Hom}_R(M, -) \) is a \( \Sigma \)-functor; in particular, there is an isomorphism of \( k \)-complexes,

\[
\text{Cone}(\text{Hom}_R(M, \beta)) \cong \text{Hom}_R(M, \text{Cone} \beta),
\]

for every morphism \( \beta \) of \( R \)-complexes.

**Proof.** First note that by 2.3.16 there is a natural isomorphism

\[
\varphi^N: \text{Hom}_R(M, \Sigma N) \rightarrow \Sigma \text{Hom}_R(M, N);
\]

namely \( \varphi^N = \delta^\text{Hom}_R(M,N) \). To prove that \( \text{Hom}_R(M, -) \) is a \( \Sigma \)-functor, it must be shown that for every morphism \( \beta: N \rightarrow N' \) of \( R \)-complexes there exists an isomorphism \( \beta \) that makes the following diagram in \( \mathcal{C}(k) \) commutative.

\[
\begin{array}{ccc}
\text{Hom}_R(M, N') & \xrightarrow{\text{Hom}_R(M, \Sigma N)} & \text{Hom}_R(M, \text{Cone} \beta) \\
\downarrow & & \downarrow \delta \\
\text{Hom}_R(M, N') & \xrightarrow{\text{Hom}_R(M, \Sigma N)} & \text{Cone} \text{Hom}_R(M, \beta) \\
\end{array}
\]

(\( \star \))

\[
\text{Hom}_R(M, N') \xrightarrow{(1)_{\text{Hom}_R(M,N')}} \text{Cone} \text{Hom}_R(M, \beta) \xrightarrow{(0)_{\text{Hom}_R(M,N)}} \Sigma \text{Hom}_R(M, N).
\]

To define \( \beta \), observe that on the level of graded modules one has

\[
\text{Hom}_R(M, \text{Cone} \beta) \cong \text{Hom}_R(M, \Sigma N),
\]

and similarly,

\[
(\text{Cone} \text{Hom}_R(M, \beta))^\Sigma = \text{Hom}_R(M, \Sigma^N) \oplus \Sigma \text{Hom}_R(M, N^2).
\]

These equalities, combined with the fact that the functor \( \text{Hom}_R(M^2, -) \) is additive and \( \varphi^N \) is an isomorphism, show that one gets an isomorphism of graded modules, \( \beta: \text{Hom}_R(M, \text{Cone} \beta) \rightarrow (\text{Cone} \text{Hom}_R(M, \beta))^\Sigma \), by setting...
for a homomorphism \( \bar{\theta} : M \to \text{Cone}\beta \). To show that \( \tilde{\beta} \) is a morphism, and hence an isomorphism of complexes, note first that the definition of \( \varphi^N \) and (2.2.3.1) yield

\[
\text{Hom}(M, \beta) \psi_{N-1}^\Sigma \text{Hom}(M, N) \varphi^N = \text{Hom}(M, \beta) \text{Hom}(M, \psi_{N-1}^\Sigma) = \text{Hom}(M, \beta \psi_{N-1}^\Sigma).
\]

Using this identity and the fact that \( \varphi^N \) is a morphism of complexes, one gets

\[
\partial_{\text{Cone}\text{Hom}(M, \beta)} \tilde{\beta}(\theta) = \left( \begin{array}{cc}
\partial_{\text{Hom}(M, N)} & \partial_{\Sigma \text{Hom}(M, N)} \\
0 & \rho_{\Sigma \text{Hom}(M, N)}
\end{array} \right) \varphi^N((1^N 0)\theta)
\]

Thus \( \tilde{\beta} \) is a morphism of complexes.

It remains to verify that the diagram (\( \ast \)) is commutative. The computation

\[
(0 1 \Sigma \text{Hom}(M, N)) \tilde{\beta}(\theta) = (0 1 \Sigma \text{Hom}(M, N)) \left( \begin{array}{c}
\varphi^N((1^N 0)\theta) \\
\varphi^N((0 1 \Sigma^N)\theta)
\end{array} \right)
\]

shows that the right-hand square in (\( \ast \)) is commutative. A similar simple computation shows that the left-hand square is commutative. \( \square \)

4.1.8 Proposition. Let \( N \) be an \( R \)-complex. The functor \( \text{Hom}_R(\cdot, N) \) is a \( \Sigma \)-functor; in particular there is an isomorphism of \( \Sigma \)-complexes,

\[
\text{Cone} \text{Hom}_R(\alpha, N) \cong \Sigma \text{Hom}_R(\text{Cone} \alpha, N),
\]

for every morphism \( \alpha \) of \( R \)-complexes.

Proof. Similar to the proof of 4.1.7. \( \square \)

4.1.9 Proposition. Let \( M \) be an \( R^\circ \)-complex. The functor \( M \otimes_R \cdot \) is a \( \Sigma \)-functor; in particular, there is an isomorphism of \( \Sigma \)-complexes,
4.1 The Mapping Cone

\[ \text{Cone} (M \otimes_R \beta) \cong M \otimes_R \text{Cone} \beta, \]

for every morphism \( \beta \) of \( R \)-complexes.

**Proof.** First note that by 2.4.12 there is a natural isomorphism

\[ \varphi^N : M \otimes_R \Sigma N \rightarrow \Sigma (M \otimes_R N); \]

namely \( \varphi^N = \varsigma^M \otimes_R \Sigma N \circ (M \otimes_R \varsigma^N) \). To prove that \( M \otimes_R - \) is a \( \Sigma \)-functor, it must be shown that for every morphism \( \beta : N \rightarrow N' \) of \( R \)-complexes there exists an isomorphism \( \breve{\beta} \) that makes the following diagram in \( \mathcal{C}(k) \) commutative.

\[
\begin{array}{ccc}
M \otimes_R N' & \xrightarrow{M \otimes ((1')^N)} & M \otimes_R \text{Cone} \beta & \xrightarrow{M \otimes ((0, 1) \Sigma N)} & M \otimes_R \Sigma N \\
\downarrow \quad \cong \quad \beta & \quad \cong \quad \varphi^N & \quad \cong \quad \varphi^N & \quad \cong \quad \varphi^N & \\
M \otimes_R N' & \xrightarrow{(1^M \otimes N')_0} & \text{Cone}(M \otimes_R \beta) & \xrightarrow{(0, 1) \Sigma (M \otimes_R N)} & \Sigma (M \otimes_R N).
\end{array}
\]

To define \( \breve{\beta} \), note that on the level of graded modules one has

\[ (M \otimes_R \text{Cone} \beta)^\natural = M^\natural \otimes_R (\text{Cone} \beta)^\natural = M^\natural \otimes_R (N'^\natural \oplus \Sigma N^\natural), \]

and

\[ (\text{Cone}(M \otimes_R \beta))^\natural = (M^\natural \otimes_R N'^\natural) \oplus (M^\natural \otimes_R N^\natural). \]

These equalities, combined with the fact that the functor \( M^\natural \otimes_R - \) is additive and \( \varphi^N \) is an isomorphism, show that one defines an isomorphism of graded modules, \( \breve{\beta} : (M \otimes_R \text{Cone} \beta)^\natural \rightarrow (\text{Cone}(M \otimes_R \beta))^\natural \), by setting

\[ \breve{\beta} \left( m \otimes \begin{pmatrix} n' \\ ^\natural 1 (n) \end{pmatrix} \right) = \begin{pmatrix} m \otimes n' \\ \varphi^N (m \otimes ^\natural 1 (n)) \end{pmatrix} \]

for an elementary tensor in \( M \otimes_R \text{Cone} \beta \). To show that \( \breve{\beta} \) is a morphism, and hence an isomorphism of complexes, note first that the definition of \( \varphi^N \) and (2.2.3.1) yield

\[ (M \otimes \beta)^{\Sigma (M \otimes_R N)} \varphi^N = (M \otimes \beta) (M \otimes_{\Sigma \beta^{-1}} N^\varphi) = M \otimes (\beta \varsigma^{-1}) \]

Using this identity and the fact that \( \varphi^N \) is a morphism of complexes, one gets
There is an isomorphism of complexes for every morphism \( \alpha \).

**4.1.10 Proposition.** Let \( \alpha \).

Thus \( \tilde{\beta} \) is a morphism of complexes.

It remains to verify that the diagram \( (\ast) \) is commutative. The computation

\[
(0 \ 1^{\Sigma(M \otimes N)}) \tilde{\beta} \left( m \otimes \left( n' \right) \right) = (0 \ 1^{\Sigma(M \otimes N)}) \left( m \otimes n' \right) \\
= \varphi^{N}(m \otimes \varsigma^{N}_{1}(n)) \\
= \varphi^{N}(M \otimes (0 \ 1^{\Sigma(N)})) \left( m \otimes \left( n' \right) \right),
\]

shows that the right-hand square in \( (\ast) \) is commutative. A similar simple computation shows that the left-hand square is commutative.

\[\square\]

**4.1.10 Proposition.** Let \( N \) be an \( R \)-complex. The functor \(- \otimes_{R} N\) is a \( \Sigma \)-functor; in particular, there is an isomorphism of \( \Sigma \)-complexes,

\[
\text{Cone}(\alpha \otimes_{R} N) \cong (\text{Cone} \alpha) \otimes_{R} N,
\]

for every morphism \( \alpha \) of \( R^{0} \)-complexes.

**Proof.** Similar to the proof of 4.1.9.

\[\square\]

**Exercises**

**E 4.1.1** Let \( f : X \rightarrow Y \) be a continuous map of topological spaces. The mapping cone, \( \text{Cone} \ f \), is defined as the quotient space of \( (X \times [0,1]) \cup Y \) with respect to the equivalence relation \( (x,0) \sim (x',0) \) and \( (x,1) \sim f(x) \) for all \( x,x' \in X \). Denote by \( S(-) \) the singular chain com-
plex functor, cf. 2.1.22 and E 2.1.7. Show that the complexes \( S(\text{Cone} f) \) and \( \text{Cone} S(f) \) are homotopy equivalent.

E 4.1.2 Let \( \alpha : M \to N \) be an isomorphism of complexes. Show that \( \text{Cone} \alpha \) is null-homotopic.

4.2 Quasi-isomorphisms

SYNOPSIS. Quasi-isomorphism; mapping cone; contractible complex; semi-simple module.

More often than not, it is the homology of a complex that one is interested in; more so than the complex itself. For example, it is the homology of the singular chain complex \( S(X) \) that yields information about “holes” in the space \( X \). Similarly, we shall later read off invariants of modules over commutative rings from the homology of Koszul complexes. We have already seen that a homotopy equivalence \( \alpha : M \to N \) induces an isomorphism \( H(\alpha) : H(M) \to H(N) \). The class of homology preserving morphisms is, however, much wider.

4.2.1 Definition. A morphism \( \alpha : M \to N \) in \( C(R) \) is called a quasi-isomorphism if the induced morphism \( H(\alpha) : H(M) \to H(N) \) is an isomorphism.

A quasi-isomorphism is marked by a ’\( \simeq \)’ next to the arrow.

REMARK. Another word for quasi-isomorphism is homology isomorphism.

4.2.2. By 2.2.27 every homotopy equivalence is a quasi-isomorphism.

Given a quasi-isomorphism of \( R \)-complexes \( \alpha : M \to N \) there need not exist a morphism \( \beta : N \to M \) with \( H(\beta) = H(\alpha)^{-1} \). Moreover, for \( R \)-complexes \( M \) and \( N \) with \( H(M) \cong H(N) \) there need not exist a quasi-isomorphism \( M \to N \) or \( N \to M \). Examples follow below.

4.2.3 Example. There is a quasi-isomorphism of \( \mathbb{Z} \)-complexes

\[
\begin{array}{ccc}
0 & \to & \mathbb{Z} \\
& \searrow & \downarrow 2 \\
& & \mathbb{Z} \\
0 & \to & 0 \\
& \searrow & \downarrow \mathbb{Z}/2\mathbb{Z} \\
& & 0 \\
\end{array}
\]

but there is not even a non-zero morphism in the opposite direction, as the zero map is the only homomorphism from \( \mathbb{Z}/2\mathbb{Z} \) to \( \mathbb{Z} \). In particular, this quasi-isomorphism is not a homotopy equivalence.

4.2.4 Example. Set \( R = \mathbb{k}[x,y] \). The complexes

\[
M = 0 \to R/(x) \xrightarrow{y} R/(x) \to 0 \\
N = 0 \to R/(y) \xrightarrow{x} R/(y) \to 0
\]
concentrated in degrees 1 and 0 have isomorphic homology \( H(M) \cong k \cong H(N) \), but there are no non-zero morphisms between them and hence no quasi-isomorphism.

**4.2.5 Example.** Let \( M \) be a smooth real manifold. It is a theorem that the embedding \( S^\infty(M) \hookrightarrow S(M) \) from 2.1.30 is a quasi-isomorphism; see [36]. Also the morphism of \( \mathbb{R} \)-complexes \( \Omega(M) \to \text{Hom}_\mathbb{R}(S^\infty(M), \mathbb{R}) \) from 2.1.30 is a quasi-isomorphism; piecing these two results together one arrives at de Rham’s theorem; see [47].

**4.2.6.** Let \( M \) be an \( R \)-complex. By 2.5.18 the embedding \( \tau^M_0 : M_0 \to M \) is a quasi-isomorphism for every \( n \leq \inf M \), and the canonical morphism \( \tau^M_0 : M \to M_0 \) is a quasi-isomorphism for every \( n \geq \sup M \).

**4.2.7 Proposition.** Let \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) be morphisms of \( R \)-complexes. If \( \alpha^u \) is a quasi-isomorphism for every \( u \in U \), then the coproduct \( \coprod_{u \in U} \alpha^u \) and the product \( \prod_{u \in U} \alpha^u \) are quasi-isomorphisms.

**PROOF.** A coproduct, or a product, of isomorphisms is an isomorphism. Homology, as a functor, preserves coproducts and products by 3.1.11 and 3.1.24, and the assertions now follow from 3.1.10 and 3.1.23.

**4.2.8 Proposition.** Let \( \{ \alpha^u : M^u \to N^u \}_{u \in U} \) be a morphism of \( U \)-direct systems in \( \mathcal{C}(R) \). If \( U \) is filtered and \( \alpha^u \) is a quasi-isomorphism for every \( u \in U \), then the colimit \( \text{colim}_{u \in U} \alpha^u \) is a quasi-isomorphism.

**PROOF.** A colimit of isomorphisms is an isomorphism. Homology, as a functor, preserves filtered colimits by 3.2.29, and the assertion now follows from 3.2.14.

**4.2.9 Proposition.** Let \( \{ \kappa^u : M^u \to M^{u-1} \}_{u \in \mathbb{Z}} \) and \( \{ \lambda^u : N^u \to N^{u-1} \}_{u \in \mathbb{Z}} \) be towers in \( \mathcal{C}(R) \), and let \( \{ \alpha^u : M^u \to N^u \}_{u \in \mathbb{Z}} \) be a morphism of towers. If for all \( u \in \mathbb{Z} \) the morphisms \( \kappa^u, \lambda^u, H(\kappa^u), \) and \( H(\lambda^u) \) are surjective and \( \alpha^u \) is a quasi-isomorphism, then the limit \( \text{lim}_{u \in \mathbb{Z}} \alpha^u \) is a quasi-isomorphism.

**PROOF.** A limit of isomorphisms in \( \mathcal{C}(R) \) is an isomorphism. The canonical morphisms \( H(\text{lim}_{u \in \mathbb{Z}} N^u) \to \text{lim}_{u \in \mathbb{Z}} H(N^u) \) and \( H(\text{lim}_{u \in \mathbb{Z}} M^u) \to \text{lim}_{u \in \mathbb{Z}} H(M^u) \) are isomorphisms by 3.3.36, and the assertion now follows from 3.3.15.

**4.2.10.** Consider a commutative diagram of \( R \)-complexes,

\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
\downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0,
\end{array}
\]

with exact rows. If two of the morphisms \( \varphi', \varphi, \) and \( \varphi'' \) are quasi-isomorphisms, then so is the third; this follows immediately from 2.2.19 and the Five Lemma.

**4.2.11.** Let \( F : \mathcal{C}(R) \to \mathcal{C}(S) \) and \( G : \mathcal{C}(R)^{\text{op}} \to \mathcal{C}(S) \) be functors induced by exact functors \( \mathcal{M}(R) \to \mathcal{M}(S) \) and \( \mathcal{M}(R)^{\text{op}} \to \mathcal{M}(S) \). It follows from 2.2.15 that \( F(\alpha) \) and \( G(\alpha) \) are quasi-isomorphisms in \( \mathcal{C}(S) \) for every quasi-isomorphism \( \alpha \) in \( \mathcal{C}(R) \).
4.2 Quasi-isomorphisms

Recall from 2.2.11 that a morphism \( \alpha : M \to N \) of complexes restricts to a morphism on cycles, \( Z(M) \to Z(N) \), and to a morphism on boundaries \( B(M) \to B(N) \).

**4.2.12 Lemma.** Let \( \alpha \) be a quasi-isomorphism of \( R \)-complexes. The following conditions are equivalent.

(i) \( \alpha \) is surjective.

(ii) \( \alpha \) is surjective on boundaries.

(iii) \( \alpha \) is surjective on cycles.

(iv) \( \alpha \) is surjective on cycles and boundaries.

**Proof.** It is immediate from 2.1.24 that a surjective morphism is surjective on boundaries, whence (i) implies (ii). An application of the Snake Lemma 2.1.39 to the diagram (2.2.12.1) shows that conditions (ii) and (iii) are equivalent and, therefore, that they both imply (iv). To prove that (iv) implies (i), apply the Snake Lemma to the following commutative diagram in \( C(R) \),

\[
\begin{array}{ccccccc}
0 & \to & Z(M) & \to & M & \xrightarrow{\partial M} & \Sigma B(M) & \to & 0 \\
\alpha & & \downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\Sigma \alpha} & & \\
0 & \to & Z(N) & \to & N & \xrightarrow{\partial N} & \Sigma B(N) & \to & 0.
\end{array}
\]

**Remark.** For a result dual to 4.2.12 see E 4.2.5.

**4.2.13.** Let \( \alpha : M \to N \) be a morphism of \( R \)-complexes and consider the mapping cone exact sequence from 4.1.3,

\[
0 \to N \xrightarrow{(1^N) \ 0} \text{Cone} \alpha \xrightarrow{(0 \ 1^{\Sigma M})} \Sigma M \to 0.
\]

By (2.2.17.1) there is an induced exact sequence,

\[
(4.2.13.1) \quad H(N) \to H(\text{Cone} \alpha) \to H(\Sigma M) \xrightarrow{\delta} \Sigma H(N) \to \Sigma H(\text{Cone} \alpha),
\]

where \( \delta \) is the connecting morphism in homology. Note that for \( [z] \in H(\Sigma M) \) one has

\[
z = \begin{pmatrix} 0 \\ 1^{\Sigma M} \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix}
\]

and

\[
\begin{align*}
\delta^\text{Cone} \alpha \circ \delta^\text{Cone} \alpha \begin{pmatrix} 0 \\ z \end{pmatrix} &= \begin{pmatrix} \delta^N \\ 0 \\ \delta^{\Sigma M} \end{pmatrix} \begin{pmatrix} \alpha \delta^{\Sigma M} \\ 0 \\ \alpha^{\Sigma M} \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix} \\
&= \begin{pmatrix} \alpha \delta^{\Sigma M}(z) \\ 0 \\ \alpha^{\Sigma M} \delta^{\Sigma M}(z) \end{pmatrix} \\
&= \Sigma \begin{pmatrix} 1^N \\ 0 \end{pmatrix} (\Sigma \alpha)(z),
\end{align*}
\]
and consequently, \( \delta([z]) = [\Sigma \alpha(z)] \) by 2.2.17. Thus, one has \( \delta = H(\Sigma \alpha) = \Sigma H(\alpha) \).

The exact sequence (4.2.13.1) establishes a crucial connection between preservation of homology and vanishing of homology.

**4.2.14 Theorem.** A morphism \( \alpha \) of \( R \)-complexes is a quasi-isomorphism if and only if the complex \( \text{Cone} \alpha \) is acyclic.

**Proof.** For a morphism \( \alpha : M \to N \) the sequence (4.2.13.1) shows that \( \text{Cone} \alpha \) is acyclic, i.e. \( H(\text{Cone} \alpha) = 0 \), if and only if \( \delta = \Sigma H(\alpha) \) is an isomorphism.

**Contractible Complexes**

**4.2.15 Definition.** An \( R \)-complex \( M \) is called contractible if the identity morphism \( 1^M \) is null-homotopic.

**Remark.** Other words for contractible are split and homotopically trivial.

**4.2.16 Example.** For every \( R \)-module \( M \), the complex \( 0 \to M \to M \to 0 \) is contractible. It follows from 4.2.22 that every contractible \( R \)-complex is a (co)product of countably many such complexes.

**4.2.17 Example.** Let \( x_1, x_2 \) be elements in \( k \) with \( (x_1, x_2) = k \) and choose \( l_1, l_2 \in k \) with \( l_1x_1 + l_2x_2 = 1 \). Set \( K = K^k(x_1, x_2) \); cf. 2.1.21. The degree 1 homomorphism \( \sigma : K \to K \) whose non-zero components \( \sigma_0 \) and \( \sigma_1 \) are given by

\[
1 \mapsto l_1e_1 + l_2e_2 \quad \text{and} \quad e_1 \mapsto -l_2e_1 \land e_2 \\
\quad e_2 \mapsto l_1e_1 \land e_2
\]

satisfies \( \partial^K \sigma + \sigma \partial^K = 1^K \), whence \( K \) is contractible.

**4.2.18.** If \( M \) is a contractible complex, then one has \( 1^{H(M)} = H(1^M) = 0 \), see 2.2.23, so \( M \) is acyclic.

**4.2.19 Lemma.** For an \( R \)-complex \( M \), the following conditions are equivalent.

(i) \( M \) is contractible.

(ii) \( \text{Hom}_R(K, M) \) is contractible for every \( R \)-complex \( K \).

(iii) \( \text{Hom}_R(K, M) \) is acyclic for every \( R \)-complex \( K \).

(iv) \( \text{Hom}_R(M, N) \) is contractible for every \( R \)-complex \( N \).

(v) \( \text{Hom}_R(M, N) \) is acyclic for every \( R \)-complex \( N \).

(vi) \( \text{Hom}_R(M, M) \) is acyclic.

(vii) \( H_0(\text{Hom}_R(M, M)) = 0 \).
4.2 Quasi-isomorphisms

**Proof.** Condition (i) implies (ii) and (iv) by 2.3.10 and 2.3.11. The implications (ii) \(\implies\) (iii) and (iv) \(\implies\) (v) are evident, see 4.2.18, and (iii) \(\implies\) (vi), (v) \(\implies\) (vi), and (vi) \(\implies\) (vii) are trivial. Finally, it follows from (vii) that \(1^M\) is a boundary in \(\text{Hom}_R(M, M)\), whence it is null-homotopic; see 2.3.3. This proves (vii) \(\implies\) (i). \(\square\)

4.2.20 Theorem. A morphism \(\alpha\) of \(R\)-complexes is a homotopy equivalence if and only if the complex \(\text{Cone} \alpha\) is contractible.

**Proof.** Let \(\alpha: M \to N\) be a morphism and set \(C = \text{Cone} \alpha\). Application of the functor \(\text{Hom}_R(C, -)\) to the degreewise split exact sequence (4.1.3.1) yields by 2.3.14 an exact sequence

\[
0 \to \text{Hom}_R(C, N) \to \text{Hom}_R(C, C) \to \text{Hom}_R(C, \Sigma M) \to 0.
\]

By 4.1.8 and 2.3.16 there are isomorphisms \(\text{Hom}_R(C, N) \cong \Sigma^{-1} \text{Cone} \text{Hom}_R(\alpha, N)\) and \(\text{Hom}_R(C, \Sigma M) \cong \text{Cone} \text{Hom}_R(\alpha, M)\). If \(\alpha\) is a homotopy equivalence, then by 2.3.11 so are the morphisms \(\text{Hom}_R(\alpha, N)\) and \(\text{Hom}_R(\alpha, M)\); in particular they are quasi-isomorphisms. Now it follows from 4.2.14 that the complexes \(\text{Hom}_R(C, N)\) and \(\text{Hom}_R(C, \Sigma M)\) are acyclic, whence \(\text{Hom}_R(C, C)\) is acyclic; cf 2.2.18. Thus, \(C\) is contractible by 4.2.19.

For the converse, assume that \(\text{Cone} \alpha\) is contractible and let \(\nu: \text{Cone} \alpha \to \text{Cone} \alpha\) be a degree 1 homomorphism with \(1^\text{Cone} \alpha = \partial^\text{Cone} \alpha + \nu \partial^\text{Cone} \alpha\). It has the form

\[
\nu = \begin{pmatrix} \nu & \sigma \\ \tau & \Sigma \mu \end{pmatrix}
\]

for degree 1 homomorphisms \(\nu: N \to N, \sigma: \Sigma M \to N, \tau: N \to \Sigma M,\) and \(\mu: M \to M\). From the equalities

\[
\begin{pmatrix} 1^N & 0 \\ 0 & 1^\Sigma M \end{pmatrix} = \begin{pmatrix} \partial^N \alpha \Sigma_{-1}^M & \nu \sigma \\ 0 & \partial^\Sigma M \end{pmatrix} \begin{pmatrix} \nu & \sigma \\ \tau & \Sigma \mu \end{pmatrix} + \begin{pmatrix} \nu & \sigma \\ \tau & \Sigma \mu \end{pmatrix} \begin{pmatrix} \partial^N \alpha \Sigma_{-1}^M & \nu \sigma \\ 0 & \partial^\Sigma M \end{pmatrix} = \begin{pmatrix} \partial^N \nu + \alpha \Sigma_{-1}^M \tau + \nu \partial^N & \partial^N \sigma + \alpha \Sigma_{-1}^M \Sigma \mu + \nu \alpha \Sigma_{-1}^M + \sigma \Sigma \mu \Sigma M & \partial^\Sigma M \Sigma \mu + \tau \alpha \Sigma_{-1}^M + \Sigma \mu \Sigma \mu \Sigma M \\ \partial^\Sigma M \Sigma \mu + \tau \alpha \Sigma_{-1}^M + \Sigma \mu \Sigma \mu \Sigma M \end{pmatrix}
\]

one gets, in particular,

\[
0 = \partial^\Sigma M \tau + \tau \partial^N,
\]

\[
1^N = \partial^N \nu + \alpha \Sigma_{-1}^M \tau + \nu \partial^N,
\]

and

\[
1^\Sigma M = \partial^\Sigma M \Sigma \mu + \tau \alpha \Sigma_{-1}^M + \Sigma \mu \Sigma \mu = \partial^\Sigma M \Sigma \mu + \Sigma (\Sigma_{-1}^M \tau (\Sigma \alpha)) + \Sigma \mu \Sigma \mu.
\]

The first equality shows that \(\tau\) is a chain map, whence \(\Sigma_{-1}^M \tau: N \to M\) is a morphism; the second and third equalities show that it is a homotopy inverse of \(\alpha\). \(\square\)

4.2.21 Corollary. If \(\alpha\) is an isomorphism in \(\mathcal{C}(R)\) of \(R\)-complexes, then the complex \(\text{Cone} \alpha\) is contractible.

4.2.22 Proposition. For an \(R\)-complex \(M\), the following conditions are equivalent.
(i) \( M \) is contractible.

(ii) There is a graded \( R \)-module \( N \) with \( M \cong \text{Cone} 1^N \).

(iii) There exist graded \( R \)-modules \( M' \) and \( M'' \) with \( M'' = M' \oplus M'' \) and \( \partial M|_{M'} = 0 \), and such that \( \partial M|_{M''} \) yields an isomorphism \( M' \cong \Sigma (\partial M(M')) \cong \Sigma M'' \).

**Proof.** Condition (ii) implies (i) by 4.2.21.

(i) \(\implies\) (iii): By assumption there is a homomorphism \( \sigma : M \to M \) of degree 1 such that \( \partial M \sigma + \sigma \partial M = 1^M \) holds. The endomorphism \( \epsilon = \sigma \partial M \) of \( M \) satisfies
\[
\epsilon^2 = (\sigma \partial M)(1^M - \partial M \sigma) = \epsilon \quad \text{and} \quad 1^M - \epsilon = \partial M \sigma,
\]
whence there is an equality \( M^i = M' \oplus M'' \) with \( M' = \text{Im} \epsilon \) and \( M'' = \text{Im} \partial M \sigma \).

Evidently, one has \( \partial M|_{M''} = 0 \) and, therefore,
\[
M'' \subseteq B(M) = \partial M(M') = \partial M \sigma \partial M(M) \subseteq \partial M \sigma (M) = M''.
\]
It follows that \( \partial M|_{M''} \) is a surjective homomorphism \( M' \to M'' \) of degree -1. To see that \( \partial M|_{M''} \) is injective, let \( m' = \epsilon(m) \) be an element in \( M'' \) with \( 0 = \partial M(m') = \partial M \sigma \partial M(m) \). Then one has \( 0 = \sigma \partial M \sigma \partial M(m) = \epsilon^2(m) = \epsilon(m) = m' \).

Thus, \( \partial M|_{M'} : M' \to \Sigma M'' \) is an isomorphism of graded modules.

(iii) \(\implies\) (ii): It is straightforward to verify that the map given by the assignment \((m', m'') \mapsto (m'', \partial M(m'))\) is an isomorphism of \( R \)-complexes \( M \to \text{Cone} 1^M \).

**Semi-simple Modules**

A graded \( R \)-module is called **semi-simple** if every graded submodule is a graded direct summand.

**4.2.23 Proposition.** Let \( M \) be an \( R \)-complex. If the graded \( R \)-module \( M^i \) is semi-simple, then there is an isomorphism of \( R \)-complexes \( M \cong H(M) \oplus \text{Cone} 1^B(M) \).

**Proof.** Set \( Z = Z(M), B = B(M), \) and \( H = H(M) \); recall from 2.2.10 the short exact sequences of \( R \)-complexes,
\[
0 \to B \to Z \xrightarrow{\tau} H \to 0 \quad \text{and} \quad 0 \to Z \xrightarrow{\rho} M \xrightarrow{\partial M} \Sigma B \to 0.
\]

By assumption \( M^i \) is semi-simple, and hence so is the graded submodule \( Z^i \). It follows that both sequences are degreewise split. In particular, there are morphisms \( \sigma : H^i \to Z^i \) and \( \tau : \Sigma B^i \to M^i \) with \( \rho \sigma = 1^H \) and \( \partial M \tau = 1^{\Sigma B} \). Thus, the map
\[
M^i \oplus \text{Cone} 1^B = H^i \oplus B^i \oplus \Sigma B^i \to M^i
\]
4.2 Quasi-isomorphisms

given by \((h, b, b') \mapsto \varepsilon \sigma(h) + \varepsilon(b) + \tau(b') = \sigma(h) + b + \tau(b')\) is an isomorphism
of graded \(R\)-modules. Moreover, it yields an isomorphism of complexes as one has
\[
\partial \mathrm{Cone}^1 (b, b') = (b', 0) \quad \text{and} \quad \partial^M (\sigma(h) + b + \tau(b')) = b'.
\]

**4.2.24 Corollary.** Assume that \(R\) is semi-simple. For every \(R\)-complex \(M\) there are
 quasi-isomorphisms \(H(M) \to M\) and \(M \to H(M)\).

**PROOF.** Immediate from 4.2.22, 4.2.21, and 4.2.18.

---

**Exercises**

**E 4.2.1** Show that a homomorphism of modules is an isomorphism if and only if it is a quasi-isomorphism when considered as a morphism of complexes.

**E 4.2.2** Show that a sequence of modules \(0 \to M' \xrightarrow{\alpha} M' \xrightarrow{\alpha'} M'' \to 0\) is exact if one of the
morphisms of complexes defined by the diagrams

\[
\begin{array}{ccc}
0 & \to & M' \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
0 & \to & M''
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
0 & \to & M' \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
0 & \to & M''
\end{array}
\]

is a quasi-isomorphism and only if they both are quasi-isomorphisms.

**E 4.2.3** Show that the \(\mathbb{Z}/4\mathbb{Z}\)-complexes \(0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \to 0\) and \(0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0\)
have isomorphic homology but that there is no quasi-isomorphism in either direction.

**E 4.2.4** Show that there are surjective quasi-isomorphisms from the Koszul complex \(K = K^R(x, y)\) to each of the complexes \(M\) and \(N\) in 4.2.4. Decide if there are quasi-isomorphisms in the opposite direction \(M \to K\) and \(N \to K\).

**E 4.2.5** Let \(\alpha\) be a quasi-isomorphism of \(R\)-complexes. Show that the following conditions are equivalent:

(i) \(\alpha\) is injective; (ii) \(\alpha\) is injective on boundaries; (iii) \(\alpha\) is injective on cokernels; (iv) \(\alpha\) is injective on boundaries and on cokernels.

**E 4.2.6** Show that the Koszul complexes \(K^2(2, 3)\) and \(K^2(4, 5)\) are acyclic. Decide if there is a non-zero quasi-isomorphism \(K^2(2, 3) \to K^2(4, 5)\) or \(K^2(4, 5) \to K^2(2, 3)\); cf. E 2.1.5.

**E 4.2.7** Let \(x_1, \ldots, x_m\) and \(y_1, \ldots, y_n\) be elements in \(k\). Show that if the Koszul complexes
\(K^k(x_1, \ldots, x_m)\) and \(K^k(y_1, \ldots, y_n)\) are quasi-isomorphic, then they are acyclic or one has
\((x_1, \ldots, x_m) = (y_1, \ldots, y_n)\) and \(m = n\).

**E 4.2.8** Consider a commutative diagram of \(R\)-complexes,

\[
\begin{array}{cccc}
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
\downarrow{\varphi'} & & \downarrow{\varphi} & & \downarrow{\varphi''} & & \downarrow{\varphi'''} & & \downarrow{\varphi'''} \\
0 & \to & N' & \to & N & \to & N'' & \to & 0
\end{array}
\]

with exact rows. Show that if two of the morphisms \(\varphi', \varphi, \text{and} \varphi''\) are homotopy equivalences, the third may not be a homotopy equivalence.

**E 4.2.9** (Cf. 4.2.16) Show that every contractible complex is a product and a coproduct of primitive contractible complexes \(0 \to M \to M \to 0\).
E 4.2.10 Show that the $\mathbb{Z}/6\mathbb{Z}$-complex $\cdots \to \mathbb{Z}/6\mathbb{Z} \xrightarrow{1} \mathbb{Z}/6\mathbb{Z} \xrightarrow{1} \mathbb{Z}/6\mathbb{Z} \to \cdots$ is contractible.

E 4.2.11 Let $x_1, \ldots, x_n$ be elements in $k$ such that $(x_1, \ldots, x_n) = k$. Show that the Koszul complex $K^k(x_1, \ldots, x_n)$ is contractible.

E 4.2.12 Show that a bounded below complex of projective modules is contractible if and only if it is acyclic.

E 4.2.13 Show that a bounded above complex of injective modules is contractible if and only if it is acyclic.

E 4.2.14 Show that a graded $R$-module $M$ is semi-simple if and only if each module $M_v$ is semi-simple.

### 4.3 Standard Isomorphisms

**Synopsis.** Commutativity and associativity of tensor product; Hom-tensor adjunction; Hom swap.

In this section, the standard isomorphisms from Sect. 1.2 are extended to morphisms of complexes.

**Identities**

For every $R$-complex $M$, there are isomorphisms in $\mathcal{C}(R)$,

\[(4.3.0.1) \quad R \otimes_R M \xrightarrow{\cong} M \text{ given by } r \otimes m \mapsto rm \text{ and} \]

\[(4.3.0.2) \quad \text{Hom}_R(R, M) \xrightarrow{\cong} M \text{ given by } \psi \mapsto \psi(1), \]

where $\psi \in \text{Hom}_k(R, M)$ and $m \in M$ are homogeneous elements. Both isomorphisms are natural in $M$. Moreover, if $M$ is a complex of $R$-$S$-bimodules, then these maps are isomorphisms in $\mathcal{C}(R-S)$.

**Commutativity**

The next construction and the proposition that follows establish a commutativity isomorphism for tensor products of complexes. It is based on, and it extends, the isomorphism from 1.2.2.

**4.3.1 Construction.** Let $M$ be an $R^e$-complex and $N$ be an $R$-complex. The commutativity isomorphism for modules 1.2.2 induces a natural isomorphism of graded $k$-modules, $M \otimes_R N \to N \otimes_R M$, with the isomorphism in degree $v$ given by

\[
(M \otimes_R N)_v = \bigoplus_{i \in \mathbb{Z}} M_i \otimes_R N_{v-i} \xrightarrow{\sum (-1)^{|v-i|} \sigma^M_{N_{v-i}}} \bigoplus_{i \in \mathbb{Z}} N_{v-i} \otimes_R M_i = (N \otimes_R M)_v,
\]
4.3 Standard Isomorphisms

This isomorphism is also denoted $\varpi^{MN}$. For homogeneous elements $m \in M$ and $n \in N$ it is given by

$$\varpi^{MN}(m \otimes n) = (-1)^{|n||m|} n \otimes m.$$

(4.3.1.1)

Note that (4.3.1.1) agrees with the definition in 1.2.2 for modules $M$ and $N$.

4.3.2 Proposition. Let $M$ be an $R^s$-complex and $N$ be an $R^s$-complex. The commutativity map defined in 4.3.1,

$$\varpi^{MN} : M \otimes_R N \rightarrow N \otimes_R M,$$

is an isomorphism in $C(\mathbb{k})$, and it is natural in $M$ and $N$. Moreover, if $M$ is in $C(Q-R^s)$ and $N$ is in $C(R-S^o)$, then $\varpi^{MN}$ is an isomorphism in $C(Q-S^o)$.

**Proof.** By construction, $\varpi^{MN}$ is an isomorphism of graded $\mathbb{k}$-modules and natural in $M$ and $N$. If $M$ is in $C(Q-R^s)$ and $N$ is in $C(R-S^o)$, then $\varpi^{MN}$ is a natural isomorphism of graded $Q-S^o$-bimodules. This follows from 1.2.2 and the construction. For homogeneous elements $m \in M$ and $n \in N$ one has

$$\varpi^{MN}(\partial^{M \otimes_R N}(m \otimes n)) = \varpi^{MN}(\partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n))$$

$$= (-1)^{|n||m|} n \otimes \partial^M(m) + (-1)^{|m|+|n|} \partial^N(n) \otimes m$$

$$= (-1)^{|n||m|} (\partial^N(n) \otimes m + (-1)^{|n|} n \otimes \partial^M(m))$$

$$= (-1)^{|n||m|} (\partial^{N \otimes_R M}(n \otimes m))$$

$$= \partial^{N \otimes_R M}(\varpi^{MN}(m \otimes n)).$$

Thus, $\varpi^{MN}$ is a morphism, and hence an isomorphism, of complexes. \qed

**ASSOCIATIVITY**

4.3.3 Construction. Let $M$ be an $R^o$-complex, $X$ be a complex of $R-S^o$-bimodules, and $N$ be an $S$-complex. The associativity isomorphism for modules 1.2.3 induces a natural isomorphism $(M \otimes_R X) \otimes_S N \rightarrow M \otimes_R (X \otimes_S N)$ of graded $\mathbb{k}$-modules. The component in degree $\nu$ is induced by $\prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \varpi^{M_{X_{i-j}}}$. Indeed, it maps

$$(M \otimes_R X) \otimes_S N) = \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} (M_j \otimes_R X_{i-j}) \otimes_S N_{\nu-i} = \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} (M_j \otimes_R X_{i-j} \otimes_S N_{\nu-i}),$$

where the isomorphism follows from 3.1.12, isomorphically to

$$(M \otimes_R (X \otimes_S N)) = \prod_{j \in \mathbb{Z}} \prod_{i \in \mathbb{Z}} (M_j \otimes_R (X_{i-j} \otimes_S N_{\nu-i})) = \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} (M_j \otimes_R (X_{i-j} \otimes_S N_{\nu-i})).$$
where the isomorphism follows from 3.1.13. The resulting isomorphism of graded modules \((M \otimes_R X) \otimes_S N \to M \otimes_R (X \otimes_S N)\) is also denoted \(\omega^{M \times N}\). On homogeneous elements \(m \in M\), \(x \in X\), and \(n \in N\) it is given by

\[
(4.3.3.1) \quad \omega^{M \times N}((m \otimes x) \otimes n) = m \otimes (x \otimes n).
\]

Note that (4.3.3.1) agrees with the definition in 1.2.3 for modules \(M\), \(X\), and \(N\).

**4.3.4 Proposition.** Let \(M\) be an \(R\)-complex, \(X\) be a complex of \(R\)-\(S\)-bimodules, and \(N\) be an \(S\)-complex. The associativity map defined in 4.3.3,

\[
\omega^{M \times N} : (M \otimes_R X) \otimes_S N \to M \otimes_R (X \otimes_S N),
\]

is an isomorphism in \(\mathcal{C}(k)\), and it is natural in \(M\), \(X\), and \(N\). Moreover, if \(M\) is in \(\mathcal{C}(Q-R)\) and \(N\) is in \(\mathcal{C}(S-T)\), then \(\omega^{M \times N}\) is an isomorphism in \(\mathcal{C}(Q-T)\).

**Proof.** By construction, \(\omega^{M \times N}\) is an isomorphism of graded \(k\)-modules and natural in \(M\), \(X\), and \(N\). If \(M\) is in \(\mathcal{C}(Q-R)\) and \(N\) is in \(\mathcal{C}(R-T)\), then \(\omega^{M \times N}\) is a natural isomorphism of graded \(Q-T\)-bimodules. This follows from 1.2.3 and the construction. A straightforward computation, similar to the one in the proof of 4.3.2, shows that \(\omega^{M \times N}\) is a morphism, and hence an isomorphism, of complexes.

**ADJUNCTION**

**4.3.5 Construction.** Let \(M\) be an \(R\)-complex, \(X\) be a complex of \(S\)-\(R\)-bimodules, and \(N\) be an \(S\)-complex. By 3.1.28 one has

\[
\text{Hom}_S(X \otimes_R M, N)_v = \prod_{h \in \mathbb{Z}} \text{Hom}_S(\prod_{j \in \mathbb{Z}} X_j \otimes_R M_{h-j}, N_{h+v})
\]

\[
\cong \prod_{h \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \text{Hom}_S(X_j \otimes_R M_{h-j}, N_{h+v})
\]

\[
= \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \text{Hom}_S(X_j \otimes_R M_i, N_{i+j+v}),
\]

and by 3.1.25 one has

\[
\text{Hom}_R(M, \text{Hom}_S(X, N))_v = \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i, \prod_{j \in \mathbb{Z}} \text{Hom}_S(X_j, N_{j+i+v}))
\]

\[
= \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \text{Hom}_R(M_i, \text{Hom}_S(X_j, N_{i+j+v})).
\]

It follows from adjunction for modules 1.2.4 that the map

\[
\text{Hom}_S(X \otimes_R M, N) \to \text{Hom}_R(M, \text{Hom}_S(X, N))
\]
with degree $v$ component induced by $\prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} (-1)^{ij} \rho^{X,M;N_i + j v}$ is a natural isomorphism of graded $k$-modules; this isomorphism is denoted by $\rho^{XMN}$. On homogeneous elements $\psi \in \text{Hom}_S(X \otimes_R M, N)$, $m \in M$, and $x \in X$ it is given by

\begin{equation}
\rho^{XMN}(\psi)(m)(x) = (-1)^{v|x|} \psi(x \otimes m).
\end{equation}

Note that (4.3.5.1) agrees with the definition in 1.2.4 for modules $M$, $X$, and $N$.

**4.3.6 Proposition.** Let $M$ be an $R$-complex, $X$ be a complex of $S$-$R^0$-bimodules, and $N$ be an $S$-complex. The adjunction map defined in 4.3.5,

\[ \rho^{XMN} : \text{Hom}_S(X \otimes_R M, N) \rightarrow \text{Hom}_R(M, \text{Hom}_S(X, N)), \]

is an isomorphism in $\mathcal{C}(k)$, and it is natural in $M$, $X$, and $N$. Moreover, if $M$ is in $\mathcal{C}(R-Q^o)$ and $N$ is in $\mathcal{C}(S-T^o)$, then $\rho^{XMN}$ is an isomorphism in $\mathcal{C}(Q-T^o)$.

**Proof.** By construction, $\rho^{XMN}$ is an isomorphism of graded $k$-modules and natural in $M$, $X$, and $N$. If $M$ is in $\mathcal{C}(R-Q^o)$ and $N$ is in $\mathcal{C}(S-T^o)$, then $\rho^{XMN}$ is an isomorphism of graded $Q-T^o$-bimodules; this follows from 1.2.4 and the construction. For homogeneous elements $\psi \in \text{Hom}_S(X \otimes_R M, N)$, $m \in M$, and $x \in X$ one has

\[ \rho^{XMN}(\partial \text{Hom}_S(X \otimes R M, N)(\psi))(m)(x) = \rho^{XMN}(\partial^N \psi - (-1)^{v|x|} \rho^{X \otimes R M}(\psi)(m)(x)
\]

\[ = (-1)^{v|x|} \partial^N \psi(x \otimes m) - (-1)^{v|x|} \rho^{X \otimes R M}(\partial^N \psi(x \otimes m))
\]

\[ = (-1)^{v|x|} \partial^N \psi(x \otimes m) - (-1)^{v|x|} \rho^{X \otimes R M}(\partial^N \psi(x \otimes m) + (-1)^{v|x|} x \otimes \partial^M(m))
\]

and

\[ (\partial \text{Hom}_R(M, \text{Hom}_S(X, N))(\rho^{XMN}(\psi)))(m)(x)
\]

\[ = (\partial \text{Hom}_S(X, N))^{\rho^{XMN}(\psi)} - (-1)^{v|x|} \rho^{XMN}(\partial^N \psi)(m)(x)
\]

\[ = \partial^N(\rho^{XMN}(\psi)(m)(x)) - (-1)^{v|x|} \rho^{XMN}(\psi)(m)(\partial^N(x))
\]

\[ - (-1)^{v|x|} \rho^{XMN}(\psi)(\partial^M(m))(x)
\]

\[ = (-1)^{v|x|} \partial^N \psi(x \otimes m) - (-1)^{v|x|+|x||m|-1} \psi(x \otimes \partial^M(m))
\]

\[ = (-1)^{v|x|} \partial^N \psi(x \otimes m) - (-1)^{v|x|+|x||m|} \psi(x \otimes \partial^M(m) + (-1)^{v|x|} x \otimes \partial^M(m)).
\]

Thus, $\rho^{XMN}$ is a morphism, and hence an isomorphism, of complexes.

\[ \square \]
4.3.7 Construction. Let $M$ be an $R$-complex, $X$ be a complex of $R$-$S^\omega$-bimodules, and $N$ be an $S^\omega$-complex. By 3.1.25 one has

$$\text{Hom}_R(M, \text{Hom}_{S^\omega}(N, X))_v = \prod_{i \in \mathbb{Z}} \text{Hom}_R(M_i, \prod_{j \in \mathbb{Z}} \text{Hom}_{S^\omega}(N_j, X_{i+j+v}))$$

$$\cong \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \text{Hom}_R(M_i, \text{Hom}_{S^\omega}(N_j, X_{i+z+j+v})),$$

and similarly,

$$\text{Hom}_{S^\omega}(N, \text{Hom}_R(M, X))_v \cong \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \text{Hom}_{S^\omega}(N_j, \text{Hom}_R(M_i, X_{i+z+j+v})).$$

It follows from swap for modules 1.2.5 that the map

$$\text{Hom}_R(M, \text{Hom}_{S^\omega}(N, X)) \to \text{Hom}_{S^\omega}(N, \text{Hom}_R(M, X))$$

with degree $v$ component induced by $\prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} (-1)^j \iota^M(N|X_{i+j+v})$ is a natural isomorphism of graded $\mathbb{k}$-modules; it is denoted by $\zeta^{MNX}$. On homogeneous elements $\psi \in \text{Hom}_R(M, \text{Hom}_{S^\omega}(N, X))$, $m \in M$, and $n \in N$ it is given by

$$(4.3.7.1) \quad \zeta^{MNX}(\psi)(n)(m) = (-1)^{|m||n|} \psi(n)(m).$$

Note that (4.3.7.1) agrees with the definition in 1.2.5 for modules $M$, $X$, and $N$.

4.3.8 Proposition. Let $M$ be an $R$-complex, $X$ be a complex of $R$-$S^\omega$-bimodules, and $N$ be an $S^\omega$-complex. The swap map defined in 4.3.7,

$$\zeta^{MNX} : \text{Hom}_R(M, \text{Hom}_{S^\omega}(N, X)) \to \text{Hom}_{S^\omega}(N, \text{Hom}_R(M, X)),$$

is an isomorphism in $\mathcal{C}(\mathbb{k})$, and it is natural in $M$, $X$, and $N$. Moreover, if $M$ is in $\mathcal{C}(R-Q^\omega)$ and $N$ is in $\mathcal{C}(T-S^\omega)$, then $\zeta^{MNX}$ is an isomorphism in $\mathcal{C}(Q-T^\omega)$.

Proof. By construction, $\zeta^{MNX}$ is an isomorphism of graded $\mathbb{k}$-modules and natural in $M$, $X$, and $N$. If $M$ is in $\mathcal{C}(R-Q^\omega)$ and $N$ is in $\mathcal{C}(T-S^\omega)$, then $\zeta^{MNX}$ is an isomorphism of graded $Q-T^\omega$-bimodules; this follows from the construction and 1.2.5. For homogeneous elements $\psi \in \text{Hom}_R(M, \text{Hom}_{S^\omega}(N, X))$, $m \in M$, and $n \in N$ one has

$$\zeta^{MNX}\left(\partial \text{Hom}_R(M, \text{Hom}_{S^\omega}(N, X))(\psi)\right)(n)(m)$$

$$= \zeta^{MNX}\left(\partial \text{Hom}_{S^\omega}(N, X)(\psi - (-1)^{\deg(\psi)} \partial M)(n)(m)\right)$$

$$= (-1)^{|m||n|} \left(\partial \text{Hom}_{S^\omega}(N, X)(\psi(m)) - (-1)^{\deg(\psi)} \partial M(m))\right)(n)$$

$$= (-1)^{|m||n|} \left(\partial X(\psi)(m) - (-1)^{\deg(\psi)} \partial M(m)\right)(n)$$

$$= (-1)^{|m||n|} \left(\partial X(\psi)(m) - (-1)^{\deg(\psi)} \partial M(m)\right)(n).$$
Thus, \( \zeta^{MNX} \) is a morphism, hence an isomorphism, of complexes.

**Exercises**

E 4.3.1 Apply 2.4.12 and commutativity 4.3.2 to give a proof of 2.4.13.

E 4.3.2 Apply 3.2.15 and commutativity 4.3.2 to give a proof of 3.2.16.

E 4.3.3 Let \( M \) be an \( R^\bullet \)-complex. Show that the functor \( M \otimes_R - : \mathcal{C}(R) \to \mathcal{C}(k) \) is left adjoint to \( \text{Hom}_k(M,-) \).

E 4.3.4 Let \( M \) be an \( R \)-complex. Show that the functor \( \text{Hom}_k(M,-) : \mathcal{C}(R) \to \mathcal{C}(k) \) is right adjoint to \( M \otimes_k - \).

E 4.3.5 Let \( F, G : \mathcal{C}(R) \to \mathcal{C}(R) \) be functors. Assume that \( F \) has a right adjoint \( \rho \), and that there is a natural isomorphism \( \zeta : F \to G \). Show that there is a canonical natural transformation \( \theta : F \to G \).

Let \( M \) and \( N \) be \( k \)-complexes. Use adjunction 4.3.6 and swap 4.3.8 to show that the result above applies with \( F = M \otimes_k - \), \( G = \text{Hom}_k(N,-) \), and \( R = k \).

E 4.3.6 Let \( F : \mathcal{C}(R) \to \mathcal{C}(R) \) and \( G : \mathcal{C}(R)^{\text{op}} \to \mathcal{C}(R) \) be functors. Assume that \( F \) has a right adjoint \( \rho \), and that there is a natural isomorphism \( \zeta : \text{GF}^{\text{op}} \to F \circ G \). Show that there is a canonical natural transformation \( \eta : F \circ G \to \text{GF}^{\text{op}} \).

Let \( M \) and \( N \) be \( k \)-complexes. Use adjunction 4.3.6 to show that the result above applies with \( F = M \otimes_k - \), \( G = \text{Hom}_k(-,N) \), and \( R = k \).

### 4.4 Evaluation Morphisms

**Synopsis.** Tensor evaluation; homomorphism evaluation; duality.

In this section, the evaluation homomorphisms from Sect. 1.4 are extended to morphisms of complexes.
4 Distinguished Morphisms

Tensor Evaluation

The next construction and the results that follow it extend 1.4.1 and 1.4.3 to complexes.

4.4.1 Construction. Let $M$ be an $R$-complex, $X$ be a complex of $R$-$S^o$-bimodules, and $N$ be an $S$-complex. There are equalities

(4.4.1.1) $\left(\text{Hom}_R(M, X) \otimes_S N\right)_v = \prod_{i \in \mathbb{Z}} \left(\prod_{h \in \mathbb{Z}} \text{Hom}_R(M_h, X_{h+i}) \right) \otimes_S N_{v-i}$

and

(4.4.1.2) $\text{Hom}_R(M, X \otimes_S N)_v = \prod_{j \in \mathbb{Z}} \text{Hom}_R(M_j, \prod_{k \in \mathbb{Z}} X_k \otimes_S N_{j+v-k})$.

To define a map from $\left(\text{Hom}_R(M, X) \otimes_S N\right)_v$ to $\text{Hom}_R(M, X \otimes_S N)_v$, it suffices, in view of 3.1.3 and 3.1.16, to define, for all integers $i$ and $j$, a map

(4.4.1.3) $\left(\prod_{h \in \mathbb{Z}} \text{Hom}_R(M_h, X_{h+i}) \right) \otimes_S N_{v-i} \longrightarrow \text{Hom}_R(M_j, \prod_{k \in \mathbb{Z}} X_{j+k} \otimes_S N_{v-k})$.

This is obtained by precomposing the tensor evaluation homomorphism 1.4.1, adjusted by a sign,

$\text{Hom}_R(M_j, X_{j+i}) \otimes_S N_{v-i} \xrightarrow{(-1)^{j(i-v)} g^{M_jX_{j+i}N_{v-i}}} \text{Hom}_R(M_j, X_{j+i} \otimes_S N_{v-i}),$

with the map induced by the projection $\prod_{h \in \mathbb{Z}} \text{Hom}_R(M_h, X_{h+i}) \rightarrow \text{Hom}_R(M_j, X_{j+i})$ and postcomposing it with the map induced by the embedding $X_{j+i} \otimes_S N_{v-i} \hookrightarrow \prod_{k \in \mathbb{Z}} X_{j+k} \otimes_S N_{v-k}$; cf. 3.1.1 and 3.1.14. The map of complexes defined hereby, $\text{Hom}_R(M, X) \otimes_S N \rightarrow \text{Hom}_R(M, X \otimes_S N)$, is denoted $\theta^{MXN}$. It follows from 1.4.1 that it is a natural morphism of graded $\mathbb{k}$-modules. On homogeneous elements $\psi \in \text{Hom}_R(M, X)$, $m \in M$, and $n \in N$ it is given by

(4.4.1.3) $\theta^{MXN}(\psi \otimes n)(m) = (-1)^{|m||n|} \psi(m) \otimes n$.

Note that (4.4.1.3) agrees with the definition in 1.4.1 for modules $M$, $X$, and $N$.

4.4.2 Proposition. Let $M$ be an $R$-complex, $X$ be a complex of $R$-$S^o$-bimodules, and $N$ be an $S$-complex. The tensor evaluation map defined in 4.4.1,

$\theta^{MXN} : \text{Hom}_R(M, X) \otimes_S N \longrightarrow \text{Hom}_R(M, X \otimes_S N),$

is a morphism in $\mathcal{C}(\mathbb{k})$, and it is natural in $M$, $X$, and $N$. Moreover, if $M$ is in $\mathcal{C}(R-Q^o)$ and $N$ is in $\mathcal{C}(S-T^o)$, then $\theta^{MXN}$ is a morphism in $\mathcal{C}(Q-T^o)$.

Proof. By construction, $\theta^{MXN}$ is a morphism of graded $\mathbb{k}$-modules and natural in $M$, $X$, and $N$. If $M$ is in $\mathcal{C}(R-Q^o)$ and $N$ is in $\mathcal{C}(S-T^o)$, then $\theta^{MXN}$ is a morphism of graded $Q-T^o$-bimodules; this follows from 1.4.1 and the construction. For homogeneous elements $\psi \in \text{Hom}_R(M, X)$, $m \in M$, and $n \in N$ one has
These two computations show that $θ$ and $\theta$ presented modules and one of the following conditions is satisfied.

4.4.3 Theorem. Let $M$ be an $R$-complex, $X$ be a complex of $R$--$S^0$-bimodules, and $N$ be an $S$-complex. The tensor evaluation morphism

$$\partial^{MXN}(\partial^{\text{Hom}_R(M,X)\otimes_S N}(\psi \otimes n))(m)$$

$$= (\partial^X \otimes_S N \partial^{MXN}(\psi \otimes n) - (-1)^{\psi} \psi \otimes \partial^N(n))(m)$$

$$= (-1)^{|m|} \partial^X \otimes_S N (\psi(m) \otimes n) - (-1)^{|\psi| + |m| + (|m|-1)} \psi \partial^M(m) \otimes n$$

$$= (-1)^{|m|} \partial^X \psi(m) \otimes n + (-1)^{|\psi| + |m|} \psi(m) \otimes \partial^N(n)$$

and

$$\partial^{MXN}(\partial^{\text{Hom}_R(M,X)\otimes_S N}(\psi \otimes n))(m)$$

$$= \partial^{MXN}(\partial^{\text{Hom}_R(M,X)}(\psi) \otimes n + (-1)^{|\psi|} \psi \otimes \partial^N(n))(m)$$

$$= \partial^{MXN}\left((\partial^X \psi - (-1)^{|\psi|} \psi \partial^M) \otimes n + (-1)^{|\psi|} \psi \otimes \partial^N(n)\right)(m)$$

$$= (-1)^{|m|} \partial^X \psi(m) \otimes n - (-1)^{|\psi| + |m|} \psi \partial^M(m) \otimes n$$

$$+ (-1)^{|\psi| + |m| + (|m|-1)} \psi(m) \otimes \partial^N(n).$$

These two computations show that $\partial^{MXN}$ is a morphism of complexes.

4.4.3 Theorem. Let $M$ be an $R$-complex, $X$ be a complex of $R$--$S^0$-bimodules, and $N$ be an $S$-complex. The tensor evaluation morphism

$$\partial^{MXN} : \text{Hom}_R(M,X) \otimes_S N \longrightarrow \text{Hom}_R(M,X \otimes_S N)$$

is an isomorphism if the complexes satisfy one of the conditions (1)--(3) and one of the conditions (a)--(c).

(1) $M$ is bounded below, and $X$ and $N$ are bounded above.
(2) $M$ is bounded above, and $X$ and $N$ are bounded below.
(3) Two of the complexes $M$, $X$, and $N$ are bounded.
(a) $M$ or $N$ is a complex of finitely generated projective modules.
(b) $M$ is a complex of projective modules and $N$ is a complex of finitely presented modules.
(c) $M$ is a complex of finitely presented modules and $N$ is a complex of flat modules.

Furthermore, $\partial^{MXN}$ is an isomorphism if $M$ or $N$ is a bounded complex of finitely presented modules and one of the following conditions is satisfied.

(d) $M$ is a complex of projective modules.
(e) $N$ is a complex of flat modules.

Proof. Under any one of the conditions (a)--(c), each evaluation homomorphism $\partial^{MXN}$ is an isomorphism of modules by 1.4.3. To prove the first assertion, it is
now sufficient to show that under each of the boundedness conditions (1)–(3), every component of $\theta^{MXN}$ is given by a direct sum of homomorphisms $\theta^{M_hX_iN_j}$.

The products and coproducts in (4.4.1.1) and (4.4.1.2) are finite under any one of the conditions (1)–(3). Indeed, under (1), assume without loss of generality that one has $M_v = 0$ for all $v < 0$ and $X_v = N_v$ for all $v > 0$; cf. 2.3.15, 2.3.16, 2.4.12, and 2.4.13. By 2.5.9 and 2.5.13 one has $(\text{Hom}_R(M, X) \otimes S N)_v = 0 = \text{Hom}_R(M, X \otimes S N)_v$ for all $v > 0$. For $v \leq 0$ equation (4.4.1.1) yields

$$(\text{Hom}_R(M, X) \otimes S N)_v = \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} \text{Hom}_R(M, X_{i+j}) \otimes S N_{v-i}$$

and from (4.4.1.2) one gets

$$\text{Hom}_R(M, X \otimes S N)_v = \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} \text{Hom}_R(M, X_{i+j} \otimes S N_{v-i})$$

In particular, the $v$th component of the morphism $\theta^{MXN}$ is given by

$$\theta^{MXN}_v = \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} (-1)^{j(v-i)} \theta^{M_hX_iN_j}$$

Parallel arguments apply under conditions (2) and (3). Thus, $\theta^{MXN}$ is an isomorphism when one of (1)–(3) and one of (a)–(c) holds.

If $M$ or $N$ is a bounded complex of finitely presented modules, then under either one of the conditions (d) and (e), each evaluation homomorphism $\theta^{M_hX_iN_j}$ is an isomorphism of modules by 1.4.3. To prove the second assertion, it is now sufficient to prove that every component of $\theta^{MXN}$ is given by a product or a coproduct of homomorphisms $\theta^{M_hX_iN_j}$.

First, let $M$ be a bounded complex of finitely presented modules and assume without loss of generality that one has $M_v = 0$ for all $v < 0$ and for all $v > u$, for some $u \geq 0$. From (4.4.1.1) one gets

$$\theta^{MXN}_v = \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} (-1)^{j(v-i)} \theta^{M_hX_iN_j}$$
\[(\Hom(R, X) \otimes_S N)_v = \prod_{i \in \mathbb{Z}} (\prod_{j=0}^{u} \Hom(R, M_j, X_{j+i}) \otimes_S N_{v-i}) \]
\[\cong \prod_{i \in \mathbb{Z}} \bigoplus_{j=0}^{u} \Hom(R, M_j, X_{j+i}) \otimes_S N_{v-i},\]
and (4.4.1.2) yields
\[\Hom(R, X \otimes_S N)_v = \prod_{j=0}^{u} \Hom(R, M_j, X_j \otimes S N_{v-i}) \]
\[\cong \prod_{i \in \mathbb{Z}} \left( \bigoplus_{j=0}^{u} \Hom(R, M_j, X_{j+i} \otimes S N_{v-i}) \right),\]
where the isomorphism follows from 3.1.32, as \(M\) is bounded and the modules \(M_j\) are finitely generated. It follows that the \(v\)th component of the morphism \(\theta^{MN}\) is given by
\[\theta^{MN}_v = \prod_{i \in \mathbb{Z}} \left( \bigoplus_{j=0}^{u} (-1)^{(v-u)j} \theta^{M_j X_{j+i} N_{v-i}} \right).\]

Finally, let \(N\) be a bounded complex of finitely presented modules and assume without loss of generality that one has \(N_v = 0\) for all \(v < 0\) and for all \(v > u\), for some \(u \geq 0\). Now (4.4.1.1) yields
\[\Hom(R, X \otimes_S N)_v = \prod_{i=v-u}^{v} (\prod_{j \in \mathbb{Z}} \Hom(R, M_j, X_{j+i}) \otimes S N_{v-i}) \]
\[\cong \prod_{j \in \mathbb{Z}, i=v-u}^{v} \Hom(R, M_j, X_{j+i}) \otimes S N_{v-i},\]
where the isomorphism follows from 3.1.31, as the modules \(N_{v-i}\) are finitely presented. Further, (4.4.1.2) yields
\[\Hom(R, X \otimes_S N)_v = \prod_{j \in \mathbb{Z}} \Hom(R, M_j, \bigoplus_{i=v-u}^{v} X_{j+i} \otimes S N_{v-i}) \]
\[\cong \prod_{j \in \mathbb{Z}, i=v-u}^{v} \Hom(R, M_j, X_{j+i} \otimes S N_{v-i}).\]

It follows that the \(v\)th component of the morphism \(\theta^{MN}\) is given by
\[\theta^{MN}_v = \prod_{j \in \mathbb{Z}, i=v-u}^{v} (-1)^{(v-u)j} \theta^{M_j X_{j+i} N_{v-i}}.\]
**HOMOMORPHISM EVALUATION**

The next construction and the results that follow it extend 1.4.4 and 1.4.6 to complexes.

**4.4.4 Construction.** Let $M$ be an $R^o$-complex, $X$ be a complex of $S^–R^o$-bimodules, and $N$ be an $S$-complex. There are equalities

\[(4.4.4.1) \quad \hom_S(X, N) \otimes_{R^o} M, = \prod_{i \in \mathbb{Z}} \left( \prod_{h \in \mathbb{Z}} \hom_S(X_h, N_{h+i}) \right) \otimes_{R^o} M_{v-i} \quad \text{and} \quad \hom_S(\hom_{R^o}(M, X), N) = \prod_{i \in \mathbb{Z}} \hom_S(\prod_{k \in \mathbb{Z}} \hom_{R^o}(M_k, X_{k+j}), N_{j+i}).\]

To define a map from \( \hom_S(X, N) \otimes_{R^o} M, \) to \( \hom_S(\hom_{R^o}(M, X), N)_v \), it suffices, in view of 3.1.3 and 3.1.6, to define, for all integers $i$ and $j$, a map

\[
(\prod_{h \in \mathbb{Z}} \hom_S(X_h, N_{h+i})) \otimes_{R^o} M_{v-i} \rightarrow \hom_S(\prod_{k \in \mathbb{Z}} \hom_{R^o}(M_k, X_{k+j}), N_{j+i}).
\]

This is achieved by precomposing the evaluation homomorphism 1.4.4, adjusted by a sign,

\[
\hom_S(X_{v-i+j}, N_{j+i}) \otimes_{R^o} M_{v-i} \rightarrow \hom_S(\hom_{R^o}(M_{v-i}, X_{v-i+j}), N_{j+i}),
\]

with the map induced by \( \prod_{h \in \mathbb{Z}} \hom_S(X_h, N_{h+i}) \rightarrow \hom_S(X_{v-i+j}, N_{j+i}) \) and postcomposing by the map induced by \( \prod_{k \in \mathbb{Z}} \hom_{R^o}(M_k, X_{k+j}) \rightarrow \hom_{R^o}(M_{v-i}, X_{v-i+j}) \). The map of complexes, \( \hom_S(X, N) \otimes_{R^o} M \rightarrow \hom_S(\hom_{R^o}(M, X), N) \), defined hereby, is denoted \( \eta^{XNM} \). It follows from 1.4.4 that it is a natural morphism of graded \( \mathcal{S} \)-modules. On homogeneous elements $\psi \in \hom_S(X, N)$, $m \in M$, and $\vartheta \in \hom_{R^o}(M, X)$ it is given by

\[(4.4.4.3) \quad \eta^{XNM}(\psi \otimes m)(\vartheta) = (-1)^{|\vartheta||m|} \psi \vartheta(m).
\]

Note that (4.4.7.1) agrees with the definition in 1.4.4 for modules $M$, $X$, and $N$.

**4.4.5 Proposition.** Let $M$ be an $R^o$-complex, $X$ be a complex of $S^–R^o$-bimodules, and $N$ be an $S$-complex. The homomorphism evaluation map defined in 4.4.4,

\[\eta^{XNM} : \hom_S(X, N) \otimes_{R^o} M \rightarrow \hom_S(\hom_{R^o}(M, X), N),\]

is a morphism in \( \mathcal{C}(\mathcal{S}) \), and it is natural in $M$, $X$, and $N$. Moreover, if $M$ is in \( \mathcal{C}(Q–R^o) \) and $N$ is in \( \mathcal{C}(S–T^o) \), then \( \eta^{XNM} \) is a morphism in \( \mathcal{C}(Q–T^o) \).

**Proof.** By construction, \( \eta^{XNM} \) is a morphism of graded \( \mathcal{S} \)-modules and natural in $M$, $X$, and $N$. If $M$ is in \( \mathcal{C}(Q–R^o) \) and $N$ is in \( \mathcal{C}(S–T^o) \), then \( \eta^{XNM} \) is a morphism of graded \( Q–T^o \)-bimodules; this follows from 1.4.1 and the construction. For homogeneous elements $\psi \in \hom_S(X, N)$, $m \in M$, and $\vartheta \in \hom_{R^o}(M, X)$ one has...
Let \( \eta \) be an \( N \)-component of \( f \). We now prove that under any one of the boundedness conditions (1)–(3), every evaluation homomorphism \( \eta \) is an isomorphism of modules by 1.4.6. To prove the first assertion, it is now sufficient to show that under any one of the boundedness conditions (1)–(3), every component of \( \eta \) is given by a direct sum of homomorphisms \( \eta_i \).

The products and coproducts in (4.4.4.1) and (4.4.4.2) are finite under any one of the conditions (1)–(3). Indeed, under (1), assume without loss of generality

\[
\begin{align*}
\eta^{XNM} \left( \partial \text{Hom}_S(X,N) \otimes_{\text{Re} M} \psi \otimes m \right) (\theta) &= \left( \partial \text{Hom}_S(X,N) \psi \otimes m - (-1)^{|\theta|} \psi \otimes \partial^{XN}(m) \right) (\theta) \\
&= (-1)^{|\theta| + |m|} \partial^{XN}(\theta) - (-1)^{|\theta|} \psi \partial^{XN}(m) \\
&= (-1)^{|\theta| + |m|} \partial^{XN}(\theta) - (-1)^{|\theta|} \psi \partial^{XN}(m) \\
&= (-1)^{|\theta| + |m|} \partial^{XN}(\theta) - (-1)^{|\theta|} \psi \partial^{XN}(m) + (-1)^{|\theta|} \theta \partial^{XN}(m) \\
\end{align*}
\]

These two computations show that \( \eta^{XNM} \) is a morphism of complexes.

**4.4.6 Theorem.** Let \( M \) be an \( R^o \)-complex, \( X \) be a complex of \( S^o-R^o \)-bimodules, and \( N \) be an \( S \)-complex. The homomorphism evaluation morphism

\[ \eta^{XNM} : \text{Hom}_S(X,N) \otimes_{\text{Re} M} \psi \rightarrow \text{Hom}_S(\text{Hom}_\text{Re}(M,X),N) \]

is an isomorphism if the complexes satisfy one of the conditions (1)–(3) and one of the conditions (a)–(b).

1. \( M \) and \( N \) are bounded below, and \( X \) is bounded above.
2. \( M \) and \( N \) are bounded above, and \( X \) is bounded below.
3. Two of the complexes \( M, X, \) and \( N \) are bounded.
   a. \( M \) is a complex of finitely generated projective modules.
   b. \( M \) is a complex of finitely presented modules and \( N \) is a complex of injective modules.

Furthermore, \( \eta^{XNM} \) is an isomorphism if \( M \) is a bounded complex of finitely presented modules and one of the following conditions is satisfied.

1. \( M \) is a complex of projective modules.
2. \( N \) is a complex of injective modules.

**Proof.** Under either condition (a) or (b), each evaluation homomorphism \( \eta^{XNM} \) is an isomorphism of modules by 1.4.6. To prove the first assertion, it is now sufficient to show that under any one of the boundedness conditions (1)–(3), every component of \( \eta^{XNM} \) is given by a direct sum of homomorphisms \( \eta^{XNM} \).

The products and coproducts in (4.4.4.1) and (4.4.4.2) are finite under any one of the conditions (1)–(3). Indeed, under (1), assume without loss of generality
that one has $M_v = 0 = N_v$ for all $v < 0$ and $X_v = 0$ for all $v > 0$; cf. 2.3.15, 2.3.16, 2.4.12, and 2.4.13. It follows that one has $(\text{Hom}_S(X,N) \otimes \Re M)_v = 0 = \text{Hom}_S(\text{Hom}_\Re(M,X),N)_v$ for all $v < 0$. For $v \geq 0$ equation (4.4.1.1) yields

$$\left(\text{Hom}_S(X,N) \otimes \Re M\right)_v = \prod_{i \leq v} \left( \prod_{j = i}^0 \text{Hom}_S(X_j,N_{j+i}) \right) \otimes \Re M_{v-i}$$

$$\cong \bigoplus_{i=0}^v \bigoplus_{j = -i}^0 \text{Hom}_S(X_j,N_{j+i}) \otimes \Re M_{v-i}$$

$$= \bigoplus_{j = -v}^v \bigoplus_{i = -j}^0 \text{Hom}_S(X_j,N_{j+i}) \otimes \Re M_{v-i},$$

and from (4.4.1.2) one gets

$$\text{Hom}_S(\text{Hom}_\Re(M,X),N)_v = \prod_{h \geq -v} \text{Hom}_S\left(\prod_{j = h}^0 \text{Hom}_\Re(M_{j-h},X_j),N_{h+v}\right)$$

$$\cong \bigoplus_{h = -v}^v \bigoplus_{j = h}^0 \text{Hom}_S(\text{Hom}_\Re(M_{j-h},X_j),N_{h+v})$$

$$= \bigoplus_{j = -v}^v \bigoplus_{h = -j}^0 \text{Hom}_S(\text{Hom}_\Re(M_{v-j},X_j),N_{j+v})$$

$$= \bigoplus_{j = -v}^v \bigoplus_{h = -j}^0 \text{Hom}_S(\text{Hom}_\Re(M_{v-j},X_j),N_{j+i}).$$

In particular, the sthv component of the morphism $\eta^{XNM}$ is

$$\eta_{v}^{XNM} = \bigoplus_{j = -v}^v \bigoplus_{i = -j}^0 (-1)^{(j-v+i)(v-i)} \eta_{X_j, N_{j+i}, M_{v-i}}.$$

Parallel arguments apply under conditions (2) and (3). Thus, $\eta^{XNM}$ is an isomorphism when one of (1)–(3) and one of (a)–(b) holds.

If $M$ is a bounded complex of finitely presented modules, then under either one of the conditions (c) and (d), each evaluation homomorphism $\eta_{X_j, N_{j+i}, M_{v-i}}$ is an isomorphism of modules by 1.4.6. To prove the second assertion, it is now sufficient to prove that every component of $\eta^{XNM}$ is given by a product of homomorphisms $\eta_{X_j, N_{j+i}, M_{v-i}}$. Assume without loss of generality that one has $M_v = 0$ for all $v < 0$ and for all $v > u$, for some $u \geq 0$. From (4.4.4.1) one gets
4.4 Evaluation Morphisms

\[(\text{Hom}_S(X, N) \otimes R^o M)_v = \prod_{i=v-u}^{v} \left( \prod_{j \in \mathbb{Z}} \text{Hom}_S(X_j, N_{j+i}) \right) \otimes R^o M_{v-i} \]

\[\cong \prod_{j \in \mathbb{Z}} \bigoplus_{i=v-u}^{v} \text{Hom}_S(X_j, N_{j+i}) \otimes R^o M_{v-i},\]

where the isomorphism follows from 3.1.30, as $M$ is a bounded complex of finitely presented modules. Further, (4.4.4.2) yields

\[\text{Hom}_S(\text{Hom}_R^o (M, X), N)_v = \prod_{h \in \mathbb{Z}} \text{Hom}_S(\prod_{j=0}^{u} \text{Hom}_R^o (M_j, X_{j+h}), N_{h+v}) \]

\[\cong \prod_{h \in \mathbb{Z}} \bigoplus_{j=h}^{h+u} \text{Hom}_S(\text{Hom}_R^o (M_{j-h}, X_j), N_{h+v}) \]

\[= \prod_{j \in \mathbb{Z}} \bigoplus_{h=j-u}^{h+u} \text{Hom}_S(\text{Hom}_R^o (M_{v-i}, X_j), N_{j+i}) \]

It follows that the $v$th component of the evaluation morphism $\eta^{XNM}$ is

\[\eta_{v}^{XNM} = \prod_{j \in \mathbb{Z}} \bigoplus_{i=v-u}^{v} (-1)^{(j-v+i)(v-i)} \eta_{i+i}^{X,N+M_{i-i}}. \]

**Biduality**

The next construction and the results that follow it extend 1.4.7 to complexes.

**4.4.7 Construction.** Let $M$ be an $R$-complex and $X$ be a complex of $R$–$S^o$-bimodules. For every $v \in \mathbb{Z}$ one has

\[\text{Hom}_{S^o}(\text{Hom}_R^o (M, X), X)_v = \prod_{i \in \mathbb{Z}} \text{Hom}_{S^o}(\prod_{j \in \mathbb{Z}} \text{Hom}_R(M_j, X_{i+j}), X_{i+v}) \]

To define a map from $M_v$ to $\text{Hom}_{S^o}(\text{Hom}_R^o (M, X), X)_v$, it suffices, in view of 3.1.16, to define, for every integer $i$, a map

\[M_v \rightarrow \text{Hom}_{S^o}(\prod_{j \in \mathbb{Z}} \text{Hom}_R(M_j, X_{i+j}), X_{i+v}) \]

This is achieved by postcomposing the biduality homomorphism 1.4.7, adjusted by a sign.
Thus, $\delta$ and $\psi$ with the map induced by the projection $\prod_{j \in \mathbb{Z}} \text{Hom}_R(M_j, X_{i+j}) \to \text{Hom}_R(M_i, X_{i+v})$. The map of complexes $M \to \text{Hom}_R(M, X)$, defined hereby, is denoted $\delta_X^M$. It follows from 1.4.7 that it is a natural morphism of graded $R$-modules. On homogeneous elements $m \in M$ and $\psi \in \text{Hom}_R(M, X)$ it is given by

\[
\delta_X^M(m)(\psi) = (-1)^{|\psi||m|}\psi(m).
\]

(4.4.7.1)

Note that (4.4.7.1) agrees with the definition in 1.4.7 for modules $M$ and $X$.

**4.4.8 Proposition.** Let $M$ be an $R$-complex and $X$ be a complex of $R$-$S^o$-bimodules. The biduality map for $M$ with respect to $X$ defined in 4.4.7,

\[
\delta_X^M : M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, X), X),
\]

is a morphism in $\mathcal{C}(R)$, and it is natural in $M$ and $X$. Moreover, if $M$ is in $\mathcal{C}(R-T^o)$, then $\delta_X^M$ is a morphism in $\mathcal{C}(R-T^o)$.

**Proof.** By construction, $\delta_X^M$ is a morphism of graded $R$-modules and natural in $M$ and $X$. If $M$ is in $\mathcal{C}(R-T^o)$, then $\delta_X^M$ is a morphism of graded $R-T^o$-bimodules; this follows from 1.4.7 and the construction above. For homogeneous elements $m \in M$ and $\psi \in \text{Hom}_R(M, X)$ one has

\[
(\partial \text{Hom}_R(\text{Hom}_R(M, X), X) \delta_X^M(m)) (\psi)
\]

\[
= (\partial^X \delta_X^M(m) - (-1)^{|\psi||m|} \delta_X^M(m) \partial \text{Hom}_R(M, X))(\psi)
\]

\[
= (-1)^{|\psi||m|} \partial^X \psi(m) - (-1)^{|m|} \delta_X^M(m) (\partial^X \psi - (-1)^{|\psi|} \psi \partial^M)
\]

\[
= (-1)^{|\psi||m|} \partial^X \psi(m) - (-1)^{|m| + |(|\psi| - 1)|} |\psi| \partial^X \psi(m) - (-1)^{|\psi|} \psi \partial^M(m))
\]

\[
= (-1)^{|\psi||m| - 1} \psi \partial^M(m)
\]

\[
= \delta_X^M(\partial^M(m))(\psi).
\]

Thus, $\delta_X^M$ is a morphism of complexes.

**4.4.9 Proposition.** Let $M$ be an $R$-complex and $X$ be a complex of $R$-$S^o$-bimodules. If one module $X_p$ is faithfully injective as an $R$-module, then the biduality morphism

\[
\delta_X^M : M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, X), X)
\]

is injective.

**Proof.** It is sufficient to show that $\delta_X^M$ is injective on homogeneous elements. Let $m \neq 0$ be homogeneous of degree $q$. By assumption, $X_p$ is faithfully injective as an $R$-module, whence there is a non-zero homomorphism from the submodule $Rm$ of $M_q$ to $X_p$. By the lifting property 1.3.24 there is then a homomorphism $\tilde{\psi}$ in $\text{Hom}_R(M_q, X_p)$ with $\tilde{\psi}(m) \neq 0$. Let $\psi : M \to X$ be the degree $p - q$ homomorphism
with $\psi_q = \tilde{\psi}$ and $\psi_v = 0$ for $v \neq q$. One now has $\delta_X^M(m)(\psi) = \psi(m) = \tilde{\psi}(m) \neq 0$, so $\delta_X^M(m)$ is non-zero.

\[ \square \]

**Exercises**

**E 4.4.1** Show that the natural transformation $M \otimes_k \text{Hom}_k(N, \cdot) \to \text{Hom}_k(N, M \otimes_k \cdot)$ in E 4.3.5 is tensor evaluation, up to an application of commutativity 4.3.2.

**E 4.4.2** Show that the natural transformation $M \otimes_k \text{Hom}_k(\cdot, N) \to \text{Hom}_k(\text{Hom}_k(\cdot, M), N)$ in E 4.3.6 is homomorphism evaluation, up to an application of commutativity 4.3.2.
Chapter 5
Resolutions

5.1 Semi-freeness

SYNOPSIS. Graded-free module; complex of free modules; semi-basis; semi-free complex; semi-free resolution; free resolution of module.

5.1.1 Proposition. For an $R$-complex $L$, the following conditions are equivalent.

(i) Each $R$-module $L_v$ is free.

(ii) The graded $R$-module $\L^\natural$ is graded-free.

PROOF. If each module $L_v$ is free with basis $E_v$, then $E = \bigcup_{v \in \mathbb{Z}} E_v$ is a graded basis for $L^\natural$. For the converse, let $E$ be a graded basis for $L^\natural$ and fix $v$. Every element in $L_v$ is a unique linear combination of elements in $E$. Only elements of degree $v$ occur with non-zero coefficients, so the elements of degree $v$ in $E$ form a basis for $L_v$.

5.1.2 Definition. An $R$-complex $L$ is called semi-free if the graded $R$-module $L^\natural$ has a graded basis $E$ that can be written as a disjoint union $E = \bigcup_{n \geq 0} E^n$ with $E^0 \subseteq \mathbb{Z}(L)$ and $\partial^L(E^n) \subseteq R(\bigcup_{i=0}^{n-1} E^i)$ for every $n \geq 1$. Such a basis is called a semi-basis.

5.1.3. Let $L$ be a graded $R$-module. A graded basis for $L$ is trivially a semi-basis for the $R$-complex $L$. Thus, a graded $R$-module is graded-free if and only if it is semi-free as an $R$-complex.

5.1.4 Example. Let $L$ be a bounded below complex of free $R$-modules and set $w = \inf L^\natural$. For every $n \geq 0$, let $E^n$ be a basis for the free module $L_{w+n}$, then $\bigcup_{n \geq 0} E^n$ is a semi-basis for $L$. Thus $L$ is semi-free.

5.1.5 Example. The Dold complex from 2.1.20 is a complex of free $\mathbb{Z}/4\mathbb{Z}$-modules. It has no semi-basis, as no graded basis for this complex contains a cycle.
**Existence of Semi-free Resolutions**

**5.1.6 Definition.** A semi-free resolution of an $R$-complex $M$ is a quasi-isomorphism $L \to M$ of $R$-complexes where $L$ is semi-free.

**5.1.7 Theorem.** Every $R$-complex $M$ has a semi-free resolution $\pi : L \xrightarrow{\sim} M$ with $L_v = 0$ for all $v < \inf M$. Moreover, $\pi$ can be chosen surjective.

The proof relies on the next construction and follows after the proof of 5.1.9.

**5.1.8 Construction.** Given an $R$-complex $M \neq 0$, we shall construct a commutative diagram in $\mathcal{C}(R)$,

\begin{equation}
\begin{array}{c}
L^0 \to L^{n-1} \\
\downarrow \pi^0 \quad \downarrow \pi^{n-1} \quad \downarrow \pi^n \\
M \\
\end{array}
\end{equation}

For $n = 0$, choose a set $Z^0$ of homogeneous cycles in $M$ whose homology classes generate $H(M)$. Let $E^0 = \{ e_z \mid |e_z| = |z|, z \in Z^0 \}$ be a graded set and define an $R$-complex $L^0$ as follows:

\begin{equation}
(L^0)^{\natural} = R\langle E^0 \rangle \quad \text{and} \quad \partial L^0 = 0.
\end{equation}

To see that the map $\pi^0 : L^0 \to M$ given by the assignment $e_z \mapsto z$ is a morphism of complexes, notice that the differential on $L^0$ is 0 and that $\pi^0$ maps to $Z(M)$, the kernel of $\partial M$.

Let $n \geq 1$ and let a morphism $\pi^{n-1} : L^{n-1} \to M$ be given. Choose a set $Z^n$ of homogeneous cycles in $L^{n-1}$ whose homology classes generate the kernel of $H(\pi^{n-1})$. Let $E^n = \{ e_z \mid |e_z| = |z| + 1, z \in Z^n \}$ be a graded set and set

\begin{equation}
(L^n)^{\natural} = (L^{n-1})^{\natural} \oplus R\langle E^n \rangle \quad \text{and} \quad \partial L^n (x + \sum_{z \in Z^n} r_z e_z) = \partial L^{n-1}(x) + \sum_{z \in Z^n} r_z z.
\end{equation}

This defines an $R$-complex. For each $z \in Z^n$ choose an element $m_z \in M$ such that $\pi^{n-1}(z) = \partial M(m_z)$. It is elementary to verify that the map $\pi^n : L^n \to M$ defined by

\begin{equation}
\pi^n (x + \sum_{z \in Z^n} r_z e_z) = \pi^{n-1}(x) + \sum_{z \in Z^n} r_z m_z
\end{equation}

is a morphism of $R$-complexes. Moreover, it agrees with $\pi^{n-1}$ on the subcomplex $L^{n-1}$ of $L^n$. That is, there is an equality of morphisms $\pi^{n-1} = \pi^n \eta^{n-1}$, where $\eta^{n-1}$ is the embedding of $L^{n-1}$ into $L^n$; cf. (5.1.8.3).

For $n < 0$ set $L^n = 0$, $\iota^n = 0$, and $\pi^n = 0$, then the family $\{ \iota^n : L^n \to L^{n+1} \}_{n \in \mathbb{Z}}$ is a telescope in $\mathcal{C}(R)$, and one has $\pi^n = \iota^n \pi^{n+1}$ for all $n \in \mathbb{Z}$. Set $L = \colim_{n \in \mathbb{Z}} L^n$. 

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by 3.2.34 there is a morphism of $R$-complexes $\pi: L \to M$, such that the diagram (5.1.8.1) is commutative.

5.1.9 Proposition. Let $M \neq 0$ be an $R$-complex. The sets, morphisms, and complexes constructed in 5.1.8 have the following properties.

(a) Each set $E^n$ consists of homogeneous elements of degree at least $n + \inf M^\flat$.

(b) Each complex $L^n$ is semi-free with semi-basis $\bigcup_{i=0}^n E^i$.

(c) The complex $L$ is semi-free with semi-basis $E = \bigcup_{n \geq 0} E^n$.

(d) The morphism $\pi: L \to M$ is a quasi-isomorphism.

(e) If $\pi^n$ is surjective for some $n \geq 0$, then $\pi$ is surjective.

Proof. Parts (a) and (b) are immediate from the definition of the sets $E^n$ and (5.1.8.3); part (e) follows from commutativity of the diagram (5.1.8.1).

(c): The morphisms $i^n$ are embeddings, so $L = \text{colim}_{n \geq 0} L^n$ is by 3.2.36 simply the union $\bigcup_{n \geq 0} L^n$; in particular, $\bigcup_{n \geq 0} E^n$ is a graded basis for $L$. By (5.1.8.3) there are containments $\partial^L(E^n) = \partial^L(E^n) \subseteq R(E^{n-1})$ for $n \geq 1$, and (5.1.8.2) yields $\partial^L(E^n) = \partial^L(E^0) = 0$, so $E^n$ consists of cycles.

(d): For each $n \geq 0$ there is a commutative diagram

$$
\begin{array}{ccc}
H(L^0) & \longrightarrow & H(L^n) \\
\downarrow H(\pi^0) & & \downarrow H(\pi^n) \\
H(M) & \leftarrow & H(\pi)
\end{array}
$$

induced from (5.1.8.1) By the choice of $Z^0$, the morphism $H(\pi^0)$ is surjective and hence so is $H(\pi)$. To see that $H(\pi)$ is injective, let $y$ be a cycle in $L$ and assume that $H(\pi)([y]) = 0$. Choose an integer $n$ such that $y \in L^{n-1}$; now one has

$$
0 = H(\pi)([y]) = [\pi(y)] = [\pi^{n-1}(y)] = H(\pi^{n-1})([y]),
$$

so $[y]$ is in $\ker H(\pi^{n-1})$. By the choice of $Z^n$ there exists an element $x \in L^{n-1}$ such that one has

$$
y = \sum_{z \in Z^n} r_z z + \partial^L(x) = \partial^L(x + \sum_{z \in Z^n} r_z z),
$$

where the second equality follows from (5.1.8.3). It follows that $[y] = 0$ in $H(L^n)$ and hence also in $H(L)$. Thus, $H(\pi)$ is injective, and $\pi$ is a quasi-isomorphism.

Proof of 5.1.7. The identity morphism $0 \to 0$ is a semi-free resolution of the zero complex. For an $R$-complex $M \neq 0$, apply the construction 5.1.8. It follows from parts (c) and (d) in 5.1.9 that $\pi: L \to M$ is a semi-free resolution. Parts (a) and (c) ensure that $L_v = 0$ holds for all $v < \inf M^\flat$. Finally, notice that choosing $Z^0$ as a set of generators for $Z(M)$ makes the morphism $\pi^0$ is surjective on cycles, and then $\pi$ is surjective on cycles by commutativity of (5.1.8.1). As $\pi$ is a quasi-isomorphism, it follows from 4.2.12 that it is surjective.
5.1.10 Proposition. Let $R \to S$ be a ring homomorphism. If $L$ is a semi-free $R$-complex, then the $S$-complex $S \otimes_R L$ is semi-free.

**Proof.** Let $E = \bigcup_{n \geq 0} E^n = \{ e_u \}_{u \in U}$ be a semi-basis for $L$. One then has $L^1 = \coprod_{e \in U} R e_u$, and by 3.1.13 there is an isomorphism $(S \otimes_R L)^1 \cong \coprod_{u \in U} S \otimes_R R e_u$. It is straightforward to verify that the $S$-module $\coprod_{u \in U} S \otimes_R R e_u$ is graded-free with basis $E' = \{ 1 \otimes e_u \}_{u \in U}$, and it follows from 2.4.1 that $E'$ is a semi-basis for $S \otimes_R L$. \square

5.1.11 Proposition. If $L$ is a semi-free $S$-complex and $L'$ is a semi-free $k$-complex, then the $S$-complex $L \otimes_k L'$ is semi-free.

**Proof.** Let $E = \bigcup_{n \geq 0} E^n$ be a semi-basis for $L$ and $F = \bigcup_{n \geq 0} F^n$ be a semi-basis for $L'$. For $n \geq 0$ set $G^n = \{ e \otimes f \mid e \in E^i, f \in F^j, i + j = n \}$. It is elementary to verify that the $S$-module $(L \otimes_k L')^n$ is graded-free with basis $G = \bigcup_{n \geq 0} G^n$, and it follows from 2.4.1 that $G$ is a semi-basis for $L \otimes_k L'$. \square

**Boundedness and Finiteness**

5.1.12 Theorem. Every $R$-complex $M$ has a semi-free resolution $L \xrightarrow{\simeq} M$ with $L_v = 0$ for all $v < \inf M$.

**Proof.** If $M$ is acyclic, then the morphism $0 \xrightarrow{\simeq} M$ is the desired resolution. If $H(M)$ is not bounded below, then any semi-free resolution of $M$ has the desired property. Assume now that $H(M)$ is bounded below and set $w = \inf M$. By 4.2.6 there is a quasi-isomorphism $M_{\geq w} \xrightarrow{\simeq} M$, and by 5.1.7 the truncated complex $M_{\geq w}$ has a semi-free resolution $L \xrightarrow{\simeq} M_{\geq w}$ with $L_v = 0$ for $v < w$. The desired semi-free resolution is the composite $L \xrightarrow{\simeq} M_{\geq w} \xrightarrow{\simeq} M$. \square

5.1.13 Lemma. Let $R$ be left Noetherian. Every bounded below degreewise finitely generated $R$-complex $M$ has a semi-free resolution $\pi: L \xrightarrow{\simeq} M$ with $L$ degreewise finitely generated and $L_v = 0$ for all $v < \inf M'$. Moreover, $\pi$ can be chosen surjective.

**Proof.** The identity morphism $0 \to 0$ is a semi-free resolution of the zero complex. Assume now that $M$ is non-zero and set $w = \inf M'$; it is an integer by the assumption on $M$. Apply the construction 5.1.8 to $M$ and notice the following.

As $M$ is left Noetherian, and $M$ is a complex of finitely generated $R$-modules, the set $Z^0$ can be chosen such that it contains only finitely many elements of each degree. Doing so ensures that $E^0 = \{ e_z \mid |e_z| = |z|, z \in Z^0 \}$ contains only finitely many elements of each degree $v$ and no elements of degree $v < w$; see 5.1.9(a).

As $M$ is left Noetherian, it follows by induction that $\text{Ker} H(\pi^{n-1})$ is degreewise finitely generated for every $n > 1$. Choosing the set $Z^n$ such that it has only finitely many elements of each degree ensures that the set $E^n = \{ e_z \mid |e_z| = |z| + 1, z \in Z^n \}$ contains only finitely many elements of each degree $v$ and no elements of degree $v < w + n$; see 5.1.9(a).
From 5.1.9 it follows that \( \pi: L \rightarrow M \) is a semi-free resolution of \( M \), and that \( E = \bigcup_{n \geq 0} E^n \) is a semi-basis for \( L \). For each \( v \in \mathbb{Z} \) the subset \( E_v \subseteq E \) of basis elements of degree \( v \) is a basis for \( L_v \), and it is finite as one has

\[
E_v = \left( \bigcup_{n \geq 0} E^n \right)_v = \bigcup_{n=0}^{v-w} (E^n)_v .
\]

Thus, each free module \( L_v \) is finitely generated, and \( L_v = 0 \) holds for all \( v < w \).

Finally, as \( R \) is left Noetherian, one can choose as \( Z^0 \) a set of generators for \( Z(M) \) with the additional property that it contains only finitely many elements of each degree. With this choice, the quasi-isomorphism \( \pi \) is surjective on cycles by 5.1.9(e) and, therefore, surjective by 4.2.12.

**5.1.14 Theorem.** Let \( R \) be left Noetherian and let \( M \) be an \( R \)-complex. If \( H(M) \) is bounded below and degreewise finitely generated, then \( M \) has a semi-free resolution \( L \rightarrow M \) with \( L \) degreewise finitely generated and \( L_v = 0 \) for all \( v < \inf M \).

**Proof.** If \( M \) is acyclic, then the morphism \( 0 \rightarrow M \) is the desired resolution. Assume now that \( M \) is not acyclic. Set \( w = \inf M \) and apply 5.1.13 to the truncated complex \( M_{>w} \) to obtain a semi-free resolution \( L \rightarrow M_{>w} \) with each module \( L_v \) finitely generated and \( L_v = 0 \) for all \( v < w \). By 4.2.6 there is a quasi-isomorphism \( M_{>w} \rightarrow M \), and the desired resolution is the composite \( L \rightarrow M_{>w} \rightarrow M \).

The requirement in the theorem that \( H(M) \) be bounded below cannot be relaxed; an example is provided in 17.2.15.

**The Case of Modules**

5.1.15. It follows from 5.1.3 that an \( R \)-module is free if and only if it is semi-free as an \( R \)-complex.

5.1.16 Theorem. For every \( R \)-module \( M \) there is an exact sequence of \( R \)-modules

\[
\cdots \rightarrow L_v \rightarrow L_{v-1} \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0
\]

where each module \( L_v \) is free.

**Proof.** Choose by 5.1.7 a surjective semi-free resolution \( \pi: L \rightarrow M \) with \( L_v = 0 \) for all \( v < 0 \). The displayed sequence of \( R \)-modules is the complex \( \Sigma^{-1} \text{Cone} \pi \); in particular, the map \( L_0 \rightarrow M \) is the homomorphism \( -\pi_0 \). The cone is acyclic because \( \pi \) is a quasi-isomorphism; see 4.2.14.

5.1.17 Definition. Let \( M \) be an \( R \)-module. Together, the surjective homomorphism \( L_0 \rightarrow M \) and the \( R \)-complex \( \cdots \rightarrow L_v \rightarrow L_{v-1} \rightarrow \cdots \rightarrow L_0 \rightarrow 0 \) in 5.1.16 is called a **free resolution of** \( M \).
5.1.18. By 5.1.4 a free resolution of an $R$-module $M$ is a semi-free resolution of $M$ as an $R$-complex.

5.1.19 Theorem. Assume that $R$ is left Noetherian. Every finitely generated $R$-module $M$ has a free resolution

$$\cdots \rightarrow L_v \rightarrow L_{v-1} \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0$$

where each module $L_v$ is finitely generated.

Proof. Choose by 5.1.13 a surjective semi-free resolution $\pi: L \rightarrow M$, where each free module $L_v$ is finitely generated and $L_v = 0$ holds for all $v < 0$. The displayed sequence is the acyclic complex $\Sigma^{-1} \text{Cone} \pi$; cf. the proof of 5.1.16.

Exercises

E 5.1.1 Show that a graded $R$-module may be free as an $R$-module without being graded-free.

E 5.1.2 A semi-free filtration of an $R$-complex $L$ is a sequence $\cdots \subseteq L_{u-1} \subseteq L_u \subseteq L_{u+1} \subseteq \cdots$ of subcomplexes, such that each quotient $L_u/L_{u-1}$ is graded-free and one has $L = \bigcup_{u \in \mathbb{Z}} L_u$, $L_{-1} = 0$, and $d^L(L_u) \subseteq L_{u-1}$ for all $u \in \mathbb{Z}$. Show that an $R$-complex is semi-free if and only if it admits a semi-free filtration.

E 5.1.3 Show that a complex $L$ of free modules is semi-free if one has $d^L(L_v) = 0$ for $v \ll 0$.

E 5.1.4 Show that a coproduct of semi-free complexes is semi-free.

E 5.1.5 Show that a direct summand of a semi-free complex may not be semi-free.

E 5.1.6 Let $L$ be a semi-free $R$-complex and $\alpha: L \rightarrow N$ be a morphism in $\mathcal{C}(R)$. Show that for every surjective quasi-ismorphism $\beta: M \rightarrow N$ there exists a morphism $\gamma: L \rightarrow M$ with $\alpha = \beta \gamma$. Hint: For each $n$ let $L^n$ be the semi-free subcomplex of $L$ with semi-basis $\bigsqcup_{i \in I} E^i$ and construct morphisms $\gamma^n: L^n \rightarrow M$ compatible with the embeddings $L^n \rightarrow L^{n+1}$.

E 5.1.7 Apply 5.1.8 to construct a semi-free resolution of the $\mathbb{Z}/4\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$.

E 5.1.8 Construct semi-free resolutions of the complexes in 4.2.4.

E 5.1.9 Show that for every $R$-complex $M$ there is a surjective morphism $L \rightarrow M$ where $L$ is a contractible complex of free $R$-modules.

E 5.1.10 Show that every morphism $\alpha: M \rightarrow N$ of $R$-complexes admits factorizations in $\mathcal{C}(R)$,

$$
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow{\varepsilon} & \cong & \downarrow{\pi} \\
X & \xrightarrow{\pi} & Y \\
\uparrow{\iota} & \cong & \uparrow{\varphi}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow{\varepsilon} & \cong & \downarrow{\varphi} \\
X & \xrightarrow{\pi} & Y \\
\uparrow{\iota} & \cong & \uparrow{\varphi}
\end{array}
$$

where $\iota$ and $\varepsilon$ are injective with semi-free cokernels; $\pi$ and $\varphi$ are surjective; and $\pi$ and $\varepsilon$ are quasi-ismorphisms. Hint: (1) Modify the first step in 5.1.8. (2) E 5.1.9.

E 5.1.11 Give alternative proofs of 5.1.16 and 5.1.19 based on 1.3.11.
5.2 Semi-projectivity

SYNOPSIS. Graded-projective module; complex of projective modules; semi-projective complex; semi-projective resolution; lifting property; projective resolution of module.

Semi-projectivity of an $R$-complex $P$ will be defined in terms of the functor $	ext{Hom}_R(P,-)$ from $\mathcal{C}(R)$ to $\mathcal{C}(k)$. First we study complexes of projective modules.

5.2.1. Lifting properties are a central theme in this section, and several key results can be interpreted in terms of the diagram

$$
\begin{array}{ccc}
P & \rightarrow & M \\
\downarrow & & \downarrow \\
M & \rightarrow & N
\end{array}
$$

where the solid arrows represent given maps of certain sorts, and a lifting property of $P$ ensures the existence of a dotted map of a specific sort such that the diagram is commutative, or commutative up to homotopy.

COMPLEXES OF PROJECTIVE MODULES

Part (iii) below can be interpreted in terms of the diagram (5.2.1.1).

5.2.2 Proposition. For an $R$-complex $P$, the following conditions are equivalent.

(i) Each $R$-module $P_v$ is projective.

(ii) The functor $	ext{Hom}_R(P,-)$ is exact.

(iii) For every homomorphism $\alpha: P \rightarrow N$ and every surjective homomorphism $\beta: M \rightarrow N$, there exists a homomorphism $\gamma: P \rightarrow M$ such that $\alpha = \beta \gamma$ holds.

(iv) Every exact sequence $0 \rightarrow M' \rightarrow M \rightarrow P \rightarrow 0$ in $\mathcal{C}(R)$ is degreewise split.

(v) The graded module $P^\bullet$ is a graded direct summand of a graded-free $R$-module.

PROOF. (i) $\Rightarrow$ (iii): The homomorphism $\alpha$ is an element in $\prod_{v \in \mathbb{Z}} \text{Hom}_R(P_v, N_{v+|\alpha|})$, and $\beta$ is an element in $\prod_{v \in \mathbb{Z}} \text{Hom}_R(M_v, N_{v+|\beta|})$. For each $v \in \mathbb{Z}$ there exists by the lifting property of the projective module $P_v$ a homomorphism $\gamma_v: P_v \rightarrow M_{v+|\alpha|-|\beta|}$ such that $\alpha_v = \beta_{v+|\alpha|-|\beta|} \gamma_v$ holds; see 1.3.17. The desired homomorphism $\gamma$ is the element $(\gamma_v)_{v \in \mathbb{Z}}$ in $\prod_{v \in \mathbb{Z}} \text{Hom}_R(P_v, M_{v+|\alpha|-|\beta|})$.

(iii) $\Rightarrow$ (ii): The functor $\text{Hom}_R(P,-)$ is left exact; see 2.3.12. Thus, if (iii) holds, then $\text{Hom}_R(P,-)$ is exact.

(ii) $\Rightarrow$ (iv): Let $\beta$ denote the morphism $M \rightarrow P$. By (ii) there exists a homomorphism $\gamma: P \rightarrow M$ such that $1^P = \text{Hom}_R(P, \beta)(\gamma) = \beta \gamma$. As $\beta$ is a morphism of complexes, also the degree of $\gamma$ must be 0, that is, $\gamma$ is a morphism of the underlying graded modules. Hence, the sequence $0 \rightarrow M' \rightarrow M \rightarrow P \rightarrow 0$ is degreewise split.


(iv) \implies (v): Choose by 5.1.7 a surjective semi-free resolution \( \pi: L \to P \) and apply (iv) to the associated exact sequence \( 0 \to \text{Ker} \pi \to L \to P \to 0 \) in \( \mathcal{C}(R) \). It follows that \( P^i \) is a graded direct summand of the graded-free \( R \)-module \( L^i \).

(v) \implies (i): Each module \( P_i \) is a direct summand of a free \( R \)-module and, therefore, projective by 1.3.17.

\( \square \)

**Remark.** The complexes described in 5.2.2 are not the projective objects in the category \( \mathcal{C}(R) \); see E 5.2.1 and E 5.2.5.

**5.2.3 Corollary.** Let \( 0 \to P' \to P \to P'' \to 0 \) be an exact sequence of \( R \)-complexes. If \( P'' \) is a complex of projective modules, then \( P \) is a complex of projective modules if and only if \( P' \) is a complex of projective modules.

**Proof.** If \( P'' \) is a complex of projective modules, then \( 0 \to P' \to P \to P'' \to 0 \) is degreewise split, and the assertion follows from 1.3.22.

**5.2.4 Definition.** A graded \( R \)-module \( P \) is called graded-projective if the \( R \)-complex \( P \) satisfies the conditions in 5.2.2.

An important application of the next lemma is to an acyclic complex \( N \) and a complex \( M \) of projective \( R \)-modules.

**5.2.5 Lemma.** Let \( M \) and \( N \) be \( R \)-complexes such that \( M \) or \( N \) is bounded below. If the complex \( \text{Hom}_R(M,v) \) is acyclic for every \( v \in \mathbb{Z} \), then \( \text{Hom}_R(M,N) \) is acyclic.

**Proof.** For every \( n \in \mathbb{Z} \) we must verify the equality \( H_n(\text{Hom}_R(M,N)) = 0 \). By 2.2.13 and 2.3.16 there is an isomorphism

\[
H_n(\text{Hom}_R(M,N)) \cong H_0(\text{Hom}_R(M,\Sigma^{-n}N)).
\]

Since the assumptions are invariant under a shift of \( N \), it suffices to argue that one has \( H_0(\text{Hom}_R(M,N)) = 0 \). Thus, we need to show that every morphism \( \alpha: M \to N \) is null-homotopic; see 2.3.12. Given a morphism \( \alpha \), we must construct a degree 1 homomorphism \( \sigma: M \to N \), such that

\[
(\star) \quad \alpha_v = \partial^N_{v+1}\sigma_v + \sigma_{v-1}\partial^M_v
\]

holds for every \( v \in \mathbb{Z} \). As \( M \) or \( N \) is bounded below, one has \( \alpha_v = 0 \) for \( v \ll 0 \). Thus, one can choose \( \sigma_v = 0 \) for \( v \ll 0 \). Now, proceed by induction. Given that (\( \star \)) holds for \( v \), it follows that \( \alpha_{v+1} - \sigma_v\partial^M_{v+1} \) is a cycle in \( \text{Hom}_R(M_{v+1},N) \) of degree \( v+1 \). Indeed, one has

\[
\partial^N_{v+1}(\alpha_{v+1} - \sigma_v\partial^M_{v+1} = (\alpha_v - \partial^N_{v+1}\sigma_v)\partial^M_{v+1} = (\sigma_{v-1}\partial^M_v)\partial^M_{v+1} = 0.
\]

As the complex \( \text{Hom}_R(M_{v+1},N) \) is acyclic, \( \alpha_{v+1} - \sigma_v\partial^M_{v+1} \) is a boundary. Thus, there exists an element \( \sigma_{v+1} \) in \( \text{Hom}_R(M_{v+1},N_{v+2}) \) with \( \partial^N_{v+2}\sigma_{v+1} = \alpha_{v+1} - \sigma_v\partial^M_{v+1} \). \( \square \)
5.2 Semi-projectivity

5.2.6 Definition. An $R$-complex $P$ is called semi-projective if $\text{Hom}_R(P, \beta)$ is a surjective quasi-isomorphism for every surjective quasi-isomorphism $\beta$ in $\mathcal{C}(R)$.

Remark. Another word for semi-projective is DG-projective.

5.2.7 Example. Let $P$ be a bounded below complex of projective $R$-modules, and let $\beta$ be a surjective quasi-isomorphism. The morphism $\text{Hom}_R(P, \beta)$ is surjective by 5.2.2. The complex $\text{Cone} \beta$ is acyclic by 4.2.14, and hence so is $\text{Hom}_R(P, \text{Cone} \beta)$ for every $v \in \mathbb{Z}$. As $P$ is bounded below, it follows from 4.1.7 and 5.2.5 that $\text{Cone} \text{Hom}_R(P, \beta) \cong \text{Hom}_R(P, \text{Cone} \beta)$ is acyclic, whence $\beta$ is a quasi-isomorphism. Thus, $P$ is semi-projective.

5.2.8 Lemma. For a semi-free $R$-complex $L$, the functor $\text{Hom}_R(L, -)$ is exact and preserves acyclicity of complexes.

Proof. Let $L$ be a semi-free $R$-complex; by 5.1.1 it is a complex of free $R$-modules, so the functor $\text{Hom}_R(L, -)$ is exact by 5.2.2. Choose a semi-basis $E = \bigcup_{i \geq 0} E^i$ for $L$. For $n < 0$ set $L^n = 0$. For $n \geq 0$ let $L^n$ be the semi-free subcomplex of $L$ with semi-basis $\bigcup_{i \geq 0} E^i$. Then one has $L = \bigcup_{n \geq 0} L^n \cong \text{colim}_{n \in \mathbb{Z}} L^n$, and for every $n \geq 0$ there is an exact sequence

$0 \to L^{n-1} \to L^n \to L^n/L^{n-1} \to 0$.

The induced differential on the subquotient $L^n/L^{n-1}$ is 0, so it is isomorphic to the graded-free $R$-module $R(E^n)$. In particular, $(\ast)$ is degreewise split by 5.2.2.

Let $A$ be an acyclic $R$-complex. For every $n \geq 0$ the complex $\text{Hom}_R(R(E^n), A)$ is acyclic; cf. 3.1.28. It follows by induction that $\text{Hom}_R(L^n, A)$ is acyclic for all $n \geq 0$. The morphisms in the tower $\{\text{Hom}_R(L^n, A) \to \text{Hom}_R(L^{n-1}, A)\}_{n \in \mathbb{Z}}$ are surjective because the sequence $(\ast)$ is degreewise split, and the Hom functor preserves degreewise split sequences; see 2.3.14. Now it follows from 3.3.19 and 3.3.37 that

$\text{Hom}_R(L, A) = \text{Hom}_R(\text{colim}_{n \in \mathbb{Z}} L^n, A) \cong \lim_{n \in \mathbb{Z}} \text{Hom}_R(L^n, A)$

is an acyclic complex.

The next result offers useful characterizations of semi-projective complexes. The lifting property in part (iii) can be interpreted in terms of the diagram (5.2.1.1).

5.2.9 Proposition. For an $R$-complex $P$, the following conditions are equivalent.

(i) $P$ is semi-projective.

(ii) The functor $\text{Hom}_R(P, -)$ is exact and preserves quasi-isomorphisms.

(iii) For every chain map $\alpha: P \to N$ and for every surjective quasi-isomorphism $\beta: M \to N$ there exists a chain map $\gamma: P \to M$ such that $\alpha = \beta \gamma$ holds.

(iv) Every exact sequence $0 \to M' \to M \to P \to 0$ in $\mathcal{C}(R)$ with $M'$ acyclic is split.
(v) $P$ is a direct summand of a semi-free $R$-complex.

(vi) $P$ is a complex of projective $R$-modules, and the functor $\text{Hom}_R(P, -)$ preserves acyclicity of complexes.

**Proof.** The implication (ii) $\implies$ (i) is evident.

(i) $\implies$ (iii): The morphism $\text{Hom}_R(P, \beta)$ is a surjective quasi-isomorphism. In particular, it is surjective on cycles by 4.2.12. Thus, in view of 2.3.3 there exists a chain map $\gamma: P \to M$ such that $\alpha = \text{Hom}_R(P, \beta)(\gamma) = \beta \gamma$.

(iii) $\implies$ (iv): By (2.2.17.1) the morphism $\beta: M \to P$ is a quasi-isomorphism, so there exists a chain map $\gamma: P \to M$ with $1^P = \beta \gamma$. As $\beta$ is of degree 0, so is $\gamma$. That is, $\gamma$ is a morphism in $C(R)$, whence the sequence is split.

(iv) $\implies$ (v): By 5.1.7 there exists a semi-free complex $L$ and a surjective quasi-isomorphism $\pi: L \to P$. Apply (iv) to the exact sequence $0 \to \ker \pi \to L \to P \to 0$.

(v) $\implies$ (vi): Immediate from 5.2.2 and 5.2.8, as the Hom functor is additive.

(vi) $\implies$ (i): The functor $\text{Hom}_R(P, -)$ is exact by 5.2.2. For a quasi-isomorphism $\beta$, the complex $\text{Cone} \beta$ is acyclic by 4.2.14. Hence the complex $\text{Hom}_R(P, \text{Cone} \beta) \cong \text{Cone} \text{Hom}_R(P, \beta)$ is acyclic; here the isomorphism follows from 4.1.7. Thus, the map $\text{Hom}_R(P, \beta)$ is a quasi-isomorphism.

5.2.10 Corollary. A semi-free $R$-complex is semi-projective.

**Remark.** A semi-projective complex of free modules may not be semi-free; see E 5.2.4.

5.2.11 Corollary. A graded $R$-module is graded-projective if and only if it is semi-projective as an $R$-complex.

**Proof.** Let $P$ be a graded $R$-module. If $P$ is semi-projective as an $R$-complex, then each module $P_v$ is projective, whence $P$ is graded-projective.

If $P$ is graded-projective, then it is a graded direct summand of a graded-free $R$-module; see 5.2.2. A graded-free $R$-module is semi-free as an $R$-complex by 5.1.3, and then $P$ is semi-free as an $R$-complex by 5.2.9.

5.2.12 Definition. A semi-projective resolution of an $R$-complex $M$ is a quasi-isomorphism $P \to M$ of $R$-complexes where $P$ is semi-projective.

The next existence result is immediate in view of 5.1.7 and 5.2.10.

5.2.13 Theorem. Every $R$-complex $M$ has a semi-projective resolution $\pi: P \to M$ with $P_v = 0$ for all $v < \inf M$. Moreover, $\pi$ can be chosen surjective.

**Properties of Semi-projective Complexes**

5.2.14 Proposition. Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence of $R$-complexes. If $P''$ is semi-projective, then $P'$ is semi-projective if and only if $P$ is semi-projective.
5.2 Semi-projectivity

PROOF. First note that since $P''$ is a complex of projective modules, it follows from 5.2.3 that $P$ is a complex of projective modules if and only if $P'$ is so. Next, let $A$ be an acyclic $R$-complex. The sequence $0 \to P' \to P \to P'' \to 0$ is degreewise split by 5.2.2, and hence so is the induced sequence

$$0 \to \text{Hom}_R(P'', A) \to \text{Hom}_R(P, A) \to \text{Hom}_R(P', A) \to 0;$$

see 2.3.14. As $P''$ is semi-projective, the complex $\text{Hom}_R(P'', A)$ is acyclic by the equivalence of (i) and (vi) in 5.2.9. It follows from 2.2.18 that $\text{Hom}_R(P, A)$ is acyclic if and only if $\text{Hom}_R(P', A)$ is acyclic. Now 5.2.9 yields the desired conclusion. □

5.2.15 Proposition. Let $\{P^u\}_{u \in U}$ be a family of $R$-complexes. The coproduct $\coprod_{u \in U} P^u$ is semi-projective if and only if each complex $P^u$ is semi-projective.

PROOF. Let $\beta : M \to N$ be a surjective quasi-iso-

morphism. There is a commutative diagram in $\mathcal{C}(k)$,

$$\begin{array}{ccc}
\text{Hom}_R(\coprod_{u \in U} P^u, M) & \xrightarrow{\text{Hom}(\coprod_{u \in U} \beta)} & \text{Hom}_R(\coprod_{u \in U} P^u, N) \\
\cong & & \cong \\
\prod_{u \in U} \text{Hom}_R(P^u, M) & \xrightarrow{\prod \text{Hom}(P^u \beta)} & \prod_{u \in U} \text{Hom}_R(P^u, N),
\end{array}$$

where the vertical maps are the canonical isomorphisms from 3.1.25. It follows that $\text{Hom}_R(\coprod_{u \in U} P^u, \beta)$ is a surjective quasi-iso-

morphism if and only if each morphism $\text{Hom}_R(P^u, \beta)$ is a surjective quasi-iso-

morphism.

Also the next result can be interpreted in terms of the diagram (5.2.1.1).

5.2.16 Proposition. Let $P$ be a semi-projective $R$-complex, let $\alpha : P \to N$ be a chain map, and let $\beta : M \to N$ be a quasi-iso-

morphism. There exists a chain map $\gamma : P \to M$ such that $\alpha \sim \beta \gamma$. Moreover, $\gamma$ is homotopic to any other chain map $\gamma'$ with $\alpha \sim \beta \gamma'$.

PROOF. Recall from 2.3.3 the characterization of (null-homotopic) chain maps as (boundaries) cycles in Hom complexes. By 5.2.9 the induced morphism $\text{Hom}_R(P, \beta)$ is a quasi-iso-

morphism, so there exists a $\gamma \in Z(\text{Hom}_R(P, M))$ such that

$$[\alpha] = H(\text{Hom}_R(P, \beta))(\gamma) = [\beta \gamma];$$

that is, $\alpha \sim \beta \gamma$ is in $B(\text{Hom}_R(P, N))$. Given another morphism $\gamma'$ such that $\alpha \sim \beta \gamma'$, one has $[\alpha] = [\beta \gamma']$ and, therefore $0 = [\beta(\gamma - \gamma')] = H(\text{Hom}_R(P, \beta))(\gamma - \gamma')$. It fol-

lows that the homology class $[\gamma - \gamma']$ is 0 as $H(\text{Hom}_R(P, \beta))$ is an isomorphism, so $\gamma - \gamma'$ is in $B(\text{Hom}_R(P, M))$. That is, $\gamma$ and $\gamma'$ are homotopic. □

REMARK. Existence and uniqueness of lifts up to homotopy, as described in 5.2.16, is an important property of semi-projective complexes, but it does not characterize them. Complexes with this property are called $K$-projective; see E 5.2.8.
5.2.17 Corollary. Let $P$ be a semi-projective $R$-complex and let $\beta: M \to P$ be a quasi-isomorphism. There exists a quasi-isomorphism $\gamma: P \to M$ with $1^P \sim \beta \gamma$.

**Proof.** By 5.2.16 there is a chain map $\gamma: P \to M$ with $1^P \sim \beta \gamma$; comparison of degrees shows that $\gamma$ is a morphism. Moreover, by 2.2.23 one has $1^H(P) = H(\beta) H(\gamma)$, whence $H(\gamma)$ is an isomorphism. \hfill $\square$

Recall from 4.2.2 that every homotopy equivalence is a quasi-isomorphism. The next result is a partial converse.

5.2.18 Corollary. A quasi-isomorphism of semi-projective $R$-complexes is a homotopy equivalence.

**Proof.** Let $\beta: P' \to P$ be a quasi-isomorphism of semi-projective $R$-complexes. By 5.2.17 there are morphisms $\gamma: P \to P'$ and $\beta': P' \to P$ such that $1^P \sim \beta \gamma$ and $1^{P'} \sim \gamma \beta'$ hold. It now follows from 2.3.5 that $\beta$ is a homotopy equivalence. \hfill $\square$

5.2.19 Proposition. Let $R \to S$ be a ring homomorphism. If $P$ is a semi-projective $R$-complex, then the $S$-complex $S \otimes_R P$ is semi-projective.

**Proof.** By adjunction 4.3.6 and (4.3.0.2) there are natural isomorphisms

$$\text{Hom}_S(S \otimes_R P, -) \cong \text{Hom}_R(P, \text{Hom}_S(S, -)) \cong \text{Hom}_R(P, -)$$

of functors from $\mathcal{C}(S)$ to $\mathcal{C}(k)$. By assumption, $\text{Hom}_R(P, -)$ is exact and preserves quasi-isomorphisms. \hfill $\square$

5.2.20 Proposition. If $P$ is a semi-projective $S$-complex and $P'$ is a semi-projective $k$-complex, then the $S$-complex $P \otimes_k P'$ is semi-projective.

**Proof.** By adjunction 4.3.6 there is a natural isomorphism,

$$\text{Hom}_S(P \otimes_k P', -) \cong \text{Hom}_k(P', \text{Hom}_S(P, -)),$$

of functors from $\mathcal{C}(S)$ to $\mathcal{C}(k)$. It follows from the assumptions on $P$ and $P'$ that the functor $\text{Hom}_k(P', \text{Hom}_R(P, -))$ is exact and preserves quasi-isomorphisms. \hfill $\square$

The Case of Modules

Notice that specialization of 5.2.2 to modules recovers 1.3.17. The next proposition is a consequence of 1.3.22 and a special case of 5.2.3.

5.2.21 Proposition. Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence of $R$-modules. Assume that $P''$ is projective, then $P'$ is projective if and only if $P$ is projective. \hfill $\square$

5.2.22. It follows from 5.2.11 that an $R$-module is projective if and only if it is semi-projective as an $R$-complex. Thus one recovers 1.3.22 from 5.2.15.
The next theorem is immediate from 5.1.16.

5.2.23 Theorem. For every $R$-module $M$ there is an exact sequence of $R$-modules

$$
\cdots \rightarrow P_v \rightarrow P_{v-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0
$$

where each module $P_v$ is projective.

5.2.24 Definition. Let $M$ be an $R$-module. Together, the surjective homomorphism $P_0 \rightarrow M$ and the $R$-complex $\cdots \rightarrow P_v \rightarrow P_{v-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0$ in 5.2.23 is called a projective resolution of $M$.

5.2.25. By 5.2.7 a projective resolution of an $R$-module $M$ is a semi-projective resolution of $M$ as an $R$-complex.

EXERCISES

E 5.2.1 Show that a graded $R$-module is graded-projective if and only if it is a projective object in the category $\mathcal{M}_R(R)$.

E 5.2.2 Show that a graded $R$-module is graded-projective if and only if it is projective as an $R$-module.

E 5.2.3 Let $R$ be left hereditary. Show that for every complex $P$ of projective $R$-modules there is a quasi-isomorphism $P \xrightarrow{\cong} H(P)$. Hint: E 1.3.19

E 5.2.4 The $\mathbb{Z}/6\mathbb{Z}$-complex $\cdots \rightarrow \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{1} \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \cdots$ is contractible; see E 4.2.10. Show that it is graded-free and semi-projective, but not semi-free.

E 5.2.5 For an $R$-complex $P$, show that the following conditions are equivalent. (i) $P$ is a projective object in the category $\mathcal{C}(R)$. (ii) $P$ is a contractible complex of projective $R$-modules. (iii) $P$ is semi-projective and acyclic. Conclude from E 5.1.9 that the category $\mathcal{C}(R)$ has enough projectives.

E 5.2.6 Show that the Dold complex from 5.1.5 is acyclic but not contractible; conclude that it is not semi-projective.

E 5.2.7 Let $P''$ be a complex of projective $R$-modules and let $0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$ be an exact sequence of $R$-complexes. Show that if two of the complexes $P$, $P'$, and $P''$ are semi-projective, then so is the third.

E 5.2.8 Show that the following conditions are equivalent for an $R$-complex $X$. (i) For every chain map $\alpha: X \rightarrow N$ and every quasi-isomorphism $\beta: M \rightarrow N$ there exists a chain map $\gamma: X \rightarrow M$, unique up to homotopy, such that $\alpha \sim \beta \gamma$. (ii) For every quasi-isomorphism $\beta$ the induced morphism $\Hom_R(X, \beta)$ is a quasi-isomorphism. (iii) For every acyclic complex $A$, the complex $\Hom_R(X, A)$ is acyclic. An $R$-complex with these properties is called $K$-projective.

E 5.2.9 Show that a complex is semi-projective if and only if it is $K$-projective, in the sense of E 5.2.8, and graded-projective. Give an example of a $K$-projective complex that is not semi-projective.
5.3 Semi-injectivity

SYNOPSIS. Character complex; graded-injective module, complex of injective modules; semi-injective complex; semi-injective resolution; lifting property; injective resolution of module.

CHARACTER COMPLEXES

Recall from 1.3.34 that \( \mathcal{E} = \text{Hom}_\mathbb{Z}(\mathbb{k}, \mathbb{Q}/\mathbb{Z}) \) is a faithfully injective \( \mathbb{k} \)-module.

5.3.1 Definition. Let \( M \) be an \( R \)-complex. The \( R^0 \)-complex \( \text{Hom}_\mathbb{k}(M, \mathcal{E}) \) is called the character complex of \( M \). The graded \( R \)-module \( \text{Hom}_\mathbb{k}(M, \mathcal{E})^i = \text{Hom}_\mathbb{k}(M, \mathcal{E})^i \) is called the character module of the graded \( R \)-module \( M^i \).

5.3.2 Lemma. If \( L \) is a semi-free \( R^0 \)-complex, then \( \text{Hom}_\mathbb{k}(L, \mathcal{E}) \) is a complex of injective \( R \)-modules, and the functor \( \text{Hom}_R(-, \text{Hom}_\mathbb{k}(L, \mathcal{E})) \) preserves acyclicity of complexes.

Proof. By 1.3.42 each module \( \text{Hom}_\mathbb{k}(L, \mathcal{E}) \) is an injective \( R \)-module. Let \( A \) be an acyclic \( R \)-complex; adjunction 4.3.6 and commutativity 4.3.2 yield isomorphisms

\[
\text{Hom}_R(A, \text{Hom}_\mathbb{k}(L, \mathcal{E})) \cong \text{Hom}_\mathbb{k}(L \otimes_R A, \mathcal{E}) \cong \text{Hom}_{R^0}(L, \text{Hom}_\mathbb{k}(A, \mathcal{E}))
\]

and \( \text{Hom}_{R^0}(L, \text{Hom}_\mathbb{k}(A, \mathcal{E})) \) is acyclic by 5.2.8 and exactness of \( \text{Hom}_\mathbb{k}(-, \mathcal{E}) \).

5.3.3 Construction. Let \( M \) be an \( R \)-complex and choose by 5.1.7 a semi-free resolution \( \pi: L \rightarrow \text{Hom}_\mathbb{k}(M, \mathcal{E}) \) with \( L_v = 0 \) for all \( v < \inf \text{Hom}_\mathbb{k}(M, \mathcal{E}) \). Pre-compose the induced morphism \( \text{Hom}_\mathbb{k}(\pi, \mathcal{E}): \text{Hom}_\mathbb{k}(\text{Hom}_\mathbb{k}(M, \mathcal{E}), \mathcal{E}) \rightarrow \text{Hom}_\mathbb{k}(L, \mathcal{E}) \) with the biduality morphism \( \delta^M: M \rightarrow \text{Hom}_\mathbb{k}(\text{Hom}_\mathbb{k}(M, \mathcal{E}), \mathcal{E}) \) to get a morphism of \( R \)-complexes

\[
e^M = \text{Hom}_\mathbb{k}(\pi, \mathcal{E})\delta^M : M \rightarrow \mathcal{E},
\]

where \( E \) is the character complex \( \text{Hom}_\mathbb{k}(L, \mathcal{E}). \)

5.3.4 Proposition. Let \( M \) be an \( R \)-complex. The morphisms and complexes constructed in 5.3.3 have the following properties.

(a) \( E \) is a complex of injective \( R \)-modules with \( E_v = 0 \) for \( v > \sup M^i \), and the functor \( \text{Hom}_R(-, E) \) preserves acyclicity of complexes.

(b) The morphism \( H(e^M) \) is injective.

(c) The morphism \( \pi \) can be chosen such that \( e^M \) is injective.

Proof. (a): By (2.2.16.1) one has \( \inf \text{Hom}_\mathbb{k}(M, \mathcal{E}) = -\sup M^i \) and, therefore, \( E_v = \text{Hom}_\mathbb{k}(L_v, \mathcal{E}) = 0 \) for all \( v > \sup M^i \). The other assertions follow from 5.3.2.

(b): One has \( H(e^M) = H(\text{Hom}_\mathbb{k}(\pi, \mathcal{E}))H(\delta^M) \), and the map \( H(\text{Hom}_\mathbb{k}(\pi, \mathcal{E})) \) is an isomorphism by 4.2.11. It follows from 2.2.15 and 4.4.7 that the map \( H(\delta^M) \) is the biduality morphism \( \delta^M : H(M) \rightarrow \mathcal{E}, \) which is injective by 4.4.9. Thus \( H(e^M) \) is injective.
(c): By 5.1.7 one can choose \( \pi \) surjective, and then it follows by exactness of \( \text{Hom}_k(\pi, \mathcal{E}) \) that the morphism \( \text{Hom}_k(\pi, \mathcal{E}) \) is injective. By 4.4.9 the biduality morphism \( \delta^M_\mathcal{E} \) is injective, and hence so is the composite \( \varepsilon^M \).

5.3 Semi-injectivity

Semi-injectivity of an \( R \)-complex \( I \) will be defined in terms of the functor \( \text{Hom}_R(\pi, \mathcal{E}) \) from \( \mathcal{C}(R) \) to \( \mathcal{C}(k) \). First we study complexes of injective modules.

**5.3.5.** Lifting properties are also central to this section; key results can be interpreted in terms of the diagram

\[
\begin{array}{ccc}
K & \longrightarrow & M \\
\downarrow & & \downarrow \\
\wr & \rightarrow & \wr \\
\end{array}
\]

(5.3.5.1)

where the solid arrows represent given maps of certain sorts, and a lifting property of \( I \) ensures the existence of a dotted map of a specific sort such that the diagram is commutative, or commutative up to homotopy.

Part (iii) below can be interpreted in terms of the diagram (5.3.5.1).

**5.3.6 Proposition.** For an \( R \)-complex \( I \), the following conditions are equivalent.

(i) Each \( R \)-module \( I_v \) is injective.

(ii) The functor \( \text{Hom}_R(\pi, I) \) is exact.

(iii) For every homomorphism \( \alpha: K \to I \) and for every injective homomorphism \( \beta: K \to M \), there exists a homomorphism \( \gamma: M \to I \) such that \( \gamma \beta = \alpha \) holds.

(iv) Every exact sequence \( 0 \to I \to M \to M'' \to 0 \) in \( \mathcal{C}(R) \) is degreewise split.

(v) The graded module \( I^\circ \) is a graded direct summand of the character module \( \text{Hom}_k(L, \mathcal{E}) \) of a graded-free \( R^\circ \)-module \( L \).

**Proof.** (i) \( \Rightarrow \) (iii): The homomorphism \( \alpha \) is an element in \( \prod_{v \in \mathbb{Z}} \text{Hom}_R(K_v, I_{v+|\alpha|}) \) and \( \beta \) is an element in \( \prod_{v \in \mathbb{Z}} \text{Hom}_R(K_v, M_{v+|\beta|}) \). For each \( v \in \mathbb{Z} \) there exists by the lifting property of the injective module \( I_{v+|\alpha|} \) a homomorphism \( \gamma_{v+|\beta|}: M_{v+|\beta|} \to I_{v+|\alpha|} \) such that \( \gamma_{v+|\beta|} \beta_v = \alpha_v \) holds; see 1.3.24. The desired homomorphism is the element \( \gamma = (\gamma_v)_{v \in \mathbb{Z}} \) in \( \prod_{v \in \mathbb{Z}} \text{Hom}_R(M_v, I_{v+|\alpha|}-|\beta|) \).

(iii) \( \Rightarrow \) (ii): The functor \( \text{Hom}_R(-, I) \) is left exact; see 2.3.12. Thus, if (iii) holds, then \( \text{Hom}_R(-, I) \) is exact.

(ii) \( \Rightarrow \) (iv): Let \( \beta \) denote the morphism \( I \to M \). Then there exists a homomorphism \( \gamma: M \to I \) such that \( I^\beta = \text{Hom}_R(\beta, I)(\gamma) = \gamma \beta \) holds. As \( \beta \) is a morphism, also the degree of \( \gamma \) must be 0; that is, \( \gamma \) is a morphism of the underlying graded modules. Hence, the sequence \( 0 \to I \to M \to M'' \to 0 \) is degreewise split.
Choose by 5.3.4 an injective morphism \( \epsilon : I \to E \), where \( E \) is the character complex \( \text{Hom}_k(L, E) \) of a semi-free \( R \)-complex \( L \). Apply (iv) to the exact sequence \( 0 \to I \to E \to \text{Coker} \epsilon \to 0 \) in \( \mathcal{C}(R) \). It follows that \( I \) is a graded direct summand of the graded module \( E^\natural = \text{Hom}_k(L^\natural, E) \), and \( L^\natural \) is a graded-free \( R \)-module by 5.1.1.

The character module of a free \( R \)-module is an injective \( R \)-module by 1.3.42. A direct summand of an injective module is injective by additivity of the \( \text{Hom} \) functor. Thus, each module \( I_v \) is an injective \( R \)-module.

**Remark.** The complexes described in 5.3.6 are not the injective objects in the category \( \mathcal{C}(R) \); see E 5.3.1 and E 5.3.3.

**5.3.7 Corollary.** Let \( 0 \to I' \to I \to I'' \to 0 \) be an exact sequence of \( R \)-complexes. If \( I' \) is a complex of injective modules, then \( I \) is a complex of injective modules if and only if \( I'' \) is a complex of injective modules.

**Proof.** If \( I' \) is a complex of injective modules, then \( 0 \to I' \to I \to I'' \to 0 \) is degreewise split, and the assertion follows from 1.3.25.

**5.3.8 Definition.** A graded \( R \)-module \( I \) is called graded-injective if the \( R \)-complex \( I \) satisfies the conditions in 5.3.6.

The prototypical application of the next lemma is to an acyclic complex \( M \) and a complex \( N \) of injective modules.

**5.3.9 Lemma.** Let \( M \) and \( N \) be \( R \)-complexes such that \( M \) or \( N \) is bounded above. If the complex \( \text{Hom}_R(M, N_v) \) is acyclic for every \( v \in \mathbb{Z} \), then \( \text{Hom}_R(M, N) \) is acyclic.

**Proof.** Parallel to the proof of 5.2.5.

**Existence of Semi-injective Resolutions**

**5.3.10 Definition.** An \( R \)-complex \( I \) is called semi-injective if \( \text{Hom}_R(\alpha, I) \) is a surjective quasi-isomorphism for every injective quasi-isomorphism \( \alpha \) in \( \mathcal{C}(R) \).

**Remark.** Another word for semi-injective is DG-injective.

**5.3.11 Example.** Let \( I \) be a bounded above complex of injective \( R \)-modules, and let \( \beta \) be an injective quasi-isomorphism. The morphism \( \text{Hom}_R(\beta, I) \) is surjective by 5.2.2, and it follows from 4.2.14, 4.1.8, and 5.3.9 that it is a quasi-isomorphism; cf. 5.2.7. Thus \( I \) is semi-injective.

**5.3.12 Definition.** A semi-injective resolution of an \( R \)-complex \( M \) is a quasi-isomorphism \( M \to I \) of \( R \)-complexes where \( I \) is semi-injective.

**Remark.** Sometimes, not here, a semi-injective resolution is called a semi-injective coresolution.

The main result of this section is the existence of semi-injective resolutions of complexes; see 5.3.18. The proof relies on the next construction.
5.3 Semi-injectivity

5.3.13 Construction. Given an $R$-complex $M$, we shall construct a commutative diagram in $\mathcal{C}(R)$,

$$
\begin{array}{c}
I & \rightarrow & I^n & \rightarrow & \cdots & \rightarrow & I^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H(I^n-1) & \rightarrow & H(I^n) & \rightarrow & \cdots & \rightarrow & H(I^0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\delta^n & \rightarrow & \delta^n & \rightarrow & \cdots & \rightarrow & \delta^0 \\
\end{array}
$$

For $n = 0$ choose by 5.3.4 an injective morphism $\delta^0: M \rightarrow I^0$, where $I^0$ is the character complex of a semi-free $R^0$-complex.

Let $n \geq 1$ and let a morphism $\iota^{n-1}: M \rightarrow I^{n-1}$ be given. Choose by 5.3.4 an injective morphism $\iota^n: \text{Coker} H(\iota^{n-1}) \rightarrow E^n$, where $E^n$ is the character complex of a semi-free $R^n$-complex. The induced morphism $Z(I^{n-1}) \rightarrow E^n$ is zero on boundaries and on $\iota^{n-1}(Z(M))$; see (5.3.13.2) below. It extends by 5.3.6 to a homomorphism $\delta^n: I^{n-1} \rightarrow E^n$ with $B(I^{n-1}) + \iota^{n-1}(Z(M))$ contained in $\text{Ker} \delta^n \cap Z(I^{n-1})$.

$$
\begin{array}{c}
Z(I^{n-1}) & \rightarrow & I^{n-1} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
H(I^{n-1}) & \rightarrow & H(I^n) & \rightarrow & \cdots & \rightarrow & H(I^0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\delta^n & \rightarrow & \delta^n & \rightarrow & \cdots & \rightarrow & \delta^0 \\
\end{array}
$$

Conversely, let $z$ be a cycle in $I^{n-1}$. If $z$ is in $\text{Ker} \delta^n$, then the element $[z] + \text{Im} H(\iota^{n-1})$ in $\text{Coker} H(\iota^{n-1})$ is in the kernel of $\iota^n$, whence the homology class $[z]$ is in the image of $H(\iota^{n-1})$. Thus, there is an equality

$$
(5.3.13.3) \quad B(I^{n-1}) + \iota^{n-1}(Z(M)) = \text{Ker} \delta^n \cap Z(I^{n-1}).
$$

Consider $\delta^n$ as a degree $-1$ homomorphism: $I^{n-1} \rightarrow \Sigma^{-1} E^n$; cf. 2.2.3. Set

$$
(5.3.13.4) \quad (I^n)^{\delta} = (I^{n-1})^{\delta} \oplus (\Sigma^{-1} E^n)^{\delta} \quad \text{and} \quad \delta^n(i + e) = \delta^n(i) + \delta^n(e).
$$

This defines an $R$-complex, as $\delta^n$ vanishes on boundaries in $I^{n-1}$. Notice that the projection $\pi^n: I^n \rightarrow I^{n-1}$ is a morphism of complexes.

For each boundary $b \in B(M)$ choose a preimage $m_b$. The assignment

$$
(5.3.13.5) \quad b \mapsto \delta^n \iota^{n-1}(m_b)
$$

is independent of the choice of preimage. Indeed, if $\bar{m}_b$ is another preimage of $b$, then $m_b - \bar{m}_b$ is a cycle in $M$, and (5.3.13.3) yields the first equality in the next computation

$$
0 = \delta^n \iota^{n-1}(m_b - \bar{m}_b) = \delta^n \iota^{n-1}(m_b) - \delta^n \iota^{n-1}(\bar{m}_b).
$$
Thus, (5.3.13.5) defines a (degree 0) homomorphism from $B(M)$ to $\Sigma^{-1}E^n$. It extends by 5.3.6 to a homomorphism $\sigma^n: M \to \Sigma^{-1}E^n$, and there is an equality

$$\sigma^n \partial^n = \partial^n \sigma^n. \tag{5.3.13.6}$$

Define a map $t^n: M \to I^n$ as follows:

$$t^n(m) = t^{n-1}(m) + \sigma^n(m). \tag{5.3.13.7}$$

The next computation shows that it is a morphism of $R$-complexes; the penultimate equality uses (5.3.13.6).

$$\partial^n t^n = \partial^n (t^{n-1} + \sigma^n) = \partial^n t^{n-1} + \partial^n \sigma^n = t^{n-1} \partial^n + \sigma^n \partial^n = t^n \partial^n. \tag{5.3.13.8}$$

For $n < 0$ set $I^n = 0$, $t^n = 0$, and $\pi^n = 0$, then the family $\{t^n: I^n \to I^{n-1}\}_{n \in \mathbb{Z}}$ is a tower in $\mathcal{C}(R)$, and one has $t^{n-1} = \pi^n t^n$ for all $n \in \mathbb{Z}$. Set $I = \lim_{n \in \mathbb{Z}} I^n$, by 3.3.29 there is a morphism of $R$-complexes $i: M \to I$, given by $m \mapsto (t^n(m))_{n \in \mathbb{Z}}$.

5.3.14 Proposition. Let $M$ be an $R$-complex. The complexes and morphisms constructed in 5.3.13 have the following properties.

(a) Each $I^n$ is a complex of injective $R$-modules with $I^n = 0$ for $n > \sup M^i$.

(b) $I$ is a complex of injective $R$-modules with $I_v = 0$ for all $v > \sup M^i$, and the functor $\text{Hom}(\_ , I)$ preserves acyclicity of complexes.

(c) The morphism $i: M \to I$ is an injective quasi-isomorphism.

Proof. Part (a) follows from 5.3.4 and (5.3.13.4).

(b): One has $I_v = 0$ for all $v > \sup M^i$ by part (a) and the definition 3.3.2 of limits. Let $0 \to K \to M \to N \to 0$ be an exact sequence in $\mathcal{C}(R)$. For every $n \geq 0$ there is an exact sequence

$$0 \to \text{Hom}_R(N, I^n) \to \text{Hom}_R(M, I^n) \to \text{Hom}_R(K, I^n) \to 0;$$

this follows from (a) and 5.3.6. Because of the degreewise split exact sequences

$$(\ast) \quad 0 \to (\Sigma^{-1}E^n)_i \to I^n \xrightarrow{\pi^n} I^{n-1} \to 0,$$

the morphisms in the tower $\{\text{Hom}_R(N, \pi^n): \text{Hom}_R(N, I^n) \to \text{Hom}_R(N, I^{n-1})\}_{n \in \mathbb{Z}}$ are surjective; see 2.3.14. It now follows from 3.3.32 that the lower row in the commutative diagram below is exact.

\[
\begin{array}{ccc}
\text{Hom}_R(M, I) & \xrightarrow{\cong} & \text{Hom}_R(K, I) \\
\downarrow & & \downarrow \\
\lim_{n \in \mathbb{Z}} \text{Hom}_R(M, I^n) & \xrightarrow{\cong} & \lim_{n \in \mathbb{Z}} \text{Hom}_R(K, I^n) \to 0
\end{array}
\]
The vertical maps are the isomorphisms from 3.3.16; the diagram shows that the functor $\text{Hom}_R(-, I)$ is exact, whence $I$ is a complex of injective modules by 5.3.6.

Let $A$ be an acyclic $R$-complex. As above the morphisms in the induced tower $\{\text{Hom}_R(A, \pi^n): \text{Hom}_R(A, \pi^n) \to \text{Hom}_R(A, \pi^{-1})\}_{n \in \mathbb{Z}}$ are surjective. By 5.3.4, the functors $\text{Hom}_R(-, (E^n)^{\vee})$ preserve acyclicity, so by induction it follows from $(\ast)$ and 2.2.18 that the functors $\text{Hom}_R(-, I^n)$ preserve acyclicity; in particular $\text{Hom}_R(A, I^n)$ is acyclic for every $n$. By 3.3.16 and 3.3.37 the complex

$$\text{Hom}_R(A, I) = \text{Hom}_R(A, \lim_{n \in \mathbb{Z}} I^n) \cong \lim_{n \in \mathbb{Z}} \text{Hom}_R(A, I^n)$$

is acyclic.

(c): As $i^0$ is injective, commutativity of (5.13.1) shows that $i$ is injective as well. By 5.3.4 the morphism $H(i^0)$ is injective, and the commutative diagram

$$\begin{array}{ccc}
H(M) & \xrightarrow{H(i)} & H(I) \\
\downarrow H(\iota^0) & & \downarrow H(\iota^0) \\
\downarrow & & \downarrow \\
H(I) & \xrightarrow{H(i^0)} & H(I') \\
\end{array}$$

which is induced from (5.13.1), shows that $H(i)$ is injective. To see that it is injective, let $z = (\pi^n)_{n \in \mathbb{Z}}$ be a cycle in $I$; the goal is to show that there exist elements $m \in Z(M)$ and $i = (\pi^n)_{n \in \mathbb{Z}}$ in $I$ with $z = \partial^i(i) + i(\iota(m))$. From (5.13.4) one gets

$$(\circ) \quad 0 = \partial^i(z) = (\partial^0(z^0), \ldots, \partial^{\pi^n}(z^{\pi^n}) + \delta^0(z^{\pi^1}), \partial^0(z^{\pi^1}) + \delta^0(z^{\pi^2}), \ldots).$$

It follows for each $n \geq 1$ that the element $z^{\pi^n}$ is a cycle in $I^{\pi^n}$ with $\delta^n(z^{\pi^n}) = 0$, whence $z^{\pi^n}$ belongs $B(I^{\pi^n} + \iota^{\pi^n}(Z(M)))$ by (5.3.13.3).

Choose elements $j^2$ in $I^2$ and $m \in Z(M)$ such that $z^2 = \partial^2(j^2) + i^2(m)$ holds. The sequence $(\pi^n)_{n \in \mathbb{Z}}$ is constructed by induction. Set $i^1 = \pi^2(j^2)$ and $\iota^1 = \pi^1(i^1)$, then there are equalities $z^1 = \pi^2(z^2) = \partial^1(i^1) + i^1(m)$ and $\iota^0 = \pi^1(z^1) = \partial^1(\iota^1) + \iota^1(m)$. Set $\pi^n = 0$ for $n < 0$. Fix $n \geq 2$ and assume that elements $i^n \in I^n$ for $u < n$ and $j^n \in I^n$ have been constructed, such that one has

$$z^n = \partial^n(j^n) + i^n(m) \quad \text{and} \quad z^n = \partial^n(\iota^n) + i^n(m) \quad \text{for} \quad u < n;$$

$$\partial^n(i^n) = i^{n-1} \quad \text{and} \quad \pi^n(i^n) = i^{n-1} \quad \text{for} \quad u < n.$$

Choose $j^n \in I^{\pi^n}$ and $m' \in Z(M)$ with $z^{\pi^n+1} = \partial^{\pi^n+1}(j^n) + i^{\pi^n+1}(m')$. The equalities

$$\partial^{\pi^n}(j^n) + i^{\pi^n}(m) = z^n = \pi^{\pi^n+1}(z^{\pi^n+1}) = \partial^{\pi^n+1}(j^n) + i^{\pi^n+1}(m')$$

show that $i^{\pi^n}(m' - m)$ is a boundary in $I^n$. It follows from commutativity of the diagram $(\circ)$ that $H(I^n)$ is injective, so $m' - m$ is in $B(M)$ and, therefore, $i^{\pi^n}(m' - m)$ is in $B(I^n)$. Thus, there exists $j^n \in I^{\pi^n+1}$ with $\partial^{\pi^n+1}(j^n) = \partial^{\pi^n+1}(j^n) + i^{\pi^n+1}(m' - m)$.
and, therefore, \( z^{n+1} = \partial^{n+1}(j'') + \iota^{n+1}(m) \). The equalities
\[
\partial^n(j'') + \iota^n(m) = z^n = \pi^{n+1}(z^{n+1}) = \partial^n(\pi^{n+1}(j'')) + \iota^n(m)
\]
show that \( j'' \in \pi^{n+1}(j'') \) is a cycle in \( I^n \) and, therefore, \( \pi^n(j'' - \pi^{n+1}(j'')) \) is a cycle in \( I^{n-1} \). Now (5.3.13.4) yields \( \delta^n(\pi^n(j'' - \pi^{n+1}(j''))) = 0 \), and it follows from (5.3.13.3) that there are elements \( i' \in I^{n-1} \) and \( c \in \mathbb{Z}(M) \) with
\[
(\$) \quad \pi^n(j'' - \pi^{n+1}(j'')) = \partial^{n-1}(i') + \iota^{n-1}(c).
\]
Choose \( i' \in I^{n+1} \) with \( \pi^n\pi^{n+1}(i') = i' \) and set \( j'' = j'' + \partial^{n+1}(i') + \iota^{n+1}(c) \). As \( \partial^{n+1}(i') + \iota^{n+1}(c) \) is a cycle in \( I^{n+1} \), the equality
\[
(#) \quad z^{n+1} = \partial^{n+1}(j^{n+1}) + \iota^{n+1}(m)
\]
holds. Set \( i'' = \pi^{n+1}(j^{n+1}) \); then (\#) yields \( z^n = \pi^{n+1}(z^{n+1}) = \partial^n(i'') + \iota^n(m) \), and the third equality below follows from (\$),
\[
\pi^n(i'') = \pi^n\pi^{n+1}(j^{n+1}) = \pi^n\pi^{n+1}(j'') + \partial^{n-1}(\pi^n\pi^{n+1}(i'') + \iota^{n-1}(c)) = \pi^n(j'') = I^{n-1}.
\]
Thus, for \( u < n + 1 \) one has
\[
(**) \quad z^n = \partial^n(i^n) + \iota^n(m) \quad \text{and} \quad i^{n-1} = \pi^n(i^n).
\]
From (\#) and (**) it now follows that the desired element \( i = (i^n)_{n \in \mathbb{Z}} \) in \( I \) with \( z = \iota(m) + \partial(i) \) exists. \( \square \)

The next result offers useful characterizations of semi-injective complexes. The lifting property in part (iii) can be interpreted in terms of the diagram (5.3.5.1).

5.3.15 Proposition. For an \( R \)-complex \( I \), the following conditions are equivalent.

(i) \( I \) is semi-injective.

(ii) The functor \( \text{Hom}_R(-, I) \) is exact and preserves quasi-isomorphisms.

(iii) For every chain map \( \alpha : K \to I \) and for every injective quasi-isomorphism \( \beta : K \to M \) there exists a chain map \( \gamma : M \to I \) such that \( \gamma \beta = \alpha \) holds.

(iv) Every exact sequence \( 0 \to I \to M \to M'' \to 0 \) in \( \mathcal{C}(R) \) with \( M'' \) acyclic is split.

(v) \( I \) is a complex of injective \( R \)-modules, and the functor \( \text{Hom}_R(-, I) \) preserves acyclicity of complexes.

PROOF. The implication (ii) \( \implies \) (i) is evident.

(i) \( \implies \) (iii): The morphism \( \text{Hom}_R(\beta, I) \) is a surjective quasi-isomorphism. In particular, it is surjective on cycles; see 4.2.12. Thus, in view of 2.3.3 there exists a chain map \( \gamma : M \to I \) such that \( \alpha = \text{Hom}_R(\beta, I)(\gamma) = \gamma \beta \) holds.

(iii) \( \implies \) (iv): By (2.2.17.1) the morphism \( \beta : I \to M \) is a quasi-isomorphism, so there exists a chain map \( \gamma : M \to I \) with \( \gamma \beta = 1_I \). As \( \beta \) is of degree 0, so is \( \gamma \). That is, \( \gamma \) is a morphism in \( \mathcal{C}(R) \), whence the sequence is split.
(iv) $\implies$ (v): Chose by 5.3.14 an injective quasi-isomorphism $\iota: I \to I'$, where $I'$ is a complex of injective modules such that the functor $\text{Hom}_R(-, I')$ preserves acyclicity of complexes. Apply (iv) to the exact sequence $0 \to I \to I' \to \text{Coker}\iota \to 0$. It follows that $I$ is a direct summand of $I'$, so by additivity of the Hom functor, $I$ is a complex of injective modules and $\text{Hom}_R(-, I)$ preserves acyclicity of complexes.

(v) $\implies$ (ii): The functor $\text{Hom}_R(-, I)$ is exact by 5.3.6. For a quasi-isomorphism $\alpha$, the complex $\text{Cone}\alpha$ is acyclic by 4.2.14; hence so is the complex $\text{Hom}_R(\text{Cone}\alpha, I) \cong \Sigma \text{Cone} \text{Hom}_R(\alpha, I)$, where the isomorphism follows from 4.1.8. Thus, $\text{Hom}_R(\alpha, I)$ is a quasi-isomorphism.

5.3.16 **Corollary.** Let $P$ be an $R^o$-complex. If $P$ is semi-projective, then the $R$-complex $\text{Hom}_R(P, E)$ is semi-injective.

**Proof.** It follows from 1.3.41 that $\text{Hom}_R(P, E)$ is a complex of injective $R$-modules. By adjunction 4.3.6 and commutativity 4.3.2 there are natural isomorphisms

$$\text{Hom}_R(-, \text{Hom}_R(P, E)) \cong \text{Hom}_R(P \otimes_R -, E) \cong \text{Hom}_R(P, \text{Hom}_R(-, E))$$

of functors from $\mathcal{C}(R)^{op}$ to $\mathcal{C}(k)$. By assumption, $\text{Hom}_R(P, \text{Hom}_R(-, E))$ preserves acyclicity of complexes. Thus, $\text{Hom}_R(P, E)$ is semi-injective.

5.3.17 **Corollary.** A graded $R$-module is graded-injective if and only if it is semi-injective as an $R$-complex.

**Proof.** Let $I$ be a graded $R$-module. If $I$ is semi-injective as an $R$-complex, then each module $I_v$ is injective and hence $I$ is graded-injective.

If $I$ is a graded-injective $R$-module, then by 5.3.6 it is a direct summand of the character module of a graded-free $R^o$-module. By 5.1.3 and 5.2.10 a graded-free module is semi-projective. Thus 5.3.16 shows that $I$ is a direct summand of a semi-projective $R$-complex and hence semi-injective by additivity of the Hom functor.

5.3.18 **Theorem.** Every $R$-complex $M$ has a semi-injective resolution $\iota: M \to I$ with $I_v = 0$ for all $v > \sup M$. Moreover, $\iota$ can be chosen injective.

**Proof.** Apply 5.3.14 to get an injective quasi-isomorphism $\iota: M \to I$. The complex $I$ has $I_v = 0$ for $v > \sup M$, and it is semi-injective by 5.3.14 and 5.3.15.

**Properties of Semi-injective Complexes**

5.3.19 **Proposition.** Let $0 \to I' \to I \to I'' \to 0$ be an exact sequence of $R$-complexes. If $I'$ is semi-injective, then $I$ is semi-injective if and only if $I''$ is semi-injective.

**Proof.** First note that since $I'$ is a complex of injective modules, it follows from 5.3.7 that $I$ is a complex of injective modules if and only if $I''$ is so. Next, let $A$ be an acyclic $R$-complex. The sequence $0 \to I' \to I \to I'' \to 0$ is degreewise split by 5.3.6, so by applying $\text{Hom}_R(A, -)$ one obtains an exact sequence; see 2.3.14. As
5.3.20 Proposition. Let \( \{I^u\}_{u \in U} \) be a family of \( R \)-complexes. The product \( \prod_{u \in U} I^u \) is semi-injective if and only if each complex \( I^u \) is semi-injective.

PROOF. Let \( \beta: K \to M \) be an injective quasi-isomorphism. There is a commutative diagram in \( \mathcal{C}(k) \),

\[
\begin{array}{ccc}
\Hom_R(M, \prod_{u \in U} I^u) & \xrightarrow{\Hom(\beta, \prod^u)} & \Hom_R(K, \prod_{u \in U} I^u) \\
\cong & & \cong \\
\prod_{u \in U} \Hom_R(M, I^u) & \xrightarrow{\prod \Hom(\beta, I^u)} & \prod_{u \in U} \Hom_R(M, I^u),
\end{array}
\]

where the vertical maps are the canonical isomorphisms from 3.1.28. It follows that \( \Hom_R(\beta, \prod_{u \in U} I^u) \) is a surjective quasi-isomorphism if and only if each morphism \( \Hom_R(\beta, I^u) \) is a surjective quasi-isomorphism.

Also the next result can be interpreted in terms of the diagram (5.3.5.1).

5.3.21 Proposition. Let \( I \) be a semi-injective \( R \)-complex, let \( \alpha: K \to I \) be a chain map, and let \( \beta: K \to M \) be a quasi-isomorphism. There exists a chain map \( \gamma: M \to I \) such that \( \gamma \beta \sim \alpha \). Moreover, \( \gamma \) is homotopic to any other chain map \( \gamma' \) with \( \gamma' \beta \sim \alpha \).

PROOF. Recall from 2.3.3 the characterization of (null-homotopic) chain maps as (boundaries) cycles in \( \Hom \) complexes. By 5.3.15 the induced morphism \( \Hom_R(\beta, I) \) is a quasi-isomorphism, so there exists a \( \gamma \in Z(\Hom_R(M, I)) \) such that

\[
[\alpha] = H(\Hom_R(\beta, I))(\gamma) = [\gamma \beta];
\]

that is, \( \alpha - \gamma \beta \) is in \( B(\Hom_R(K, I)) \). Given another morphism \( \gamma' \) such that \( \gamma' \beta \sim \alpha \), one has \([\alpha] = [\gamma' \beta] \) and, therefore \( 0 = [(\gamma - \gamma') \beta] = H(\Hom_R(\beta, I))(\gamma - \gamma') \). It follows that the homology class \( [\gamma - \gamma'] \) is 0 as \( H(\Hom_R(\beta, I)) \) is an isomorphism, so \( \gamma - \gamma' \) is in \( B(\Hom_R(M, I)) \). That is, \( \gamma \) and \( \gamma' \) are homotopic.

REMARK. Existence and uniqueness of lifts up to homotopy, as described in 5.3.21, is an important property of semi-injective complexes, but it does not characterize them. Complexes with this property are called \( K \)-injective; see E 5.3.9.

5.3.22 Corollary. Let \( I \) be a semi-injective \( R \)-complex and let \( \beta: I \to M \) be a quasi-isomorphism. There exists a quasi-isomorphism \( \gamma: M \to I \) such that \( \gamma \beta \sim I^1 \).

PROOF. By 5.3.21 there exists a chain map \( \gamma: M \to I \) with \( \gamma \beta \sim I^1 \); comparison of degrees shows that \( \gamma \) is a morphism. Moreover, by 2.2.23 one has \( H(\gamma) H(\beta) = I^{1[H(I)]} \), whence \( H(\gamma) \) is an isomorphism.
Recall from 4.2.2 that every homotopy equivalence is a quasi-isomorphism. The next result is a partial converse and akin to 5.2.18.

5.3.23 Corollary. A quasi-isomorphism of semi-injective \( R \)-complexes is a homotopy equivalence.

**Proof.** Let \( \beta : I \rightarrow I' \) be a quasi-isomorphism of semi-injective \( R \)-complexes. By 5.3.22 there are morphisms \( \gamma : I' \rightarrow I \) and \( \beta' : I \rightarrow I' \) such that \( \gamma \beta \sim 1_I \) and \( \beta' \gamma \sim 1_{I'} \) hold. It now follows from 2.3.5 that \( \beta \) is a homotopy equivalence. \( \Box \)

5.3.24 Proposition. Let \( R \rightarrow S \) be a ring homomorphism. If \( I \) is a semi-injective \( R \)-complex, then the \( S \)-complex \( \text{Hom}_R(S, I) \) is semi-injective.

**Proof.** By adjunction 4.3.6 and (4.3.0.1) there are natural isomorphisms

\[
\text{Hom}_S(-, \text{Hom}_R(S, I)) \cong \text{Hom}_R(S \otimes_S -, I) \cong \text{Hom}_R(-, I)
\]

of functors from \( C(S) \) to \( C(k) \). By assumption, \( \text{Hom}_R(-, I) \) is exact and preserves quasi-isomorphisms. \( \Box \)

5.3.25 Proposition. If \( I \) is a semi-injective \( S \)-complex and \( P \) is a semi-projective \( k \)-complex, then the \( S \)-complex \( \text{Hom}_k(P, I) \) is semi-injective.

**Proof.** By swap 4.3.8 there is a natural isomorphism

\[
\text{Hom}_S(-, \text{Hom}_k(P, I)) \cong \text{Hom}_k(P, \text{Hom}_S(-, I))
\]

of functors from \( C(S) \) to \( C(k) \). It follows from the assumptions on \( I \) and \( P \) that the functor \( \text{Hom}_k(P, \text{Hom}_S(-, I)) \) is exact and preserves quasi-isomorphisms. \( \Box \)

**Boundedness**

5.3.26 Theorem. Every \( R \)-complex \( M \) has a semi-injective resolution \( M \rightarrowto I \) with \( I_v = 0 \) for all \( v > \text{sup} M \).

**Proof.** If \( M \) is acyclic, then the morphism \( M \rightarrowto 0 \) is the desired resolution. If \( H(M) \) is not bounded above, then any semi-injective resolution of \( M \) has the desired property. Assume now that \( H(M) \) is bounded above and set \( u = \text{sup} M \). By 4.2.6 there is a quasi-isomorphism \( M \rightarrowto M_{<u} \). By 5.3.18 the truncated complex \( M_{<u} \) has a semi-injective resolution \( M_{<u} \rightarrowto I \) with \( I_v = 0 \) for \( v > u \). The desired semi-injective resolution is the composite \( M \rightarrowto M_{<u} \rightarrowto I \). \( \Box \)
The Case of Modules

From 5.3.4 one immediately gets the following dual to 1.3.11.

**5.3.27 Lemma.** For every $R$-module $M$ there is an injective homomorphism of $R$-modules $M \to I$ where $I$ is injective.

A homomorphism $M \to I$ as above is called an injective preenvelope of $M$.

Specialization of 5.3.6 to modules gives the next result. Part (ii), which recovers the lifting property 1.3.24, can be interpreted in terms of the diagram (5.3.5.1).

**5.3.28 Proposition.** For an $R$-module $I$, the following conditions are equivalent.

(i) $I$ is injective.

(ii) For every homomorphism $\alpha: K \to I$ and for every injective homomorphism $\beta: K \to M$, there exists a homomorphism $\gamma: M \to I$ such that $\gamma \beta = \alpha$ holds.

(iii) Every exact sequence $0 \to I \to M \to M'' \to 0$ of $R$-modules is split.

(iv) $I$ is a direct summand of the character module of a free $R^a$-module.

Specialization of 5.3.6 to modules yields the following.

**5.3.29 Corollary.** Let $0 \to I' \to I \to I'' \to 0$ be an exact sequence of $R$-modules. Assume that $I'$ is injective, then $I$ is injective if and only if $I''$ is injective.

**5.3.30.** It follows from 5.3.17 that an $R$-module is injective if and only if it is semi-injective as an $R$-complex. Thus one recovers 1.3.25 from 5.3.20.

**5.3.31 Theorem.** For every $R$-module $M$ there is an exact sequence of $R$-modules

\[ 0 \to M \to I_0 \to \cdots \to I_v \to I_{v-1} \to \cdots \]

where each module $I_v$ is injective.

**PROOF.** Choose by 5.3.18 a semi-injective resolution $\iota: M \to I$ with $I_v = 0$ for all $v > 0$ and $I_0$ injective. The displayed sequence of $R$-modules is the complex $\text{Cone} \iota$; in particular, the map $M_0 \to I_0$ is the homomorphism $t_0$. The cone is acyclic because $\iota$ is a quasi-isomorphism; see 4.2.14.

**5.3.32 Definition.** Let $M$ be an $R$-module. Together, the injective homomorphism $M \to I_0$ and the $R$-complex $0 \to I_0 \to \cdots \to I_v \to I_{v-1} \to \cdots$ in 5.3.31 is called an injective resolution of $M$.

**5.3.33.** By 5.3.11 an injective resolution of an $R$-module $M$ is a semi-injective resolution of $M$ as an $R$-complex.
5.4 Minimality

EXERCISES

E 5.3.1 Show that a graded $R$-module is graded-injective if and only if it is an injective object in the category $\mathcal{M}_R(R)$.

E 5.3.2 Show that a graded $R$-module is graded-injective if it is injective as an $R$-module. Is the converse true?

E 5.3.3 For an $R$-complex $I$, show that the following conditions are equivalent. (i) $I$ is an injective object in the category $\mathcal{C}(R)$. (ii) $I$ is a contractible complex of injective $R$-modules. (iii) $I$ is semi-injective and acyclic.

E 5.3.4 Show that a graded $R$-module is graded-injective if it is injective as an $R$-module. Is the converse true?

E 5.3.5 For an $R$-complex $I$, show that the following conditions are equivalent. (i) $I$ is an injective object in the category $\mathcal{C}(R)$. (ii) $I$ is a contractible complex of injective $R$-modules. (iii) $I$ is semi-injective and acyclic.

E 5.3.6 Show that the category $\mathcal{C}(R)$ has enough injectives. That is, for every $R$-complex $M$ there is an injective morphism $M \to I$ where $I$ is a contractible complex of injective $R$-modules; cf. E 5.3.3.

E 5.3.7 Show that the Dold complex from 5.1.5 is an acyclic complex of injective modules. Show that it is not contractible and conclude that it is not semi-injective.

E 5.3.8 Give a proof of 5.3.9.

E 5.3.9 Show that the following conditions are equivalent for an $R$-complex $Y$. (i) For every chain map $\alpha: K \to Y$ and every quasi-isomorphism $\beta: K \to M$ there exists a chain map $\gamma: M \to Y$, unique up to homotopy, such that $\gamma \beta \sim \alpha$. (ii) For every quasi-isomorphism $\beta$ the induced morphism $\text{Hom}_R(\beta, Y)$ is a quasi-isomorphism. (iii) For every acyclic complex $A$, the complex $\text{Hom}_R(A, Y)$ is acyclic.

An $R$-complex with these properties is called K-injective.

E 5.3.10 Show that a complex is semi-injective if and only if it is K-injective, in the sense of E 5.3.9, and graded-injective. Give an example of a K-injective complex that is not semi-injective.

E 5.3.11 Show that every morphism $\alpha: M \to N$ of $R$-complexes admits factorizations in $\mathcal{C}(R)$,

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow{\varepsilon} & \sim & \downarrow{\varphi} \\
X & & Y
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow{\iota} & \sim & \downarrow{\pi} \\
X & & Y
\end{array}
\]

where $\varepsilon$ and $\iota$ are injective; $\varphi$ and $\pi$ are surjective with semi-injective kernels; and $\varphi$ and $\pi$ are quasi-isomorphisms. Hint: (1) E 5.3.4. (2) Modify the first step in 5.3.13

E 5.3.12 Give an alternative proof of 5.3.31 based on 5.3.27.

5.4 Minimality

SYNOPSIS. Minimal complex; injective envelope; minimal semi-injective resolution; Nakayama’s lemma; projective cover; minimal semi-projective resolution; semi-perfect ring; perfect ring.

5.4.1 Lemma. Let $0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a degree-wise split exact sequence of $R$-complexes.
(a) The complex \( K \) is contractible if and only if \( \beta \) is a homotopy equivalence, and in that case the sequence splits in \( C(R) \).

(b) The complex \( N \) is contractible if and only if \( \alpha \) is a homotopy equivalence, and in that case the sequence splits in \( C(R) \).

**Proof.** It follows from 2.3.14 that the following sequences are exact.

\[
\begin{align*}
0 & \to \text{Hom}_R(K,K) \to \text{Hom}_R(K,M) \xrightarrow{\text{Hom}_R(K,\beta)} \text{Hom}_R(K,N) \to 0, \\
0 & \to \text{Hom}_R(N,M) \xrightarrow{\text{Hom}_R(\beta,M)} \text{Hom}_R(M,M) \to \text{Hom}_R(K,M) \to 0, \quad \text{and} \\
0 & \to \text{Hom}_R(N,K) \to \text{Hom}_R(N,M) \xrightarrow{\text{Hom}_R(N,\beta)} \text{Hom}_R(N,N) \to 0.
\end{align*}
\]

If \( \beta \) is a homotopy equivalence, then so is \( \text{Hom}_R(K,\beta) \) by 2.3.10; in particular, it is a quasi-isomorphism. It follows from (a) and (2.2.17.1) that \( \text{Hom}_R(K,K) \) is acyclic, whence \( K \) is contractible by 4.2.19.

Conversely, if \( K \) is contractible, then \( \text{Hom}_R(K,M) \) and \( \text{Hom}_R(N,K) \) are acyclic by 4.2.19. It follows from (c) that \( \text{Hom}_R(N,\beta) \) is a quasi-isomorphism. It is surjective on cycles by 4.2.12, so there exists a morphism \( \gamma : N \to M \) with \( \beta \gamma = 1^N \). In particular, the original sequence splits in \( C(R) \). Moreover, it follows from (c) that \( \text{Hom}_R(\beta,M) \) is a quasi-isomorphism, whence there exists a morphism \( \gamma' : N \to M \) such that \( \gamma' \beta \sim 1^M \). Thus \( \beta \) is a homotopy equivalence; see 2.3.5. This proves part (a), and a similar argument proves (b).

5.4.2 **Definition.** An \( R \)-complex \( M \) is called **minimal** if every homotopy equivalence \( e : M \to M \) is an isomorphism.

5.4.3 **Example.** The \( \mathbb{Z} \)-complex \( I = 0 \to Q \to Q/\mathbb{Z} \to 0 \) is minimal, as there are no non-zero homomorphisms \( Q/\mathbb{Z} \to Q \) and hence no non-zero homomorphisms \( I \to I \) of degree 1. It follows that every homotopy equivalence \( I \to I \) is an isomorphism.

5.4.4 **Example.** Let \( k \) be a field and consider the local ring \( R = k[x]/(x^2) \) of **dual numbers**. The \( R \)-complex

\[
X = \cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots
\]

is minimal. Indeed, every homomorphism \( R \to R \) is given by multiplication by an element in \( R \). A homotopy equivalence \( \alpha : X \to X \) is, therefore, given by a sequence of ring elements \( (a_v)_{v \in \mathbb{Z}} \); so is its homotopy inverse \( \beta = (b_v)_{v \in \mathbb{Z}} \) and the homotopy \( \varphi = (r_v)_{v \in \mathbb{Z}} \) from \( 1^X \) to \( a \beta \). For every \( v \in \mathbb{Z} \) one has \( 1 - a_v b_v = x r_v + r_{v-1} x \). The element \( a_v b_v = 1 - x(r_v + r_{v-1}) \) is not in the maximal ideal \( (x) \), so it is invertible, whence \( a_v \) is invertible and \( a \) is an isomorphism.

5.4.5 **Proposition.** For an \( R \)-complex \( M \), the following conditions are equivalent.

(i) \( M \) is minimal.

(ii) Every morphism \( e : M \to M \) with \( e \sim 1^M \) is an isomorphism.

(iii) Every homotopy equivalence \( \alpha : K \to M \) has a right inverse.
Every homotopy equivalence $\varepsilon: M \to M$ has a right inverse.

(iv) Every homotopy equivalence $\beta: M \to N$ has a left inverse.

(iv') Every homotopy equivalence $\varepsilon: M \to M$ has a left inverse.

When these conditions hold, the complexes $\text{Ker}\alpha$ and $\text{Coker}\beta$ are contractible for every homotopy equivalence $\alpha: K \to M$ and every homotopy equivalence $\beta: M \to N$.

**Proof.** First notice that if $\alpha: K \to M$ is a homotopy equivalence and (iii) holds, then there is a split exact sequence $0 \to \text{Ker}\alpha \to K \xrightarrow{\alpha} M \to 0$, and $\text{Ker}\alpha$ is contractible by 5.4.1. Similarly, if $\beta: M \to N$ is a homotopy equivalence and (iv) holds, then $\text{Coker}\beta$ is contractible.

(i) $\implies$ (ii): If the endomorphisms $\varepsilon: M \to M$ and $1^M$ are homotopic, then one has $\varepsilon^2 \sim \varepsilon \sim 1^M$; see 2.3.5. It follows that $\varepsilon$ is its own homotopy inverse; in particular, $\varepsilon$ is a homotopy equivalence, so by (i) it is an isomorphism.

(ii) $\implies$ (iii): Let $\alpha: K \to M$ be a homotopy equivalence with homotopy inverse $\gamma: M \to K$. By (ii) the morphism $\varepsilon = \alpha\gamma$ is an isomorphism, and it follows that $\gamma\varepsilon^{-1}$ is a right inverse of $\alpha$.

(iii') $\implies$ (i): Let $\varepsilon: M \to M$ be a homotopy equivalence. It has a right inverse, so the sequence $0 \to \text{Ker}\varepsilon \xrightarrow{1^M} M \xrightarrow{\varepsilon} M \to 0$ is split exact. Consider the split exact sequence $0 \to M \xrightarrow{\varepsilon} M \xrightarrow{\text{Ker}\varepsilon} 0$, where $\tau = 1^{\text{Ker}\varepsilon}$ and $\varepsilon\sigma = 1^M$. As already noted, $\text{Ker}\varepsilon$ is contractible, whence $\sigma$ is a homotopy equivalence by 5.4.1 and surjective by (iii'). Thus $\sigma$ is an isomorphism and, therefore, $\varepsilon$ is an isomorphism.

(ii) $\implies$ (iv): Let $\beta: M \to N$ be a homotopy equivalence with homotopy inverse $\gamma: N \to M$. By (ii) the morphism $\varepsilon = \gamma\beta$ is an isomorphism, and it follows that $\varepsilon^{-1}\gamma$ is a left inverse of $\beta$.

(iv') $\implies$ (i): Let $\varepsilon: M \to M$ be a homotopy equivalence. It has a left inverse, so the sequence $0 \to M \xrightarrow{\varepsilon} M \xrightarrow{\text{Coker}\varepsilon} 0$ is split exact. Consider the split exact sequence $0 \to \text{Coker}\varepsilon \xrightarrow{1^{\text{Coker}\varepsilon}} M \xrightarrow{\tau} M \to 0$, where $\pi\sigma = 1^{\text{Coker}\varepsilon}$ and $\tau\varepsilon = 1^M$. As already noted, $\text{Coker}\varepsilon$ is contractible, whence $\tau$ is a homotopy equivalence by 5.4.1 and injective by (iv'). Thus $\tau$ is an isomorphism and, therefore, $\varepsilon$ is an isomorphism.

5.4.6 Corollary. Let $M$ and $M'$ be minimal $R$-complexes.

(a) Every homotopy equivalence $\alpha: M \to M'$ is an isomorphism.

(b) If there exist contractible $R$-complexes $C$ and $C'$ such that $M \oplus C$ and $M' \oplus C'$ are homotopy equivalent, then the complexes $M$ and $M'$ are isomorphic.

**Proof.** (a): It follows from 5.4.5 that $\alpha$ has a right inverse as well as a left inverse, whence it is an isomorphism.

(b): It follows from 5.4.1 that the embedding $M \to M \oplus C$ and the projection $M' \oplus C' \to M'$ are homotopy equivalences. Therefore, the composite map

$$M \hookrightarrow M \oplus C \xrightarrow{\sim} M' \oplus C' \to M'$$

is a homotopy equivalence, and hence an isomorphism by (a).
INJECTIVE ENVELOPES

5.4.7 Definition. Let $M$ be a graded $R$-module. A graded submodule $N$ of $M$ is called essential if $M' \cap N \neq 0$ holds for every graded submodule $M' \neq 0$ of $M$.

Remark. Another word for essential submodule is large submodule.

5.4.8 Example. Every non-zero ideal in $\mathbb{Z}$ is essential.

5.4.9. A graded direct summand $N$ of a graded $R$-module $M$ is essential if and only if $N = M$.

5.4.10 Lemma. Let $M$ be a graded $R$-module with graded submodules $N$ and $N'$.

(a) Assume that there is an inclusion $N \subseteq N'$. If $N$ is essential in $N'$, and $N'$ is essential in $M$, then $N$ is essential in $M$.

(b) If $N$ and $N'$ are essential in $M$, then $N \cap N'$ is essential in $M$.

(c) Let $\alpha : M \rightarrow X$ be a homomorphism of graded $R$-modules. If $\alpha$ is injective and $N$ is essential in $M$, then $\alpha(N)$ is essential in $\alpha(M)$.

Proof. (a): Let $M' \neq 0$ be a graded submodule of $M$. The submodule $N \cap M' = N \cap (N' \cap M')$ is non-zero, as $N'$ is essential in $M$ and $N$ is essential in $N'$.

(b): Let $M' \neq 0$ be a graded submodule of $M$. The submodule $(N \cap N') \cap M' = N \cap (N' \cap M')$ is non-zero, as $N'$ and $N$ are both essential in $M$.

(c): Immediate as $\alpha : M \rightarrow \alpha(M)$ is an isomorphism.

5.4.11 Definition. An injective envelope of a graded $R$-module $M$ is an injective morphism $\iota : M \rightarrow E$ of graded $R$-modules where $E$ is graded-injective and $\text{Im } \iota$ is essential in $E$.

5.4.12 Example. The embedding $\mathbb{Z} \rightarrow \mathbb{Q}$ is an injective envelope of the $\mathbb{Z}$-module $\mathbb{Z}$.

5.4.13 Proposition. Let $M$ be a graded $R$-module. If $\iota : M \rightarrow E$ and $\iota' : M \rightarrow E'$ are injective envelopes, then there exists an isomorphism $\gamma : E \rightarrow E'$ with $\gamma \iota = \iota'$.

Proof. By graded-injectivity of $E'$ there is a morphism $\gamma : E \rightarrow E'$ with $\gamma \iota = \iota'$; see 5.3.6. Since one has $\text{Ker } \gamma \cap \text{Im } \iota = 0$ and $\text{Im } \iota$ is essential in $E$, the map $\gamma$ is injective. In particular, $\text{Im } \gamma \cong E$ is graded-injective, whence there is an equality $E' = \text{Im } \gamma \oplus C$ by 5.3.6. As $\text{Im } \gamma$ is contained in $\text{Im } \iota$ one has $\text{Im } \gamma \cap C = 0$; consequently, $C$ is zero as $\text{Im } \iota'$ is essential in $E'$. It follows that $\gamma$ is surjective and hence an isomorphism.

5.4.14 Theorem. Every graded $R$-module has an injective envelope.

Proof. Let $M$ be a graded $R$-module. By 5.3.4 there exists an injective morphism of graded $R$-modules $\iota : M \rightarrow I$, where $I$ is graded-injective. The set of graded submodules of $I$ that contains $\iota(M)$ as an essential submodule is inductively ordered by inclusion. Hence, by Zorn’s lemma, it has a maximal element $E$. It remains to prove
that $E$ is graded-injective. By another application of Zorn’s lemma, choose a graded submodule $Z$ of $I$ maximal with the property $Z \cap E = 0$. Denote by $\zeta$ the morphism $E \to I \to I/Z$. It is sufficient to prove that $\zeta$ is an isomorphism. Indeed, the composite of $I \to I/Z$ and $\zeta^{-1}: I/Z \to E$ will then be a left inverse of the embedding $E \to I$. As $I$ is graded-injective, so is the graded direct summand $E$.

To prove that $\zeta$ is an isomorphism, notice first that it is injective as one has $\text{Ker} \zeta = Z \cap E = 0$. It follows from maximality of $Z$ that $\text{Im} \zeta$ is essential in $I/Z$. By the lifting property of the graded-injective module $I$ there is a morphism $\alpha: I/Z \to I$ such that $\alpha \zeta: E \to I$ is the inclusion; see 5.3.6. As $\text{Ker} \alpha \cap \text{Im} \zeta = 0$ and $\text{Im} \zeta$ is essential in $I/Z$, it follows that $\alpha$ is injective. Thus 5.4.10 yields that $\alpha \zeta(E) = E$ is essential in $\alpha(I/Z)$, whence $\iota(M)$ is essential in $\alpha(I/Z)$. By maximality of $E$ one gets $\alpha(I/Z) = E$, and thus $\alpha: I/Z \to E$ satisfies $\alpha \zeta = 1^E$. It follows that the essential submodule $\text{Im} \zeta$ is a direct summand of $I/Z$, so one has $\text{Im} \zeta = I/Z$.

5.4.15 Corollary. Let $I$ be a graded-injective $R$-module. For every graded submodule $Z$ of $I$ there exist graded-injective submodules $E$ and $V$ of $I$, such that $Z$ is essential in $E$ and there is an equality $I = E \oplus V$ of graded $R$-modules.

**Proof.** By 5.4.14 the graded module $Z$ has an injective envelope $\iota: Z \to E'$. By the lifting property of $I$ there is a morphism $\alpha: E' \to I$ such that $\alpha \iota: Z \to I$ is the inclusion; see 5.3.6. As $\alpha \iota \cap \text{Im} \iota = 0$ and $\text{Im} \iota$ is essential in $E'$, it follows that $\alpha$ is injective. Set $E = \text{Im} \alpha$; note that $E \cong E'$ is graded-injective and hence a direct summand of $I$ by 5.3.6. It follows from 5.4.10 that $\alpha(Z) = Z$ is essential in $E$, so the assertion holds with $V = I/E$.

### Minimal Complexes of Injective Modules

5.4.16 Lemma. Let $I$ be a complex of injective $R$-modules. If the graded submodule $Z(I)\gamma$ of $I^\gamma$ is essential, then $I$ is minimal.

**Proof.** Assume that $Z(I)\gamma$ is essential in $I^\gamma$. Let $\varepsilon: I \to I$ be an endomorphism homotopic to $1^I$; to prove that $I$ is minimal, it suffices by 5.4.5 to show that $\varepsilon$ is an isomorphism. By assumption there exists a homomorphism $\sigma: I \to I$ of degree 1 with $1^I - \varepsilon = \partial^I \sigma + \sigma \partial^I$. Set $X = Z(I) \cap \text{Ker} \varepsilon$; for every $x$ in $X$ one has $x = \partial^I \sigma(x)$ and, therefore, $\sigma(X) \cap Z(I) = 0$. As $Z(I)\gamma$ is essential in $I^\gamma$, it follows that $\sigma(X)$ is 0, and then $X$ is zero. From the definition of $X$ it follows that $\varepsilon$ is injective.

The exact sequence $0 \to I \xrightarrow{\varepsilon} I \to \text{Coker} \varepsilon \to 0$ is degreewise split by 5.3.6. Since $\varepsilon$ is homotopic to $1^I$, it is a homotopy equivalence, and it follows from 5.4.1 that the sequence is split in $\text{C}(R)$. Let $\varrho: I \to I$ be a morphism such that $\varrho \varepsilon = 1^I$. It follows that $\varrho$ is homotopic to $1^I$, so by the argument above $\varrho$ is injective and, therefore, it is an isomorphism. Hence, also $\varepsilon$ is an isomorphism.

5.4.17 Theorem. Let $I$ be a complex of injective $R$-modules. There is an equality $I = I' \oplus I''$, where $I'$ and $I''$ are complexes of injective $R$-modules, $I'$ is minimal, and $I''$ is contractible. Moreover, the following assertions hold.
(a) The complex \( I' \) is unique in the following sense: if one has \( I = I' \oplus I'' \), where \( I' \) is minimal and \( I'' \) is contractible, then \( I' \) is isomorphic to \( I' \).

(b) \( I \) is minimal if and only if \( Z(I)^2 \) is essential in \( I^2 \).

(c) If \( I \) is semi-injective, then \( I' \) and \( I'' \) are semi-injective.

PROOF. By 5.4.15 there is an equality \( I^3 = E \oplus V \) of graded \( R \)-modules, where \( Z(I)^2 \) is essential in \( E \). As one has \( V \cap Z(I)^2 = 0 \), the differential induces an isomorphism \( V \cong \Sigma \partial_I(V) \), in particular \( \partial_I(V) \) is a graded-injective \( R \)-module. As \( \partial_I(V) \) is contained in \( Z(I)^2 \) and hence in \( E \), there is a graded \( R \)-module \( U \) such that \( E = U \oplus \partial_I(V) \); see 5.3.6. The differential \( \delta_I \) restricts to a homomorphism from the graded-injective module \( V \oplus \partial_I(V) \) to itself. Denote by \( I'' \) the subcomplex of \( I \) given by \( V \oplus \partial_I(V) \) and notice that it is contractible with the contracting homotopy given by the inverse of the isomorphism \( \partial : V \to \Sigma \partial_I(V) \). Now it follows from 5.4.1 that there is an equality \( I = I' \oplus I'' \) in \( C(R) \), where \( I' \) is isomorphic to the quotient complex \( \bar{I} = I/I'' \). Since \( I \) is a complex of injective modules, so is \( I' \). To prove that the complex \( I' \cong \bar{I} \) is minimal, it suffices by 5.4.16 to prove that \( Z(I') \) is essential in \( \bar{I} \). Denote by \( \pi \) the canonical map \( I \to \bar{I} \) and note that its restriction to \( U \) is injective. Let \( x \neq 0 \) be an element in \( \bar{I} \) and choose an element \( u \in U \) with \( \pi(u) = x \). As \( Z(I)^2 \) is essential in \( E = U \oplus \partial_I(V) \), there exists an element \( r \) in \( R \) such that \( ru \neq 0 \) is in \( Z(I) \) and, therefore \( rx = \pi(ru) \neq 0 \) is in \( Z(I) \). Thus \( Z(I) \) is essential in \( \bar{I} \). This proves the existence of complexes \( I' \) and \( I'' \) with the desired properties. The assertions (a) and (c) follow from 5.4.6 and 5.3.20, respectively.

(b): The “if” part is 5.4.16. For the converse, assume that \( I \) is minimal. By the arguments above, there is an equality of \( R \)-complexes \( I = I' \oplus I'' \), where \( I'' \) is contractible and \( Z(I') \) is essential in \( I'' \); in particular \( I' \) is minimal. The surjection \( I \to I' \) is a homotopy equivalence by 5.4.1 and hence an isomorphism by 5.4.6. Thus, one has \( I = I' \) so \( Z(I)^2 \) is essential in \( I^2 \).

minimal semi-injective resolutions

5.4.18 Proposition. For an \( R \)-complex \( I \), the following conditions are equivalent.

(i) \( I \) is semi-injective and minimal.

(ii) Every quasi-isomorphism \( I \to M \) has a left inverse.

PROOF. Assume that \( I \) is semi-injective and minimal. If \( \beta : I \to M \) is a quasi-isomorphism, then there exists by 5.3.22 a morphism \( \gamma : M \to I \) with \( \gamma \beta \sim I' \). Set \( e = \gamma \beta \); by assumption \( e \) has an inverse, so \( e^{-1} \gamma \beta \) is a left inverse for \( \beta \).

Assume that every quasi-isomorphism \( I \to M \) has a left inverse. In particular, every homotopy equivalence \( I \to I \) has a left inverse and hence \( I \) is minimal by 5.4.5. It follows from 5.3.15 that \( I \) is semi-injective.

The semi-injective resolutions described in the next theorem are called minimal.
\textbf{5.4.19 Theorem.} Every $R$-complex $M$ has a semi-injective resolution $M \xrightarrow{\cong} I$ with $I$ minimal and $I_v = 0$ for all $v > \text{sup}M$.

\textsc{Proof.} By 5.3.26 there is a semi-injective resolution $\iota: M \xrightarrow{\cong} I'$ with $I'_v = 0$ for $v > \text{sup}M$. By 5.4.17 one has $I' = I \oplus I''$, where $I$ is semi-injective and minimal, and $I''$ is contractible. Let $\pi$ be the projection $I' \rightarrow I$; it is a quasi-isomorphism, as $I''$ is acyclic. Now the composite $\pi_\ast: M \xrightarrow{\cong} I$ is the desired resolution. \hfill $\square$

\textsc{Nakayama’s Lemma}

\textbf{5.4.20 Definition.} Let $M$ be a graded $R$-module. A graded submodule $N$ of $M$ is called \textit{superfluous} if $N + M' \neq M$ holds for every graded submodule $M' \neq M$ of $M$.

\textsc{Remark.} Another word for superfluous submodule is \textit{small} submodule.

\textbf{5.4.21 Example.} The only superfluous ideal in $\mathbb{Z}$ is 0. The maximal ideal $2\mathbb{Z}/4\mathbb{Z}$ is superfluous in $\mathbb{Z}/4\mathbb{Z}$.

\textbf{5.4.22.} A graded direct summand $N$ of a graded $R$-module $M$ is superfluous if and only if $N = M$.

\textbf{5.4.23 Lemma.} Let $M$ be a graded $R$-module with graded submodules $N$ and $N'$.

(a) Assume that there is an inclusion $N \subseteq N'$. If $N$ is superfluous in $N'$, then $N$ is superfluous in $M$.

(b) If $N$ and $N'$ are superfluous in $M$, then $N + N'$ is superfluous in $M$.

(c) If $\alpha: M \rightarrow X$ is a homomorphism of graded $R$-modules and $N$ is superfluous in $M$, then $\alpha(N)$ is superfluous in $X$.

\textsc{Proof.} (a): If $M'$ is a graded submodule of $M$ such that $N + M' = M$ holds, then one has $N' = (N + M') \cap N' = N + (M' \cap N')$. Thus, if $N$ is superfluous in $N'$, then one has $M' \cap N' = N'$ and, therefore, $N' \subseteq M'$. In particular, $N$ is then a submodule of $M'$, whence one has $M' = N + M' = M$.

(b): If $M'$ is a graded submodule of $M$ such that $(N + N') + M' = N + (N' + M')$ is $M$, then one has $N' + M' = M$ because $N$ is superfluous in $M$, and then $M = M'$ because $N'$ is superfluous in $M$ as well.

(c): By part (a) it suffices to show that $\alpha(N)$ is superfluous in $\alpha(M)$, so assume without loss of generality that $\alpha$ is surjective. If $X'$ is a submodule of $X$ such that $\alpha(N) + X' = X$ holds, then one has $N + \alpha^{-1}(X') = M$ and, therefore, $\alpha^{-1}(X') = M$ as $N$ is superfluous in $M$. Thus, one has $X' = \alpha(\alpha^{-1}(X')) = \alpha(M) = X$. \hfill $\square$

The next result is known as Nakayama’s lemma.

\textbf{5.4.24 Lemma.} Let $\mathfrak{z}$ denote the Jacobson radical of $R$. For a left ideal $\alpha$ in $R$ the following conditions are equivalent.

(i) The left ideal $\alpha$ is a superfluous submodule of $R$. 

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(ii) There is an inclusion \( a \subseteq J \).

(iii) For every finitely generated \( R \)-module \( M \neq 0 \) one has \( aM \neq M \).

(iv) For every graded \( R \)-module \( M \) and every graded submodule \( N \subseteq M \) such that the quotient \( (M/N)_v \) is non-zero and finitely generated for some \( v \in \mathbb{Z} \), one has \( N + aM \neq M \).

(v) For every graded degreewise finitely generated \( R \)-module \( M \), the submodule \( aM \) is superfluous.

**Proof.** Condition (i) is a special case of (v).

(i) \(\Rightarrow\) (ii): If \( a \) is not contained in \( J \), then one has \( a \not\subseteq M \) for at least one maximal left ideal \( M \). Thus one has \( a + M = R \), and hence \( a \) can not be superfluous in \( R \).

(ii) \(\Rightarrow\) (iii): Let \( M \neq 0 \) be a finitely generated \( R \)-module and choose a set of generators \( \{m_1, \ldots, m_t\} \) for \( M \) with \( t \) least possible. Assume towards a contradiction that one has \( aM = M \). Then there exist elements \( a_1, \ldots, a_t \) in \( a \) such that \( m_1 = \sum_{i=1}^t a_im_i \) holds. Since \( a_1 \) is in \( J \), the element \( 1 - a_1 \) is invertible, whence \( m_1 \) is a linear combination of \( m_2, \ldots, m_t \), which contradicts the minimality of \( t \).

(iii) \(\Rightarrow\) (iv): Assume that \( (M/N)_v \) is non-zero and finitely generated. By (iii) one has \( a(M/N)_v \neq (M/N)_v \), and, therefore \( (N + aM)_v \neq M_v \).

(iv) \(\Rightarrow\) (v): For every proper graded submodule \( N \subset M \) it follows from (iv) that \( N + aM \neq M \) holds. Thus, \( aM \) is superfluous in \( M \).

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**Projective Covers**

**5.4.25 Definition.** A projective cover of a graded \( R \)-module \( M \) is a surjective morphism \( \pi: P \rightarrow M \) of graded \( R \)-modules where \( P \) is graded-projective and \( \text{Ker} \pi \) is superfluous in \( P \).

**5.4.26 Example.** Let \( P \) be a graded-projective \( R \)-module. The identity morphism \( 1^P \) is a projective cover of \( P \). Moreover, if \( P \) is degreewise finitely generated, then it follows from Nakayama’s lemma 5.4.24 that the canonical map \( P \rightarrow P/aP \) is a projective cover for every left ideal \( a \) contained in the Jacobson radical of \( R \).

**5.4.27 Lemma.** Let \( M \) be a graded \( R \)-module with a projective cover \( \pi: P \rightarrow M \). Let \( \pi': P' \rightarrow M \) be a surjective morphism with \( P' \) graded-projective. There is a morphism \( \gamma: P \rightarrow P' \) with \( \pi = \pi' \gamma \), and for every such morphism the following hold.

(a) The morphism \( \gamma \) has a left inverse.

(b) If \( \pi' \) is a projective cover, then \( \gamma \) is an isomorphism.

For the direct summand \( P'' = \gamma(P) \) of \( P' \), the restriction \( \pi'|_{P''}: P'' \rightarrow M \) is a projective cover, and there is an equality of graded modules, \( P' = P'' \oplus K \), where \( K \) is contained in \( \text{Ker} \pi' \).

**Proof.** By 5.2.2 there exist morphisms \( \gamma: P \rightarrow P' \) and \( \gamma': P' \rightarrow P \) with \( \pi = \pi' \gamma \) and \( \pi' = \pi \gamma' \). Set \( \varepsilon = \gamma' \gamma \). Then one has \( \pi \varepsilon = \pi \), so there is an equality \( P = \varepsilon(P) + \text{Ker} \pi \).
As $\text{Ker} \pi$ is superfluous in $P$, it follows that $\varepsilon$ is surjective. Since $P$ is graded-projective, $\text{Ker} \varepsilon$ is a direct summand in $P$; again by 5.4.22. By 5.4.23, the module $\text{Ker} \varepsilon$ is superfluous as it is contained in $\text{Ker} \pi$, whence it is zero. Thus, $\varepsilon$ is an isomorphism, and $\varepsilon^{-1} \gamma'$ is a left inverse of $\gamma$. This proves part (a), and with $P'' = \gamma(P)$ and $K = \text{Ker} \gamma'$ it follows that there is an equality $P' = P'' \oplus K$. The submodule $P'' \cap \text{Ker} \pi'$ is superfluous in $P''$, because the isomorphism $\gamma'_{|P''}: P'' \to P$ maps it to $\text{Ker} \pi$, which is superfluous in $P$. Thus, $\pi'_{|P''}$ is a projective cover. Moreover, the equality $\pi' = \pi \gamma$ implies that $K$ is contained in $\text{Ker} \pi'$.

To prove part (b), note that one has $P'' + \text{Ker} \pi' = P'$. Thus, if $\pi'$ is a projective cover, then $P' = P''$ holds, whence $\gamma$ is an isomorphism.

5.4.28 Proposition. Let $M$ be a graded $R$-module. If $\pi: P \to M$ and $\pi': P' \to M$ are projective covers, then there exists an isomorphism $\gamma: P \to P'$ with $\pi' \gamma = \pi$. Moreover, if $M$ is degreewise finitely generated, then so are $P$ and $P'$.

Proof. The existence of an isomorphism $\gamma$ with $\pi' \gamma = \pi$ is part of 5.4.27. It also follows from 5.4.27 that if $P \to M$ is a projective cover and $P' \to M$ is any surjective morphism with $P'$ graded-projective, then $P$ is isomorphic to a graded direct summand in $P'$. If $M$ is degreewise finitely generated, then $P'$ can be chosen degreewise finitely generated by 2.5.6, and hence the direct summand $P$ is degreewise finitely generated as well.

5.4.29 Lemma. Let $\mathfrak{J}$ denote the Jacobson radical of $R$ and let $P \neq 0$ be a graded-projective $R$-module. One has $\mathfrak{J}P \neq P$, and every superfluous submodule of $P$ is contained in $\mathfrak{J}P$. In particular, if $M$ is a graded $R$-module and $\pi: P \to M$ is a projective cover, then the induced morphism $\tilde{\pi}: P/\mathfrak{J}P \to M/\mathfrak{J}M$ is an isomorphism.

Proof. By 5.2.2 the module $P$ is a graded direct summand of a graded free $R$-module $L$. Let $E$ be a graded basis for $L$. Fix a homogeneous element $p \neq 0$ in $P$; it is a unique linear combination of some basis elements, $p = \sum_{i=1}^{m} r_i e_i$. Let $\varepsilon$ be the composition of canonical morphisms $L \to P \to L$. For each $i \in \{1, \ldots, m\}$ write $\varepsilon(e_i) = \sum_{j=1}^{n+1} a_{ij} e_j$, where also $e_{m+1}, \ldots, e_n$ are elements in $E$. Suppose the equality $P = \mathfrak{J}P$ holds, then all the elements $a_{ij}$ belong to $\mathfrak{J}$. The equality $p = \varepsilon(p)$ yields

\[
\sum_{i=1}^{m} r_i e_i = \sum_{i=1}^{m} r_i \sum_{j=1}^{n} a_{ij} e_j .
\]

Let $A$ be the $m \times m$ matrix with entries $a_{ij}$ for $1 \leq i, j \leq m$, and let $I_m$ denote the $m \times m$ identity matrix. From (*) one gets the equality $(r_1, \ldots, r_m)(I_m - A) = 0$. It is elementary to verify that the Jacobson radical of the matrix ring $M_{m \times m}(R)$ contains (in fact, equality holds) the ideal $M_{m \times m}(\mathfrak{J})$. Since $A$ has entries in $\mathfrak{J}$, it follows that $I_m - A$ is invertible in $M_{m \times m}(R)$. Hence $(r_1, \ldots, r_m)$ is the zero row and one has $p = 0$, a contradiction.

Let $N$ be a superfluous graded submodule of $P$ and thereby of $L$; cf. 5.4.23. It follows that $N$ is contained in every maximal submodule of $L$. In particular, $N$ is contained in the module $\mathfrak{J}e' + R\langle E \setminus \{e'\} \rangle$ for every $e' \in E$. Indeed, this is the
intersection of the maximal submodules \( \mathfrak{M} e' + R(E \setminus \{ e' \}) \), where \( \mathfrak{M} \) is a maximal left ideal in \( R \). Thus, for an element \( x = \sum_{e \in E} r_e e \) in \( N \) one has \( r_e \in \mathfrak{M} \) for all \( e \in E \).

It follows that \( N \) is contained in \( \mathfrak{J}L \cap P = \mathfrak{J}P \), where the equality holds because \( P \) is a direct summand of \( L \).

The last assertion is now immediate as the kernel of a cover \( P \twoheadrightarrow M \) is superfluous in \( P \) and hence contained in \( \mathfrak{J}P \).

\[ \square \]

**Remark.** For a graded-projective \( R \)-module \( P \), the submodule \( \mathfrak{J}P \) itself may not be superfluous; see E 5.4.11.

### Semi-perfect Modules

**5.4.30 Definition.** A graded \( R \)-module \( M \) is called **semi-perfect** if every homomorphic image of \( M \) has a projective cover.

**5.4.31 Example.** If \( R \) is semi-simple, then every \( R \)-module is projective by 1.3.26, and it follows that every \( R \)-module is semi-perfect.

The \( \mathbb{Z} \)-module \( \mathbb{Z} \) has a projective cover, namely \( 1_{\mathbb{Z}} \), but it is not semi-perfect, since the quotient \( \mathbb{Z}/n\mathbb{Z} \) has no projective cover for \( n > 1 \). Indeed, suppose that \( \pi: P \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \) is a projective cover and let \( \pi': \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \) be the canonical map. By 5.4.27 there is an isomorphism of \( \mathbb{Z} \)-modules, \( \mathbb{Z} \cong P \oplus K \), and since \( \mathbb{Z} \) is indecomposable, it follows that \( K = 0 \), \( P = \mathbb{Z} \), and \( \pi = \pi' \). However, \( \pi': \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \) is not a projective cover since its kernel is not superfluous; see 5.4.21.

**5.4.32 Lemma.** Let \( \mathfrak{J} \) denote the Jacobson radical of \( R \) and let \( M \) be a graded \( R \)-module. If \( M \) is semi-perfect, then the following assertions hold.

(a) If \( M \) is non-zero, then \( \mathfrak{J}M \neq M \) holds.

(b) The submodule \( \mathfrak{J}M \) is superfluous in \( M \).

(c) If \( P \twoheadrightarrow M \) is a projective cover, then \( P \) is semi-perfect.

(d) The graded \( R/\mathfrak{J} \)-module \( M/\mathfrak{J}M \) is semi-simple.

Let \( \alpha: L \twoheadrightarrow M \) be a morphism of graded \( R \)-modules and denote by \( \tilde{\alpha} \) the induced morphism \( L/\mathfrak{J}L \twoheadrightarrow M/\mathfrak{J}M \) of \( R/\mathfrak{J} \)-modules.

(e) If \( \tilde{\alpha} \) is surjective, then \( \alpha \) is surjective.

(f) If \( \tilde{\alpha} \) is bijective and \( L \) and \( M \) are graded-projective, then \( \alpha \) is bijective.

**Proof.** Let \( \pi: P \twoheadrightarrow M \) be a projective cover.

(a): If \( \mathfrak{J}M = M \) holds, then one has \( \pi(\mathfrak{J}P) = M \), whence there is an equality \( \text{Ker} \pi + \mathfrak{J}P = P \). As \( P \) is graded-projective and \( \text{Ker} \pi \) is superfluous in \( P \), it follows from 5.4.29 that \( P \) is zero, whence \( M = 0 \).

(b): Let \( M' \) be a graded submodule of \( M \) such that the equality \( \mathfrak{J}M + M' = M \) holds. The quotient module \( N = M/M' \) then satisfies \( \mathfrak{J}N = N \). By assumption the module \( N \) is semi-perfect, so part (a) yields \( N = 0 \), whence one has \( M' = M \).

(c): To prove that \( P \) is semi-perfect, it must be show that \( P/N \) has a projective cover for every graded submodule \( N \) of \( P \). Let \( N \) be such a submodule and
set $K = \text{Ker} \pi$, then $P/(N + K)$ is a homomorphic image of $P/K \cong M$, so there
is a projective cover $\pi' : P' \to P/(N + K)$. As $P'$ is graded-projective, there exists
by 5.2.2 a morphism $\gamma : P' \to P/N$ with $\pi' = \beta \gamma$, where $\beta$ is the canonical
 morphism $P/N \to P/(N + K)$. To see that $\gamma$ is a projective cover, notice first that
$\text{Ker} \beta = (N + K)/N$ is a superfluous submodule of $P/N$, because it is the image
of the superfluous submodule $K$ of $P$; see 5.4.23. On the other hand, since $\pi'$ is surjective,
one has $\gamma(P') + \text{Ker} \beta = P/N$, so $\gamma$ is surjective. Finally, $\text{Ker} \gamma$ is contained
in $\text{Ker} \pi'$, which is superfluous in $P'$.

(d): The morphism of graded $R/\mathfrak{J}$-modules $P/\mathfrak{J}P \to M/\mathfrak{J}M$, induced by $\pi$, is
surjective; it is, therefore, sufficient to prove that $P/\mathfrak{J}P$ is semi-simple. To that end,
let $X/\mathfrak{J}P$ be a proper graded submodule of $P/\mathfrak{J}P$; the goal is to show that this
submodule is a graded direct summand. The module $P/X$ has a projective cover
$\chi : F \to P/X$ by part (c). Let $\beta$ be the canonical morphism $P \to P/X$. By 5.4.27
there is a morphism $\gamma : F \to P$ with $\chi = \beta \gamma$ and $P = \gamma(F) \oplus K$, where $K$ is contained
in $\text{Ker} \beta = X$ and the restriction of $\beta$ to $\gamma(F)$ is a projective cover. Replacing $\chi$ with
$\beta|_{\gamma(F)}$ one has $P = F \oplus K$ and, therefore,

\begin{equation}
\frac{P}{\mathfrak{J}P} = \frac{F + \mathfrak{J}P}{\mathfrak{J}P} \oplus \frac{X}{\mathfrak{J}P}.
\end{equation}

The submodule $F \cap X$ is contained in $\text{Ker} \chi$, so it is superfluous in $F$ and hence
contained in $\mathfrak{J}F \subseteq \mathfrak{J}P$ by 5.4.29. Thus, the sum in $\ast$ is direct; indeed, one has

$$
\frac{F + \mathfrak{J}P \cap X}{\mathfrak{J}P} = \frac{(F + \mathfrak{J}P) \cap X}{\mathfrak{J}P} = \frac{(F \cap X) + \mathfrak{J}P}{\mathfrak{J}P} = 0.
$$

(e): Set $C = \text{Coker} \alpha$ and consider the exact sequence $L \xrightarrow{\alpha} M \to C \to 0$ of
graded $R$-modules. It induces an exact sequence $L/\mathfrak{J}L \xrightarrow{\alpha} M/\mathfrak{J}M \to C/\mathfrak{J}C \to 0$
with $C/\mathfrak{J}C = 0$ by assumption. The graded module $C$ is semi-perfect, because it is a
homomorphic image of $M$. Thus, part (a) yields $C = 0$, whence $\alpha$ is surjective.

(f): It follows from part (e) that $\alpha$ is surjective. Set $K = \text{Ker} \alpha$ and consider the exact
sequence $0 \to K \xrightarrow{\alpha} L \xrightarrow{\alpha} M \to 0$ of graded $R$-modules. By assumption, $L$
and $M$ are graded-projective, so the sequence splits and $K$ is graded-projective by
5.2.2 and 5.2.3. Now the induced sequence $0 \to K/\mathfrak{J}K \to L/\mathfrak{J}L \xrightarrow{\alpha} M/\mathfrak{J}M \to 0$
is exact. By assumption $\alpha$ is injective, so one has $K/\mathfrak{J}K = 0$. Lemma 5.4.29 now
yields $K = 0$, so $\alpha$ is injective.

5.4.33 Lemma. Let $\mathfrak{J}$ denote the Jacobson radical of $R$ and set $k = R/\mathfrak{J}$. Consider
every $k$-module as an $R$-module via the canonical homomorphisms $\kappa : R \to k$.

(a) Let $e \in R$ be an idempotent and set $u = \kappa(e)$. The map $\kappa_u : Re \to ku$ given by
re $\mapsto \kappa(r)u$ is a projective cover of the $R$-module $ku$, and one has $\text{Ker} \kappa_u = \mathfrak{J}e$.

(b) If a graded $k$-module generated by a homogeneous element $u$ has a projective
cover as a graded $R$-module, then it has one of the form $\kappa_u : \Sigma^{[\alpha]} Re_u \to ku$,
where $e_u$ is an idempotent in $R$. 

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(c) Let $M$ be a graded $R$-module such that the graded $k$-module $M/\mathfrak{J}M$ is semi-simple. Let $U$ be a set of homogeneous elements with $M/\mathfrak{J}M = \coprod_{u \in U} ku$. Assume that each $R$-module $ku$ has a projective cover $\kappa_u : \Sigma^{[u]} Re_u \rightarrow ku$ with $e_u$ an idempotent in $R$. The $R$-module $F = \coprod_{u \in U} \Sigma^{[u]} Re_u$ is graded-projective, and the morphism $\kappa : F \rightarrow M/\mathfrak{J}M$ is surjective with $\text{Ker} \kappa = \mathfrak{J}F$.

PROOF. (a): Let $e$ be an idempotent in $R$. The ideal $Re$ is a projective $R$-module as one has $R = Re \oplus R(1-e)$. The inclusion $\mathfrak{J}e \subseteq \text{Ker} \kappa_u$ holds by the definition of $\kappa_u$. To prove the reverse inclusion, let $re$ be an element in $\text{Ker} \kappa_u$. The equalities $0 = \kappa(r)u = \kappa(re)$ in $ku$ show that $re$ is in $\mathfrak{J}$, and since $e$ is an idempotent one has $re = (re)e \in \mathfrak{J}e$. By Nakayama’s lemma 5.4.24, the Jacobson radical $\mathfrak{J}$ is superfluous in $R$, so $\mathfrak{J}e = \text{Ker} \kappa_u$ is superfluous in $Re$ by 5.4.23. Thus, $\kappa_u$ is a projective cover.

(b): Let $\pi : P \rightarrow ku$ be a projective cover of $ku$ as a graded $R$-module. Set $P' = \Sigma^{[u]} R$ and let $p$ denote the generator 1 in $P'$. Let $\pi' : P' \rightarrow ku$ be the surjective morphism of graded $R$-modules that maps $p$ to $u$. By 5.4.27 there is a morphism $\gamma : P \rightarrow P'$ with $\pi = \pi' \gamma$, $P'' = \gamma(P)$, and $P' = P'' \oplus K$, such that $\kappa_u = \pi'|_{P''} : P'' \rightarrow ku$ is a projective cover. Restrict the codomain of $\gamma$ so that it becomes an isomorphism $\gamma : P \rightarrow P''$ and let $\nu : P' \rightarrow P$ denote the canonical morphism with $\nu \gamma = 1_P$. One has $\nu u(p) = ep$ for some $e \in R$. The identity $\nu u = \gamma \nu = (\nu u)(\gamma u)$ yields $ep = e^2 p$, so $e$ is an idempotent. The equality $R = Re \oplus R(1-e)$ now yields $P'' = \Sigma^{[u]} Re$, so $\kappa_u$ is the desired projective cover.

(c): Each graded $R$-module $\Sigma^{[u]} Re_u$ is graded-projective as $e_u$ is an idempotent. It follows from 5.2.11 and 5.2.15 that $F$ is graded-projective, and $\kappa$ is surjective by construction. By 3.1.6 and (a) one has $\text{Ker} \kappa = \coprod_{u \in U} \text{Ker} \kappa_u = \coprod_{u \in U} \mathfrak{J}e_u = \mathfrak{J}F$. \hfill $\square$

5.4.34 Theorem. Let $M$ be a semi-perfect graded $R$-module. If $P \rightarrow M$ is a projective cover, then $P$ is isomorphic to a module of the form $\coprod_{u \in U} \Sigma^{[u]} Re_u$, where each $e_u$ is an idempotent in $R$.

PROOF. Let $\mathfrak{J}$ denote the Jacobson radical of $R$ and set $k = R/\mathfrak{J}$. By parts (c) and (d) in 5.4.32 the graded $R$-module $P$ is semi-perfect, and the $k$-module $P/\mathfrak{J}P$ is semi-simple. By a standard application of Zorn’s lemma, choose a set $U$ of homogeneous elements with $P/\mathfrak{J}P = \coprod_{u \in U} ku$. Each graded module $ku$ is a homomorphic image of $P$ and, therefore, it has a projective cover. By 5.4.33 it then has a projective cover of the form $\Sigma^{[u]} Re_u \rightarrow ku$, where $e_u$ is an idempotent in $R$. Now it follows, still from 5.4.33, that there is a surjective morphism $\kappa : F \rightarrow P/\mathfrak{J}P$ with $\text{Ker} \kappa = \mathfrak{J}F$, where $F$ is the graded-projective $R$-module $\coprod_{u \in U} \Sigma^{[u]} Re_u$. By graded-projectivity of $F$, there is a morphism $\gamma : F \rightarrow P$ such that the composite $F \xrightarrow{\gamma} P \rightarrow P/\mathfrak{J}P$ equals $\kappa$; see 5.2.2. The induced morphism $\tilde{\kappa} : F/\mathfrak{J}F \rightarrow P/\mathfrak{J}P$ is an isomorphism; it follows that also $\tilde{\gamma} : F/\mathfrak{J}F \rightarrow P/\mathfrak{J}P$ is an isomorphism, so $\gamma$ is an isomorphism by 5.4.32(f). \hfill $\square$
5.4 Minimality

### 5.4.35 Definition
Let $\mathfrak{J}$ denote the Jacobson radical of $R$. The ring $R$ is called **semi-perfect** if $R/\mathfrak{J}$ is semi-simple and idempotents lift from $R/\mathfrak{J}$ to $R$.

**Remark.** Because semi-simplicity is a left/right symmetric property so is semi-perfection. A ring $R$ with $R/\mathfrak{J}$ semi-simple is sometimes called **semi-local**. A commutative ring is semi-local if and only if it has finitely many maximal ideals.

### 5.4.36 Example
If $R$ is local with unique maximal ideal $m$, then $k = R/m$ is a division ring, so 1 and 0 are the only idempotents in $k$. Moreover, $k$ is simple, so $R$ is semi-perfect.

If $R$ is left (or right) Artinian with Jacobson radical $\mathfrak{J}$, then $R/\mathfrak{J}$ is semi-simple. Moreover, if $r$ is in $R$ and $[r]_\mathfrak{J}$ is an idempotent, then $r - r^2$ is in $\mathfrak{J}$. Since $\mathfrak{J}$ is nilpotent, there is an integer $n \geq 1$ with $(r - r^2)^n = 0$. Powers of $r$ commute, so one has $0 = (r(1-r))^n = r^n(1-r)^n = r^n - r^{n+1}x$ for an element $x$ with $rx = xr$. It follows that one has $(rx)^n = r^nx^n = r^{n+1}x^{n+1} = (rx)^{n+1}$, so the element $e = (rx)^n$ is an idempotent in $R$. One has $[r]_\mathfrak{J} = [r^n]_\mathfrak{J} = [r^{n+1}x]_\mathfrak{J} = [r^{n+1}]_\mathfrak{J}$ and hence $[e]_\mathfrak{J} = [rx]_\mathfrak{J}^n = [r]_\mathfrak{J}$. Thus $R$ is semi-perfect.

**Remark.** A ring $R$ is semi-perfect with $R/\mathfrak{J}$ simple if and only if it is isomorphic to a matrix ring $M_{n \times n}(S)$ where $S$ is local; see [33, thm. (23.10)]. A commutative ring is semi-perfect if an only if it is a finite product of commutative local rings; see [33, thm. (23.11)].

### 5.4.37 Lemma
Let $\mathfrak{J}$ denote the Jacobson radical of $R$ and let $M$ be a graded $R$-module. If $R$ is semi-perfect, then there exists a graded-projective $R$-module $F$ and a surjective morphism $\bar{x}: F \rightarrow M/\mathfrak{J}M$ with $\text{Ker} \bar{x} = \mathfrak{J}F$. Moreover, if $M$ is degreewise finitely generated, then one can choose $F$ degreewise finitely generated.

**Proof.** Set $k = R/\mathfrak{J}$ and consider the graded $k$-module $M/\mathfrak{J}M$. By assumption, $k$ is semi-simple, so by a standard application of Zorn’s lemma one can choose a set $U$ of homogeneous elements with $M/\mathfrak{J}M = \bigsqcup_{u \in U} \Sigma|u| ku$. As every cyclic $k$-module is a direct sum of simple modules generated by idempotents, one can assume that each $u$ is an idempotent in $k$. By assumption, each $u$ now lifts to an idempotent $e_u$ in $R$, so by 5.4.33 the canonical surjection $\bar{x}_u: R e_u \rightarrow ku$ is a projective cover of $ku$, and the desired morphism is $\bar{x} = \bigsqcup_{u \in U} \bar{x}_u$. Finally, if $M$ is degreewise finitely generated, then one can choose $U$ with only finitely many elements of each degree.

### 5.4.38 Theorem
The following conditions are equivalent.

(i) The ring $R$ is semi-perfect.

(ii) Every graded degreewise finitely generated $R$-module has a projective cover.

(iii) Every graded degreewise finitely generated $R$-module is semi-perfect.

(iv) The $R$-module $R$ is semi-perfect.

Moreover, if $R$ is semi-perfect and $P \rightarrow M$ is a projective cover of a graded degreewise finitely generated $R$-module, then $P$ is degreewise finitely generated.
Proof. Let \( \mathfrak{J} \) denote the Jacobson radical of \( R \) and set \( k = R/\mathfrak{J} \). The implication \((ii) \implies (iii)\) follows from the definition of semi-perfect modules 5.4.30, and \((iv)\) is a special case of \((iii)\).

\((i) \implies (ii)\): Let \( M \) be a graded degreewise finitely generated \( R \)-module. By 5.4.37 there exists a graded-projective and degreewise finitely generated \( R \)-module \( F \) and surjective morphism \( \pi : F \to M/\mathfrak{J}M \) with \( \ker \pi = \mathfrak{J}F \). Let \( \beta \) be the canonical morphism \( M \to M/\mathfrak{J}M \); by 5.2.2 there exists a morphism \( \gamma : F \to M \) with \( \beta \gamma = \pi \). Thus, one has \( M = \gamma(F) + \mathfrak{J}M \). It follows from Nakayama’s lemma 5.4.24 that \( \mathfrak{J}M \) is superfluous in \( M \), so \( \gamma \) is surjective. To see that \( \gamma \) is a projective cover, it remains to verify that \( \ker \gamma \) is superfluous in \( F \). This follows as \( \ker \gamma \) is contained in \( \ker \pi = \mathfrak{J}F \), which is a superfluous submodule by Nakayama’s lemma. As a projective cover is unique up to isomorphism, see 5.4.28, the final assertion in the theorem also follows.

\((iv) \implies (i)\): By 5.4.32(d) the \( k \)-module \( k \) is semi-simple, so \( k \) is a semi-simple ring. From 5.4.32(b) it follows that \( \mathfrak{J} \) is superfluous in \( R \), whence the canonical map \( R \to k \) is a projective cover. Let \( u \) be an idempotent in \( k \); the goal is to lift \( u \) to \( R \). There is an equality \( k = ku \oplus k(1 - u) \). As \( R \) is a semi-perfect \( R \)-module, its homomorphic images \( ku \) and \( k(1 - u) \) have projective covers \( \pi : P \to ku \) and \( \pi' : P' \to k(1 - u) \). The morphism \( \pi \oplus \pi' \) is a projective cover of \( k \). By 5.4.27 there is an isomorphism \( \gamma : P \oplus P' \to R \) such that \( \gamma \) followed by the canonical map \( R \to k \) is \( \pi \oplus \pi' \). Thus, the modules \( P \) and \( P' \) are isomorphic to ideals \( \epsilon \) and \( \epsilon' \) in \( R \), and one has \( R = \epsilon \oplus \epsilon' \). Choose elements \( e \in \epsilon \) and \( e' \in \epsilon' \) with \( 1 = e + e' \) in \( R \). It is elementary to verify that \( e \) and \( e' \) are orthogonal idempotents and that \( e \) maps to \( u \) in \( k \).

Remark. Every finitely generated flat module over a semi-perfect ring is projective; see E 5.4.15. It is a result of Kaplansky [27] that the finiteness hypothesis in the next corollary can be omitted.

5.4.39 Corollary. Let \( R \) be local. Every degreewise finitely generated graded-projective \( R \)-module is graded-free.

Proof. Let \( P \) be a degreewise finitely generated and graded-projective \( R \)-module. The identity \( 1^P \) is a projective cover of \( P \). By 5.4.38 the graded module \( P \) is semi-perfect, so by 5.4.34 it has the form \( \coprod_{u \in \Sigma} R e_u \), where each \( e_u \) is an idempotent in \( R \). As \( R \) is local, 1 and 0 are the only idempotents in \( R \), so \( P \) is graded-free.

Perfect Rings

5.4.40 Definition. A left ideal \( a \) in \( R \) is called left T-nilpotent if for every sequence of elements \((a_i)_{i \in \mathbb{N}}\) in \( a \) there is an integer \( n \geq 1 \) such that \( a_1 a_2 \cdots a_n = 0 \) holds.

Every nilpotent left ideal is left T-nilpotent; in particular the Jacobson radical of a left (or right) Artinian ring is left T-nilpotent. A T-nilpotent ideal has properties similar to the Jacobson radical as captured by Nakayama’s lemma.

5.4.41 Lemma. For a left ideal \( a \) in \( R \), the following conditions are equivalent.

(i) The left ideal \( a \) is left T-nilpotent.
For every element $x$ follows by induction that there exists a sequence $R_{5.4.37}$ that there exists a graded-projective $5.4.45$ Theorem. The following conditions are equivalent.

- $\beta$ is nilpotent, whence idempotents lift from $5.4.36$. Thus, one has $\beta a \subseteq b$ and, therefore, $\beta(\alpha M) = 0$, so $\alpha M \neq M$ holds.
- $(ii) \implies (v)$: Let $a$ be a graded $R$-module $M/M'$. Apply $(ii)$ to the non-zero $R$-module $M/M'$.
- $(v) \implies (i)$: Let a sequence $(a_i)_{i \in \mathbb{N}}$ of elements in $a$ be given. For $j > i \geq 1$ let $a^{(j)}$ be the homothety given by right multiplication on $R$ with $a_1 \cdots a_{j-1}$ and set $a^{(j)} = 1^R$. These maps form a direct system of $R$-modules; let $A$ denote its colimit. For every element $a$ in $A$ there is a ring element $r$ and an integer $i \geq 1$ with $a = a^{(i)}(r)$, where $a^{(i)}$ is the canonical morphism $R \to A$; see 3.2.25. From the equalities $a^{(i+1)}(ra_i) = ra_i a^{(i+1)}(1)$ one gets $\alpha A = A$. By $(v)$ the submodule $\alpha A$ is superfluous in $A$, so the colimit $A$ is zero. In particular, $a^{(i)}(1)$ is zero, so by 3.2.25 one has $a^{(1)}(1) = a_1 \cdots a_{j-1} = 0$ some $j > 1$. Thus, $a$ is left $T$-nilpotent.

5.4.42 Definition. Let $\mathfrak{J}$ be the Jacobson radical of $R$. The ring $R$ is called left perfect if $R/\mathfrak{J}$ is semi-simple and $\mathfrak{J}$ is left $T$-nilpotent.

5.4.43 Example. As a nilpotent ideal is left $T$-nilpotent, every left (or right) Artinian ring is left perfect.

5.4.44. If the Jacobson radical $\mathfrak{J}$ of $R$ is left $T$-nilpotent, then every element in $\mathfrak{J}$ is nilpotent, whence idempotents lift from $R/\mathfrak{J}$ to $R$; see 5.4.36. Thus, every left perfect ring is semi-perfect.

5.4.45 Theorem. The following conditions are equivalent.

- $(i)$ The ring $R$ is left perfect.
- $(ii)$ Every graded $R$-module has a projective cover.
- $(iii)$ Every graded $R$-module is semi-perfect.

Proof. Let $\mathfrak{J}$ denote the Jacobson radical of $R$ and set $k = R/\mathfrak{J}$. The implication $(ii) \implies (iii)$ follows from the definition 5.4.30 of semi-perfect modules.

$(i) \implies (ii)$: Let $M$ be a graded $R$-module. Since $R$ is semi-perfect, it follows from 5.4.37 that there exists a graded-projective $R$-module $F$ and a surjective morphism.
\( \kappa : F \to M / \mathfrak{J} M \) with \( \text{Ker} \kappa = \mathfrak{J} F \). Let \( \beta \) be the canonical morphism \( M \to M / \mathfrak{J} M \); by 5.2.2 there is a morphism \( \gamma : F \to M \) with \( \beta \gamma = \kappa \). Thus, one has \( M = \gamma (F) + \mathfrak{J} M \).

It follows from 5.4.41 that \( \mathfrak{J} M \) is superfluous in \( M \), so \( \gamma \) is surjective. To see that \( \gamma \) is a projective cover, it remains to verify that \( \text{Ker} \gamma \) is superfluous in \( F \). This follows as \( \text{Ker} \gamma \) is contained in \( \text{Ker} \kappa = \mathfrak{J} F \), which is a superfluous submodule by 5.4.41.

(iii) \( \implies \) (i): It follows from 5.4.32(d) that \( k \) is semi-simple as a \( k \)-module; i.e. it is a semi-simple ring. For every graded \( R \)-module \( M \) the submodule \( \mathfrak{J} M \) is superfluous in \( M \) by 5.4.32(b), so \( \mathfrak{J} \) is left \( T \)-nilpotent by 5.4.41. Thus, \( R \) is left perfect. \( \square \)

Another homological characterization of perfect rings is given in 5.5.26.

**Minimal Complexes of Projective Modules**

**5.4.46 Lemma.** Let \( \mathfrak{J} \) be the Jacobson radical of \( R \) and let \( P \) be a complex of projective \( R \)-modules. If \( P^0 \) is semi-perfect and one has \( \partial^P (P) \subseteq \mathfrak{J} P \), then \( P \) is minimal.

**Proof.** Let \( \varepsilon : P \to P \) be a morphism with \( \varepsilon \sim 1^P \); it suffices by 5.4.5 to prove that \( \varepsilon \) is an isomorphism. By assumption there exists a homomorphism \( \sigma : P \to P \) of degree 1 such that \( 1^P - \varepsilon = \partial^P \sigma + \sigma \partial^P \) holds. For every \( p \in P \), the element \( p - \varepsilon (p) \) belongs to \( \mathfrak{J} P + \sigma (\mathfrak{J} P) = \mathfrak{J} P \). It follows that the induced morphism \( \hat{\varepsilon} : P / \mathfrak{J} P \to P / \mathfrak{J} P \) is the identity \( 1^{P/\mathfrak{J} P} \), so \( \varepsilon \) is an isomorphism by 5.4.32(f). \( \square \)

**5.4.47 Theorem.** Let \( \mathfrak{J} \) denote the Jacobson radical of \( R \) and let \( P \) be a complex of projective \( R \)-modules such that \( P^0 \) is semi-perfect. There is an equality \( P = P' \oplus P'' \), where \( P' \) and \( P'' \) are complexes of projective \( R \)-modules, \( P' \) is minimal, and \( P'' \) is contractible. Moreover, the following assertions hold.

(a) The complex \( P' \) is unique in the following sense: if one has \( P = F' \oplus F'' \), where \( F' \) is minimal and \( F'' \) is contractible, then \( F' \) is isomorphic to \( P' \).

(b) \( P \) is minimal if and only if \( B(P) \) is superfluous in \( P^0 \) if and only if the inclusion \( \partial^P (P) \subseteq \mathfrak{J} P \) holds.

(c) If \( P \) is semi-projective, then \( P' \) and \( P'' \) are semi-projective.

**Proof.** Set \( k = R / \mathfrak{J} \); the graded \( k \)-module \( (P / \mathfrak{J} P)^\mathfrak{J} \) is semi-simple by 5.4.32(d). Set \( B = B(P / \mathfrak{J} P) \) and \( H = H(P / \mathfrak{J} P) \); by 4.2.23 there is a split exact sequence of \( k \)-complexes,

\[
0 \to H \to P / \mathfrak{J} P \to \text{Cone} 1^B \to 0 .
\]

Because it is a homomorphic image of the semi-perfect graded \( R \)-module \( P^0 \), the graded module \( B^\mathfrak{J} \) is semi-perfect. In particular, \( B^\mathfrak{J} \) has a projective cover \( \kappa : F \to B^\mathfrak{J} \).

Set \( P'' = \text{Cone} 1^F \) and \( C = \text{Cone} 1^B \); both complexes are contractible by 4.2.21. The projective cover \( \kappa \) induces a surjective morphism \( \chi = \kappa \oplus \Sigma \chi : P'' \to C \). It is elementary to verify that \( \chi : P'' \to C^\mathfrak{J} \) is a projective cover. By \( * \) the graded module
C^\circ is a homomorphic image of P^\circ, so it is semi-perfect and, therefore, P'^\circ is semi-perfect by 5.4.32(c).

Let \pi denote the canonical map P \rightarrow P/3P. The complexes Hom_R(P,P'^\circ) and Hom_R(P,C) are contractible by 4.2.19 so Hom_R(P,\chi) is a quasi-isomorphism. Moreover, Hom_R(P,\chi) is surjective by 5.2.2 and hence surjective on cycles; see 4.2.12. Thus, there exists a morphism \gamma: P \rightarrow P'^\circ with \chi \gamma = \pi \tau. Let \pi' be the restriction of \pi to the subcomplex P' = Ker \gamma and consider the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & P' \rightarrow P \rightarrow P'^\circ \rightarrow 0 \\
\downarrow \pi' & & \downarrow \pi \\
0 & \rightarrow & H \rightarrow P/3P \rightarrow C \rightarrow 0.
\end{array}
\]

The bottom row is the split exact sequence (*), and by construction the top row is exact at P' and at P. To see that it is exact at P'^\circ, notice that \tilde{\chi}: P'^\circ/3P'^\circ \rightarrow C/3C is bijective by 5.4.29 and \pi is the identity morphism 1_P/3P, while \tilde{\pi} = \pi is surjective. It follows that \tilde{\gamma}: P/3P \rightarrow P'^\circ/3P'^\circ is surjective, and then \gamma is surjective by 5.4.32(f).

As P'^\circ is a graded-projective R-module, the top row in (‡) is degreewise split by 5.2.2 and, therefore, split by 5.4.1 as P'^\circ is contractible. Thus, one has P = P' \oplus P'^\circ.

To see that P' is minimal, note that (‡) now yields a commutative diagram,

\[
\begin{array}{ccc}
0 & \rightarrow & P'/3P' \rightarrow P/3P \rightarrow P'^\circ/3P'^\circ \rightarrow 0 \\
\downarrow \pi' & & \downarrow \pi \\
0 & \rightarrow & H \rightarrow P/3P \rightarrow C \rightarrow 0,
\end{array}
\]

with exact rows. It follows from the Five Lemma that \pi' is an isomorphism. The differential on H is zero, so \partial_{P'}(P') is contained in 3P'. As the top row in (‡) splits, the graded-projective R-module P'^\circ is a homomorphic image of P^\circ and hence semi-perfect. It follows from 5.4.46 that P' is a minimal complex, which finishes the proof of the first assertion. Parts (a) and (c) follow from 5.4.6 and 5.2.15, respectively.

(b): Since P^\circ is graded-projective and semi-perfect, it follows from 5.4.29 and 5.4.32(b) that B(P)^\circ = (\partial P(P))^\circ is superfluous in P^\circ if and only if the inclusion \partial P(P) \subseteq 3P holds. In view of 5.4.46 it remains to prove that \partial P(P) \subseteq 3P holds if P is minimal. Assume that P is minimal; by the arguments above one has P = P' \oplus P'^\circ, where P'^\circ is contractible and \partial P(P') is contained in 3P'. The surjection P \rightarrow P' is a homotopy equivalence by 5.4.1 and hence an isomorphism by 5.4.6. Thus \partial P(P) is contained in 3P.

\[\square\]

MINIMAL SEMI-PROJECTIVE RESOLUTIONS

5.4.48 Proposition. For an R-complex P, the following conditions are equivalent.

\[\square\]
(i) \( P \) is semi-projective and minimal.

(ii) Every quasi-isomorphism \( M \to P \) has a right inverse.

**Proof.** Assume that \( P \) is semi-projective and minimal. If \( \alpha: M \to P \) is a quasi-isomorphism, then there exists by 5.2.17 a morphism \( \gamma: P \to M \) with \( \alpha \gamma \sim 1_P \). Set \( e = \alpha \gamma \); by assumption, \( e \) has an inverse, so \( \gamma e^{-1} \) is a right inverse for \( \alpha \).

Assume that every quasi-isomorphism \( M \to P \) has a right inverse. In particular, every homotopy equivalence \( P \to P \) has a right inverse and hence \( P \) is minimal by 5.4.5. It follows from 5.2.9 that \( P \) is semi-projective.

The semi-projective resolutions detailed in the next theorems are called *minimal*.

**5.4.49 Theorem.** Let \( R \) be left perfect. Every \( R \)-complex \( M \) has a semi-projective resolution \( P \xrightarrow{\sim} M \) with \( P \) minimal and \( P_v = 0 \) for \( v < \inf M \).

**Proof.** Let \( M \) be an \( R \)-complex, by 5.2.13 there is a semi-projective resolution \( \pi': P' \xrightarrow{\sim} M \) with \( P'_v = 0 \) for \( v < \inf M \). By 5.4.45 the graded \( R \)-module \( P^\bullet \) is semi-perfect, so by 5.4.47 one has \( P' = P \oplus P'' \), where \( P \) is minimal and semi-projective, and \( P'' \) is contractible. Let \( \iota \) be the embedding \( P \to P' \); it is a quasi-isomorphism as \( P'' \) is acyclic. Thus \( \pi': P \xrightarrow{\sim} M \) is the desired resolution.

**5.4.50 Theorem.** Let \( R \) be left Noetherian and semi-perfect. Every \( R \)-complex \( M \) with \( H(M) \) bounded below and degreewise finitely generated has a semi-projective resolution \( P \xrightarrow{\sim} M \) with \( P \) minimal and degreewise finitely generated and with \( P_v = 0 \) for all \( v < \inf M \).

**Proof.** By 5.1.14 there is a semi-projective resolution \( \pi': P' \xrightarrow{\sim} M \) with \( P' \) degree-wise finitely generated and \( P'_v = 0 \) for \( v < \inf M \). By 5.4.38 the graded \( R \)-module \( P^\bullet \) is semi-perfect, so by 5.4.47 one has \( P' = P \oplus P'' \), where \( P \) is minimal and semi-projective, and \( P'' \) is contractible. Let \( \iota \) be the embedding \( P \to P' \); it is a quasi-isomorphism as \( P'' \) is acyclic. Thus \( \pi': P \xrightarrow{\sim} M \) is the desired resolution.

**Exercises**

**E 5.4.1** Let \( R \) be an integral domain with field of fractions \( Q \). Show that the embedding \( R \to Q \) is an injective envelope of the \( R \)-module \( R \).

**E 5.4.2** Let \( p \) be a prime and denote by \( \mathbb{Z}(p^\infty) \) the subgroup \( \mathbb{Z}[1/p]/\mathbb{Z} \) of \( \mathbb{Q}/\mathbb{Z} \) (the so-called Prüfer \( p \)-group). Show that the canonical map \( \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}(p^\infty) \) is an injective envelope of the \( \mathbb{Z} \)-module \( \mathbb{Z}/p\mathbb{Z} \).

**E 5.4.3** Let \( \iota: M \to I \) be an injective preenvelope of an \( R \)-module. Show that \( \iota \) is an injective envelope if and only if every endomorphism \( \gamma: I \to I \) with \( \gamma \iota = \iota \) is an automorphism.

**E 5.4.4** Let \( \iota: M \to I \) and \( \iota': M' \to I' \) be injective envelopes of graded \( R \)-modules. Show that the direct sum \( \iota \oplus \iota': M \oplus M' \to I \oplus I' \) is an injective envelope.

**E 5.4.5** Assume that \( R \) is left Noetherian and let \( \{ \iota^n: M^n \to I^n \}_{n \in \mathbb{N}} \) be a family of injective envelopes. Show that the coproduct \( \bigsqcup_{n \in \mathbb{N}} \iota^n: \bigsqcup_{n \in \mathbb{N}} M^n \to \bigsqcup_{n \in \mathbb{N}} I^n \) is an injective envelope. **Hint:** See [54, thm. 1.4.6].

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5.5 Semi-flatness

SYNOPSIS. Graded-flat module; complex of flat modules; semi-flat complex; perfect ring.

Semi-flatness of an $R$-complex $F$ will be defined in terms of the functor $- \otimes_R F$ from $\mathcal{C}(R^\alpha)$ to $\mathcal{C}(\mathbb{k})$. First we study complexes of flat modules.

**Complexes of Flat Modules**

**5.5.1 Proposition.** For an $R$-complex $F$, the following conditions are equivalent.

(i) Each $R$-module $F_i$ is flat.

(ii) The functor $- \otimes_R F$ is exact.

(iii) For every exact sequence $0 \to M' \to M \to F \to 0$ in $\mathcal{C}(R)$ the exact sequence $0 \to \text{Hom}_\mathbb{k}(F, \mathbb{E}) \to \text{Hom}_\mathbb{k}(M, \mathbb{E}) \to \text{Hom}_\mathbb{k}(M', \mathbb{E}) \to 0$ is degreewise split.

(iv) The character complex $\text{Hom}_\mathbb{k}(F, \mathbb{E})$ is a complex of injective $R^\alpha$-modules.
PROOF. Conditions (i) and (iv) are equivalent by 1.3.41.

(ii) \iff (iv): By adjunction 4.3.6 and commutativity 4.3.2 there is a natural isomorphism of functors from \( \mathcal{C}(\mathcal{R}^\circ)^{op} \) to \( \mathcal{C}(\mathcal{R}) \),

\[ \text{Hom}_{\mathcal{R}^\circ}(-, \text{Hom}_{\mathcal{R}}(F, \mathcal{E})) \cong \text{Hom}_{\mathcal{R}}(- \otimes_{\mathcal{R}} F, \mathcal{E}). \]

By 5.3.6 the functor on the left-hand side is exact if and only if \( \text{Hom}_{\mathcal{R}}(F, \mathcal{E}) \) is a complex of injective \( \mathcal{R}^\circ \)-modules. As \( \mathcal{E} \) is faithfully injective, the functor on the right-hand side is exact if and only if \( - \otimes_{\mathcal{R}} F \) is exact.

(iv) \implies (iii): Immediate from 5.3.6.

(iii) \implies (i): Choose by 5.1.7 a surjective semi-free resolution \( L \xrightarrow{i} F \) and consider the associated short exact sequence \( 0 \to \text{Ker} \pi \to L \to F \to 0 \). By 5.3.2 the complex \( \text{Hom}_{\mathcal{R}}(L, \mathcal{E}) \) consists of injective \( \mathcal{R}^\circ \)-modules, so it follows from 5.3.6 and split exactness of the sequence

\[ 0 \to \text{Hom}_{\mathcal{R}}(F, \mathcal{E}) \oplus \to \text{Hom}_{\mathcal{R}}(L, \mathcal{E}) \oplus \to \text{Hom}_{\mathcal{R}}(\text{Ker} \pi, \mathcal{E}) \oplus \to 0 \]

that each module \( \text{Hom}_{\mathcal{R}}(F, \mathcal{E}) \) is an injective \( \mathcal{R}^\circ \)-module, whence each \( F_v \) is a flat \( \mathcal{R} \)-module by 1.3.41. \( \square \)

5.5.2 Corollary. Let \( 0 \to F' \to F \to F'' \to 0 \) be an exact sequence of \( \mathcal{R} \)-complexes. If \( F'' \) is a complex of flat modules, then \( F \) is a complex of flat modules if and only if \( F' \) is a complex of flat modules.

PROOF. Apply 5.5.1 and 5.3.7 to the exact sequence of \( \mathcal{R}^\circ \)-complexes

\[ 0 \to \text{Hom}_{\mathcal{R}}(F'', \mathcal{E}) \to \text{Hom}_{\mathcal{R}}(F, \mathcal{E}) \to \text{Hom}_{\mathcal{R}}(F', \mathcal{E}) \to 0. \] \( \square \)

5.5.3 Corollary. Let \( 0 \to M' \to M \to F \to 0 \) be an exact sequence in \( \mathcal{C}(\mathcal{R}) \). If \( F \) is a complex of flat modules, then the sequence \( 0 \to N \otimes_{\mathcal{R}} M' \to N \otimes_{\mathcal{R}} M \to N \otimes_{\mathcal{R}} F \to 0 \) is exact for every \( \mathcal{R}^\circ \)-complex \( N \).

PROOF. Assume that \( F \) is a complex of flat \( \mathcal{R} \)-modules, and let \( N \) be an \( \mathcal{R}^\circ \)-complex. By tensor product is right exact, it is sufficient to show that the induced morphism \( N \otimes_{\mathcal{R}} M' \to N \otimes_{\mathcal{R}} M \) is injective. There is a commutative diagram in \( \mathcal{C}(\mathcal{R}) \),

\[ \text{Hom}_{\mathcal{R}^\circ}(N, \text{Hom}_{\mathcal{R}}(M, \mathcal{E})) \xrightarrow{\phi_{MN}} \text{Hom}_{\mathcal{R}^\circ}(N, \text{Hom}_{\mathcal{R}}(M', \mathcal{E})) \to 0 \]

\[ \begin{array}{ccc}
\text{Hom}_{\mathcal{R}}(M \otimes_{\mathcal{R}} N, \mathcal{E}) & \xrightarrow{\rho_{NM}} & \text{Hom}_{\mathcal{R}}(M' \otimes_{\mathcal{R}} N, \mathcal{E}) \\
\end{array} \]

The vertical maps are adjunction isomorphisms. The upper row is exact by 5.5.1 and the fact that the functor \( \text{Hom}_{\mathcal{R}}(N, -) \) preserves degreewise split-exactness of sequences; cf. 2.3.14. By commutativity of the diagram, the lower row is also exact. As \( \mathcal{E} \) is faithfully injective, this implies that the sequence \( 0 \to M' \otimes_{\mathcal{R}^\circ} N \to M \otimes_{\mathcal{R}^\circ} N \) is exact, and commutativity 4.3.2 finishes the proof. \( \square \)
5.5.4 Definition. A graded $R$-module $F$ is called graded-flat if the $R$-complex $F$ satisfies the conditions in 5.5.1.

A standard application of the next lemma is to an acyclic complex $M$ and a complex $N$ of flat $R$-modules.

5.5.5 Lemma. Let $M$ be an $R^o$-complex and let $N$ be an $R$-complex, such that $M$ is bounded above or $N$ is bounded below. If the complex $M \otimes_R N_v$ is acyclic for every $v \in \mathbb{Z}$, then $M \otimes_R N$ is acyclic.

Proof. By adjunction 4.3.6 there is an isomorphism of $k$-complexes

$$\text{Hom}_k(M \otimes_R N, \mathbb{E}) \cong \text{Hom}_R(N, \text{Hom}_k(M, \mathbb{E})).$$

Recall from (2.2.16.1) that $\text{Hom}_k(M, \mathbb{E})$ is bounded below if $M$ is bounded above. As $\mathbb{E}$ is a faithfully injective $k$-module, the claim follows immediately from 5.2.5. 

Existence of Semi-flat Complexes

5.5.6 Definition. An $R$-complex $F$ is called semi-flat if $\beta \otimes_R F$ is an injective quasi-isomorphism for every injective quasi-isomorphism $\beta$ in $\mathcal{C}(R^o)$.

Remark. Another word for semi-flat is DG-flat.

5.5.7 Example. Let $F$ be a bounded below complex of flat $R$-modules and let $\beta$ be an injective quasi-isomorphism in $\mathcal{C}(R^o)$. The morphism $\beta \otimes_R F$ is injective by 5.5.1. The complex $\text{Cone}\beta$ is acyclic by 4.2.14, and hence so is $(\text{Cone}\beta) \otimes_R F_v$ for every $v \in \mathbb{Z}$. As $F$ is bounded below, it follows from 4.1.10 and 5.5.5 that the complex $\text{Cone}(\beta \otimes_R F) \cong (\text{Cone}\beta) \otimes_R F$ is acyclic, whence $\beta$ is a quasi-isomorphism. Thus, $F$ is semi-flat.

5.5.8 Theorem. Let $F$ be an $R$-complex and let $I$ be a semi-injective $k$-complex. If $F$ is semi-flat, then the $R^o$-complex $\text{Hom}_k(F,I)$ is semi-injective, and the converse holds if $I$ is a faithfully injective $k$-module.

Proof. By commutativity 4.3.2 and adjunction 4.3.6 there is a natural isomorphism of functors,

$$\text{Hom}_k(\,\cdot\otimes_R F, I) \cong \text{Hom}_{R^o}(\,\cdot, \text{Hom}_k(F,I)),$$

from $\mathcal{C}(R^o)^{op}$ to $\mathcal{C}(k)$. Let $\beta$ be an injective quasi-isomorphism in $\mathcal{C}(R^o)$. If $F$ is semi-flat, then $\text{Hom}_k(\beta \otimes_R F, I)$ is a surjective quasi-isomorphism by 5.5.6 and 5.3.10. Thus, $\text{Hom}_{R^o}(\beta, \text{Hom}_k(F,I))$ is a surjective quasi-isomorphism, whence the $R^o$-complex $\text{Hom}_k(F,I)$ is semi-injective. Conversely, if $\text{Hom}_k(F,I)$ is semi-injective and $I$ is a faithfully injective $k$-module, then $\text{Hom}_{R^o}(\beta, \text{Hom}_k(F,I))$ and...
hence \( \text{Hom}_k(\beta \otimes_R F, I) \) is a surjective quasi-isomorphism. If \( I \) is faithfully injective, it now follows that \( \beta \otimes_R F \) is injective and a quasi-isomorphism; cf. 4.2.11.

The next result gives useful characterizations of semi-flat complexes.

5.5.9 Proposition. For an \( R \)-complex \( F \), the following conditions are equivalent.

(i) \( F \) is semi-flat.

(ii) The functor \(- \otimes_R F\) is exact and preserves quasi-isomorphisms.

(iii) The character complex \( \text{Hom}_k(F, \mathcal{E}) \) is a semi-injective \( R^0 \)-complex.

(iv) \( F \) is a complex of flat \( R \)-modules and the functor \(- \otimes_R F\) preserves acyclicity of complexes.

PROOF. The implication (ii) \( \Rightarrow \) (i) is trivial; (i) and (iii) are equivalent by 5.5.8.

(iii) \( \Rightarrow \) (iv): It follows from 5.3.15 that \( \text{Hom}_k(F, \mathcal{E}) \) is a complex of injective \( R^0 \)-modules, so \( F \) is a complex of flat \( R \)-modules by 5.5.1. Let \( A \) be an acyclic \( R^0 \)-complex; by adjunction 4.3.6 and commutativity 4.3.2 there is an isomorphism \( \text{Hom}_{k^r}(A, \text{Hom}_k(F, \mathcal{E})) \cong \text{Hom}_k(A \otimes_R F, \mathcal{E}) \). The left-hand complex is acyclic by 5.3.15, so it follows by faithfulness of the functor \( \text{Hom}_k(-, \mathcal{E}) \) that \( A \otimes_R F \) is acyclic.

(iv) \( \Rightarrow \) (i): Let \( \beta \) be an injective quasi-isomorphism. The morphism \( \beta \otimes_R F \) is then injective by 5.5.1. The complex \( \text{Cone} \beta \) is acyclic by 4.2.14, and hence so is the complex \( (\text{Cone} \beta) \otimes_R F \cong \text{Cone}(\beta \otimes_R F) \), where the isomorphism comes from 4.1.10. It follows that \( \beta \otimes_R F \) is a quasi-isomorphism.

Now the next result is immediate in view of 5.3.16.

5.5.10 Corollary. Every semi-projective \( R \)-complex is semi-flat.

5.5.11 Corollary. A graded \( R \)-module is graded-flat if and only if it is semi-flat as an \( R \)-complex.

PROOF. If \( F \) is semi-flat as a complex, then each module \( F_v \) is flat by 5.5.9, whence \( F \) is graded-flat. If \( F \) is graded-flat, then the character module \( \text{Hom}_k(F, \mathcal{E}) \) is graded-injective by 5.5.1 and hence semi-injective as an \( R^0 \)-complex; see 5.3.17. As an \( R \)-complex, \( F \) is then semi-flat.

Properties of Semi-flat Complexes

5.5.12 Proposition. Let \( 0 \to F' \to F \to F'' \to 0 \) be an exact sequence of \( R \)-complexes. If \( F'' \) is semi-flat, then \( F \) is semi-flat if and only if \( F' \) is semiflat.

PROOF. Apply 5.5.9 and 5.3.19 to the exact sequence of \( R^0 \)-complexes

\[
0 \to \text{Hom}_k(F'', \mathcal{E}) \to \text{Hom}_k(F, \mathcal{E}) \to \text{Hom}_k(F', \mathcal{E}) \to 0.
\]

5.5.13 Proposition. Let \( \{F^a\}_{a \in U} \) be a family of \( R \)-complexes. The coproduct \( \bigsqcup_{a \in U} F^a \) is semi-flat if and only if each complex \( F^a \) is semi-flat.
5.5 Semi-flatness

**Proof.** Let $\beta: K \to M$ be an injective quasi-isomorphism in $\mathcal{C}(R^\omega)$. There is a commutative diagram in $\mathcal{C}(k)$,

$$
\begin{array}{ccc}
\prod_{u \in U} (K \otimes_R F^u) & \xrightarrow{\prod (\beta \otimes F^u)} & \prod_{u \in U} (M \otimes_R F^u) \\
\xrightarrow{\cong} & & \xrightarrow{\cong} \\
K \otimes_R \left( \prod_{u \in U} F^u \right) & \xrightarrow{\beta \otimes \left( \prod F^u \right)} & M \otimes_R \left( \prod_{u \in U} F^u \right),
\end{array}
$$

where the vertical maps are the canonical isomorphisms (3.1.13.1). In view of 3.1.6 and 3.1.11 it follows that $\beta \otimes_R \left( \prod_{u \in U} F^u \right)$ is an injective quasi-isomorphism if and only if each map $\beta \otimes_R F^u$ is an injective quasi-isomorphism.

**5.5.14 Proposition.** Let $\{ \mu^u: F^u \to F^v \}_{u \leq v}$ be a $U$-direct system of semi-flat $R$-complexes. If $U$ is filtered, then $\text{colim}_{u \in U} F^u$ is semi-flat.

**Proof.** Let $\beta: K \to M$ be an injective quasi-isomorphism in $\mathcal{C}(R^\omega)$. There is a commutative diagram in $\mathcal{C}(k)$,

$$
\begin{array}{ccc}
\text{colim}_{u \in U} (K \otimes_R F^u) & \xrightarrow{\text{colim}(\beta \otimes F^u)} & \text{colim}_{u \in U} (M \otimes_R F^u) \\
\xrightarrow{\cong} & & \xrightarrow{\cong} \\
K \otimes_R \left( \text{colim}_{u \in U} F^u \right) & \xrightarrow{\beta \otimes \left( \text{colim} F^u \right)} & M \otimes_R \left( \text{colim}_{u \in U} F^u \right),
\end{array}
$$

where the vertical maps are the canonical isomorphisms (3.2.16.1). In view of 3.2.27 and 3.2.29 it follows that $\beta \otimes_R \left( \text{colim} F^u \right)$ is an injective quasi-isomorphism if each map $\beta \otimes_R F^u$ is an injective quasi-isomorphism.

Contrary to the situation for semi-projective complexes, see 5.2.18, a quasi-isomorphism of semi-flat $R$-complexes need not be a homotopy equivalence.

**5.5.15 Example.** It follows from 1.3.10, 1.3.11, and 5.1.18 that the $\mathbb{Z}$-module $Q$ has a semi-free resolution $\pi: L \xrightarrow{\pi} Q$ with $L_n = 0$ for $n \neq 0, 1$. Both $\mathbb{Z}$-complexes $Q$ and $L$ are semi-flat. Suppose $\gamma: Q \to L$ were a homotopy inverse of $\pi$, then one would have $1^Q \sim \pi \gamma$, and hence $1^Q = \pi \gamma$ as $\partial^Q = 0$. This would imply that $Q$ is a direct summand of $L_0$ and hence a free $\mathbb{Z}$-module, but it is not.

**5.5.16 Proposition.** Let $\alpha: F \to F'$ be a quasi-isomorphism between semi-flat $R$-complexes. For every $R^\omega$-complex $M$, the induced morphism $M \otimes_R \alpha$ is a quasi-isomorphism.

**Proof.** By 5.3.15 and 5.5.8 the morphism $\text{Hom}_R(\alpha, \mathbb{Z})$ is a quasi-isomorphism of semi-injective $R^\omega$-complexes and hence a homotopy equivalence by 5.3.23. Therefore, the upper horizontal map in the following commutative diagram is also a homotopy equivalence; see 2.3.10.
The diagram shows that $\text{Hom}_k(M \otimes R \alpha, E)$ is a quasi-isomorphism, and by faithful injectivity of $E$ it follows that $M \otimes R \alpha$ is a quasi-isomorphism; cf. 4.2.11.

5.5.17 Proposition. Let $R \to S$ be a ring homomorphism. If $F$ is a semi-flat $R$-complex, then the $S$-complex $S \otimes_R F$ is semi-flat.

Proof. By associativity 4.3.4 and (4.3.0.1) there are natural isomorphisms,

$$- \otimes_S (S \otimes_R F) \cong (- \otimes_S S) \otimes_R F \cong - \otimes_R F,$$

of functors from $\mathcal{C}(S^0)$ to $\mathcal{C}(k)$. By assumption, the functor $- \otimes_R F$ is exact and preserves quasi-isomorphisms.

5.5.18 Proposition. If $F$ is a semi-flat $S$-complex and $F'$ is a semi-flat $k$-complex, then the $S$-complex $F \otimes_k F'$ is semi-flat.

Proof. By associativity 4.3.4 there is a natural isomorphism of functors

$$- \otimes_S (F \otimes_k F') \cong (- \otimes_S F) \otimes_k F',$$

from $\mathcal{C}(S^0)$ to $\mathcal{C}(k)$. By the assumptions on $F$ and $F'$, the functor $(- \otimes_S F) \otimes_k F'$ is exact and preserves quasi-isomorphisms.

The Case of Modules

Specialization of 5.5.1 and 5.5.2 to modules yields the next two results.

5.5.19 Proposition. For an $R$-module $F$, the following conditions are equivalent.

(i) $F$ is flat.

(ii) For every exact sequence $0 \to M' \to M \to F \to 0$ of $R$-modules, the exact sequence $0 \to \text{Hom}_k(F, E) \to \text{Hom}_k(M, E) \to \text{Hom}_k(M', E) \to 0$ is split.

(iii) The character module $\text{Hom}_k(F, E)$ is an injective $R^0$-module.

5.5.20 Corollary. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of $R$-modules. Assume that $F''$ is flat, then $F'$ is flat if and only if $F$ is flat.
5.5 Semi-flatness

5.5.21. It follows from 5.5.11 that an $R$-module is flat if and only if it is semi-flat as an $R$-complex.

The next two results now follow immediately from 5.5.13 and 5.5.14.

5.5.22 Proposition. Let $\{F^u\}_{u \in U}$ be a family of $R$-modules. The coproduct $\bigsqcup_{u \in U} F^u$ is flat if and only if each module $F^u$ is flat.

5.5.23 Proposition. Let $\{\mu^u : F^u \rightarrow F^v\}_{u \leq v}$ be a $U$-direct system of flat $R$-modules. If $U$ is filtered, then $\text{colim}_{u \in U} F^u$ is flat.

**Perfect Rings**

5.5.24 Lemma. Assume that every flat $R$-module is projective. For every sequence $(a_i)_{i \in \mathbb{N}}$ in $R$ there exists $n \geq 1$ such that for every $j > n$ there is an equality of right ideals, $(a_1 \cdots a_{j-1})R = (a_1 \cdots a_{j-1}a_j)R$.

**Proof.** For $1 \leq i < j$ let $a^{ij}_i$ be the homothety given by right multiplication on $R$ with $a_i \cdots a_{j-1}$ and set $a^{ij} = 1^R$. These maps form a direct system of $R$-modules; let $A$ denote its colimit. By 5.5.23 the module $A$ is flat; hence it is projective by the assumption on $R$. Set $L = R^{(\mathbb{N})}$ and let $\iota' : R \rightarrow L$ be the embedding into the $i^{th}$ component; note that $\{\iota'(1)\}_{i \in \mathbb{N}}$ is a basis for $L$. Set $f_i = \iota'(1) - a_id^{i+1}(1)$ for every $i \in \mathbb{N}$. The elements $\{f_i\}_{i \in \mathbb{N}}$ in $L$ are linearly independent as one has

$$r_1f_1 + r_2f_2 + \cdots + r_m f_m = r_1\iota'^1(1) + (r_2 - r_1a_1)\iota'^2(1) + \cdots + (r_m - r_{m-1}a_{m-1})\iota'^m(1) - r_m a_m t^{m+1}(1)$$

for $r_1, r_2, \ldots, r_m$ in $R$. Denote by $F$ the free submodule of $L$ generated by $\{f_i\}_{i \in \mathbb{N}}$. Since one has $\iota'(r) - \iota'a^{ij}_i(r) = rf_i + r_ia_if_{i+1} + \cdots + ra_i \cdots a_{j-2} f_{j-1}$, it follows from 3.2.2 that there is an exact sequence $0 \rightarrow F \rightarrow L \rightarrow A \rightarrow 0$. As $A$ is projective, the embedding $F \rightarrow L$ has a right inverse $\pi : L \rightarrow F$; cf. 1.3.17. Write $\pi(\iota'(1)) = \sum_{j \geq 1} b_j f_j$; then one has

$$f_i = \pi(f_i) = \pi(\iota'(1) - a_id^{i+1}(1)) = \sum_{j \geq 1} (b_{ij} - a_i b_{(i+1)j}) f_j$$

and, therefore, $b_{ii} - a_i b_{(i+1)i} = 1$ and $b_{ij} - a_i b_{(i+1)j} = 0$ for all $j \neq i$. Thus, for every $j > 1$ there are equalities,

$$b_{1j} = a_1 b_{2j} = a_1a_2 b_{3j} = \cdots = a_1a_2 \cdots a_{j-1} b_{jj} = a_1a_2 \cdots a_{j-1}(1 + a_j b_{(j+1)j})$$

and hence $a_1 \cdots a_{j-1} = b_{1j} - a_1 \cdots a_{j-1} a_j b_{(j+1)j}$. Since there exists an $n$ such that one has $b_{1j} = 0$ for all $j > n$, the desired assertion follows.

5.5.25 Lemma. If the Jacobson radical of $R$ is zero and every descending chain of principal right ideals in $R$ becomes stationary, then $R$ is semi-simple.
PROOF. First note that every right ideal \( a \neq 0 \) in \( R \) contains a minimal right ideal \( b \); indeed, take \( b \) minimal among the non-zero principal right ideals contained in \( a \). Furthermore, every minimal right ideal \( b \) in \( R \) has a complement. Indeed, as the Jacobson radical of \( R \) is zero, one has \( b \not\subseteq \mathfrak{M} \) for some maximal right ideal \( \mathfrak{M} \); and since \( b \) is minimal, \( b \cap \mathfrak{M} = 0 \) follows. Consequently, \( R = b \oplus \mathfrak{M} \) holds.

Now, let \( b_1 \) be a minimal right ideal in \( R \) and write \( R = b_1 \oplus a_1 \) for some right ideal \( a_1 \). If \( a_1 = 0 \) then the \( R^a \)-module \( R = b_1 \) is simple. Otherwise, let \( b_2 \) be a minimal right ideal contained in \( a_1 \), and write \( a_1 = b_2 \oplus a_2 \) for some right ideal \( a_2 \); then one has \( R = b_1 \oplus b_2 \oplus a_2 \). If \( a_2 = 0 \) then the \( R^a \)-module \( R = b_1 \oplus b_2 \) is semi-simple. If \( a_2 \neq 0 \) one can continue the process, which after \( n \) iterations yields minimal right ideals \( b_1, b_2, \ldots, b_n \) and right ideals \( a_1 \supseteq a_2 \supseteq \cdots \supseteq a_n \) such that \( R = b_1 \oplus \cdots \oplus b_n \oplus a_n \). Each right ideal \( a_n \) is principal, as it is a direct summand of \( R \), so the process terminates with \( a_n = 0 \) for some \( n \). Thus the \( R^a \)-module \( R = b_1 \oplus \cdots \oplus b_n \) is semi-simple. \( \Box \)

5.5.26 Theorem. The ring \( R \) is left perfect if an only if every flat \( R \)-module is projective.

PROOF. Let \( \mathfrak{J} \) be the Jacobson radical of \( R \).

"Only if": Let \( F \) be a flat \( R \)-module. By 5.4.45 it has projective cover, so there is a projective \( R \)-module \( P \) with a superfluous submodule \( K \) such that \( P/K \) is flat. By 1.3.38 one has \( \mathfrak{J}P = \mathfrak{J} \cap K \). As \( K \) is a superfluous submodule of \( P \) it is contained in \( \mathfrak{J}P \) by 5.4.29. Thus one has \( \mathfrak{J}K = K \) and, therefore, \( K = 0 \) by 5.4.41.

"If": To show that \( \mathfrak{J} \) is left T-nilpotent, let \( (a_i)_{i \in \mathbb{N}} \) be a sequence in \( \mathfrak{J} \). By 5.5.24 there exist \( n \geq 1 \) and \( r \in R \) such that one has \( a_1 \cdots a_n = a_1 \cdots a_n a_{n+1} r \), and therefore \( a_1 \cdots a_n (1 - a_{n+1} r) = 0 \). Since \( a_{n+1} \) is in \( \mathfrak{J} \), the element \( 1 - a_{n+1} r \) is a unit, and it follows that \( a_1 \cdots a_n = 0 \). It remains to show that the ring \( k = R/\mathfrak{J} \) is semi-simple; to this end apply 5.5.25. Clearly, the Jacobson radical of \( k \) is zero. A descending chain of principal right ideals in \( k \) has the form \( (a_1)k \supseteq (a_1a_2)k \supseteq (a_1a_2a_3)k \supseteq \cdots \) with \( a_i \in R \). It follows from 5.5.24 that the descending chain \( (a_1)R \supseteq (a_1a_2)R \supseteq (a_1a_2a_3)R \supseteq \cdots \) in \( R \) becomes stationary, and hence so does the chain in \( k \). \( \Box \)

REMARK. The statement of the previous theorem is part of Bass’ Theorem P [10] from 1960. With the existence of flat covers for all modules, which was only proved in 2001 by Bican, El Bashir, and Enochs [13], a short proof of the “if” part in 5.5.26 became available. Indeed, every \( R \)-module \( M \) has a flat cover \( \pi: F \to M \), so if every flat \( R \)-module is projective, then \( \pi \) is a projective cover.

EXERCISES

E 5.5.1 Show that a graded \( R \)-module is graded-flat if and only if it is flat as an \( R \)-module.

E 5.5.2 Show that the Dold complex from 2.1.20 is not semi-flat.

E 5.5.3 Find an exact, but not split, sequence \( 0 \to M' \to M \to M'' \to 0 \) of \( R \)-modules with \( M'' \) not flat such that \( 0 \to \text{Hom}_R(M'', E) \to \text{Hom}_R(M, E) \to \text{Hom}_R(M', E) \to 0 \) is split.

E 5.5.4 Let \( k \) be a field and set \( R = k[x] \). Show that \( \mathfrak{a} = k[(0)] \) is an ideal in \( R \) and that the complex \( F = 0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0 \) is semi-flat; cf. E 1.3.32. Show that \( F \) is not contractible.
Appendix A
Triangulated Categories

In this appendix, $\mathcal{T}$ is an additive category equipped with an additive autoequivalence $\Sigma$. We describe the conditions for $(\mathcal{T}, \Sigma)$ to form a triangulated category; the first step is to settle on a collection of triangles.

A.1 Definition. A candidate triangle in $\mathcal{T}$ is a diagram

$$
\begin{array}{c}
M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M,
\end{array}
$$

such that the composites $\beta\alpha$, $\gamma\beta$, and $(\Sigma\alpha)\gamma$ are all zero. An morphism $(\varphi, \psi, \chi)$ of candidate triangles is a commutative diagram in $\mathcal{T}$,

$$
\begin{array}{c}
M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \\
M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} \Sigma M';
\end{array}
$$

it is called an isomorphism if $\varphi$, $\psi$, and $\chi$ are isomorphisms in $\mathcal{T}$.

A.2. For a collection $\triangle$ of candidate triangles in $\mathcal{T}$, consider the next conditions.

(TR0) For every $M$ in $\mathcal{T}$, the candidate triangle

$$
\begin{array}{c}
M \xrightarrow{1_M} M \xrightarrow{0} \Sigma M
\end{array}
$$

is in $\triangle$. Every candidate triangle that is isomorphic to one from $\triangle$ is in $\triangle$.

(TR1) Every morphism $\alpha: M \to N$ in $\mathcal{T}$ fits in a candidate triangle from $\triangle$,

$$
\begin{array}{c}
M \xrightarrow{\alpha} N \xrightarrow{0} X \xrightarrow{\gamma} \Sigma M.
\end{array}
$$

(TR2) For every candidate triangle $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\triangle$, the following two candidate triangles belong to $\triangle$ as well,

$$
\begin{array}{c}
N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N \\
\Sigma^{-1} X \xrightarrow{-\Sigma^{-1}\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X.
\end{array}
$$
Consider two candidate triangles,

\[ M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \quad \text{and} \quad N \xrightarrow{-\beta} X \xrightarrow{-\gamma} \Sigma M \xrightarrow{-\Sigma \alpha} \Sigma N, \]

If one belongs to \( \triangle \) then so does the other.

For every commutative diagram

\[ M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \]
\[ M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} \Sigma M', \]

where the rows are candidate triangles in \( \triangle \), there exists a (not necessarily unique) morphism \( \chi : X \to X' \), such that \((\varphi, \psi, \chi)\) is a morphism of candidate triangles.

For every commutative diagram (A.2.1), where the rows are candidate triangles in \( \triangle \), there exists a (not necessarily unique) morphism \( \chi : X \to X' \) such that \((\varphi, \psi, \chi)\) is a morphism of candidate triangles, and such that the following candidate triangle belongs to \( \triangle \),

\[ M' \oplus \left( \begin{array}{cc} a' & \psi \\ 0 & -\beta \end{array} \right) \oplus N' \oplus \left( \begin{array}{cc} \beta' & x \\ 0 & -\gamma \end{array} \right) \oplus X' \oplus \left( \begin{array}{cc} \gamma' & \Sigma \varphi \\ 0 & -\Sigma \alpha \end{array} \right) \oplus \Sigma M' \]
\[ \oplus N \oplus \Sigma M \oplus \Sigma N \]

The candidate triangle (A.2.2) is called the mapping cone of \((\varphi, \psi, \chi)\).

Condition (TR4') is evidently stronger than (TR3), and it is proved in A.4 below that (TR2) and (TR2') are equivalent under assumption of (TR0). The conditions in A.2 supply the axioms for a triangulated category.

A.3 Definition. A triangulated category is an additive category \( \mathcal{T} \) equipped with an additive autoequivalence \( \Sigma \) and a collection \( \triangle \) of candidate triangles, called distinguished triangles, such that (TR0), (TR1), (TR2'), and (TR4') are satisfied.

Remark. If the collection \( \triangle \) in \((\mathcal{T}, \Sigma)\) satisfies only (TR0), (TR1), (TR2'), and (TR3), then \( \mathcal{T} \) is called pretriangulated. It can be proved that for a pretriangulated category the so-called octahedral axiom, which is usually denoted (TR4), is equivalent to (TR4'); see Neeman [39]. That is, a triangulated category is a pretriangulated category that satisfies the octahedral axiom. This perspective goes back to Verdier’s thesis on derived categories [51] from the mid 1960s—it was published 30 years late and only after Verdier’s death. Indeed, triangulated categories in algebra were originally defined through axiomatization of the properties of derived categories; the axioms being (TR0), (TR1), (TR2), (TR3), and (TR4); usually with (TR0) included in (TR1). The contemporary formulation of the definition in A.3 follows Neeman’s monograph [40].

A.4 Lemma. Let \( \triangle \) be a collection of candidate triangles in \((\mathcal{T}, \Sigma)\) such that (TR0) is satisfied. Condition (TR2) is then satisfied if and only if (TR2') is satisfied.
PROOF. Assume that (TR2’) is satisfied. Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ be a candidate triangle in $\triangle$. Consider the following isomorphism of candidate triangles,

\[
\begin{array}{c}
N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \xrightarrow{-\Sigma \alpha} \Sigma N \\
\downarrow \cong \downarrow \cong \\
N \xrightarrow{-\beta} X \xrightarrow{-\gamma} \Sigma M \xrightarrow{-\Sigma \alpha} \Sigma N.
\end{array}
\]

By (TR2’) the lower row in $(\ast)$ belongs to $\triangle$, and hence so does the upper row by (TR0). To show that the candidate triangle

$\Sigma^{-1} X \xrightarrow{-\Sigma^{-1} \gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X$

is in $\triangle$, is by (TR2’) equivalent to showing that $M \xrightarrow{-\alpha} N \xrightarrow{-\beta} X \xrightarrow{\gamma} \Sigma M$ is in $\triangle$; and that follows from (TR0) and the next isomorphism of candidate triangles,

\[
\begin{array}{c}
M \xrightarrow{-\alpha} N \xrightarrow{-\beta} X \xrightarrow{\gamma} \Sigma M \\
\downarrow \cong \downarrow \cong \\
M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M.
\end{array}
\]

Similar arguments show that (TR2) implies (TR2’).}

A.5 Proposition. Let $(\mathcal{T}, \Sigma)$ be a triangulated category. The opposite category $(\mathcal{T}^{\text{op}}, \Sigma^{-1})$ is triangulated in the following canonical way: A candidate triangle $M \to N \to X \to \Sigma^{-1} M$ in $\mathcal{T}^{\text{op}}$ is distinguished if and only if the corresponding diagram $\Sigma^{-1} M \to X \to N \to M$ is a distinguished triangle in $\mathcal{T}$.

PROOF. It is evident that a diagram in $\mathcal{T}^{\text{op}}$ is a candidate triangle if and only if the corresponding diagram in $\mathcal{T}$ is a candidate triangle. Let $\triangle$ be the collection of distinguished triangles in $\mathcal{T}$. It is elementary to verify that the collection of diagrams $M \to N \to X \to \Sigma^{-1} M$ in $\mathcal{T}^{\text{op}}$ such that the corresponding diagram in $\mathcal{T}$ belongs to $\triangle$ satisfies the axioms in A.3. As an example, we provide the details for (TR0).

Let $M$ be an object in $\mathcal{T}^{\text{op}}$, and hence in $\mathcal{T}$. The candidate triangle

\[
\begin{array}{c}
M \xrightarrow{1M} M \longrightarrow 0 \longrightarrow \Sigma^{-1} M
\end{array}
\]

in $\mathcal{T}^{\text{op}}$ is distinguished if and only if the corresponding candidate triangle in $\mathcal{T}$,

\[
\begin{array}{c}
\Sigma^{-1} M \longrightarrow 0 \longrightarrow M \xrightarrow{1M} M
\end{array}
\]

belongs to $\triangle$. By (TR2’), applied twice, $(\ddagger)$ is in $\triangle$ if and only if the following candidate triangle is in $\triangle$,

\[
\begin{array}{c}
M \xrightarrow{1M} M \longrightarrow \Sigma 0 \longrightarrow \Sigma M.
\end{array}
\]
There is an isomorphism \( \Sigma 0 \cong 0 \) in \( \mathcal{T} \), so by (TR0) the triangle \((\circ)\) is in \( \Delta \), whence \((\ast)\) is distinguished in \( \mathcal{T}^{\text{op}} \). Next, let

\[
\begin{array}{c}
M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} \Sigma^{-1} M' \\
\cong \psi \quad \cong \phi \quad \cong \chi \quad \cong \Sigma^{-1} \psi \\
M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma^{-1} M
\end{array}
\]

be an isomorphism of candidate triangles in \( \mathcal{T}^{\text{op}} \), and assume that the bottom row is distinguished. In the corresponding diagram in \( \mathcal{T} \),

\[
\begin{array}{c}
\Sigma^{-1} M \xrightarrow{\gamma} X \xrightarrow{\beta} N \xrightarrow{\alpha} M \\
\cong \Sigma^{-1} \psi \quad \cong \chi \quad \cong \phi \quad \cong \psi \\
\Sigma^{-1} M' \xrightarrow{\gamma'} X' \xrightarrow{\beta'} N' \xrightarrow{\alpha'} M'
\end{array}
\]

the top row belongs to \( \Delta \), and by (TR0) so does the bottom row. Hence, the top row in \((\S)\) is a distinguished triangle in \( \mathcal{T}^{\text{op}} \). \( \square \)

**A.6 Definition.** Let \((\mathcal{T}, \Sigma_{\mathcal{T}})\) and \((\mathcal{U}, \Sigma_{\mathcal{U}})\) be triangulated categories. A **triangulated functor** \( F: \mathcal{T} \rightarrow \mathcal{U} \) is an additive functor with a natural isomorphism \( \phi: F \Sigma_{\mathcal{T}} \rightarrow \Sigma_{\mathcal{U}} F \) such that for every distinguished triangle in \( \mathcal{T} \),

\[
M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma_{\mathcal{T}} M,
\]

the induced candidate triangle in \( \mathcal{U} \),

\[
F(M) \xrightarrow{F(\alpha)} F(N) \xrightarrow{F(\beta)} F(X) \xrightarrow{\phi M \circ F(\gamma)} \Sigma_{\mathcal{U}} F(M),
\]

is distinguished.

**A.7 Definition.** Let \((\mathcal{T}, \Sigma)\) be a triangulated category. A **triangulated subcategory** of \( \mathcal{T} \) is a full additive subcategory \( \mathcal{S} \) that satisfies the following conditions.

1. If \( N \) and \( N' \) are isomorphic objects in \( \mathcal{T} \), then \( N \) is in \( \mathcal{S} \) if and only if \( N' \) is in \( \mathcal{S} \).
2. An object \( N \) is in \( \mathcal{S} \) if and only if \( \Sigma N \) is in \( \mathcal{S} \).
3. For every distinguished triangle \( M \rightarrow N \rightarrow X \rightarrow \Sigma M \) in \( \mathcal{T} \), such that the objects \( M \) and \( N \) are in \( \mathcal{S} \), also \( X \) is in \( \mathcal{S} \).

Note that if \( \mathcal{S} \) is a triangulated subcategory of \((\mathcal{T}, \Sigma)\), then \((\mathcal{S}, \Sigma)\) is a triangulated category.

**A.8 Proposition.** Let \( \mathcal{T} \) be an additive category equipped with an additive autoequivalence \( \Sigma \). Consider a commutative diagram in \( \mathcal{T} \),
where \((\varphi, \psi, \chi)\) and \((\tilde{\varphi}, \tilde{\psi}, \tilde{\chi})\) are morphisms of candidate triangles, while \((\mu^1, \nu^1, \kappa^1)\) and \((\mu^2, \nu^2, \kappa^2)\) are isomorphisms of candidate triangles. The mapping cone candidate triangles of \((\varphi, \psi, \chi)\) and \((\tilde{\varphi}, \tilde{\psi}, \tilde{\chi})\) are isomorphic.

**Proof.** The commutative diagram

\[
\begin{array}{ccc}
\tilde{M}^2 & \xrightarrow{(\tilde{\alpha}^2 \tilde{\psi} \ 0 \ -\tilde{\beta}^1)} & \tilde{N}^2 \\
\oplus & \xrightarrow{(\tilde{\mu}^2 \ 0 \ \nu^1)} & \tilde{X}^2 \\
\tilde{M}^1 & \xrightarrow{(\varphi \ 0 \ -\beta^1)} & \tilde{N}^1 \\
\oplus \tilde{\phi} & \xrightarrow{\varphi \ 0} & \tilde{X}^1 \\
M^2 & \xrightarrow{(\alpha^2 \psi \ 0 \ -\beta^1)} & N^1 \\
\oplus \mu^2 & \xrightarrow{\alpha^2 \psi \ 0} & X^1 \\
\Sigma M^1 & \xrightarrow{(\gamma^1 \Sigma \phi \ 0 \ -\gamma^1)} & \Sigma N^1 \\
\oplus \Sigma \alpha^1 & \xrightarrow{\gamma^1 \Sigma \phi \ 0} & \Sigma X^1 \\
\Sigma M^2 & \xrightarrow{(\kappa^1 \Sigma \mu^1 \ 0 \ -\kappa^1)} & \Sigma N^1 \\
\oplus \Sigma \beta^1 & \xrightarrow{\kappa^1 \Sigma \mu^1 \ 0} & \Sigma X^1 \\
\Sigma \phi & \xrightarrow{\Sigma \phi \ 0} & \Sigma X^1 \\
\Sigma M^2 & \xrightarrow{(\Sigma \phi \ 0 \ -\Sigma \delta^1)} & \Sigma N^1 \\
\oplus \Sigma \beta^1 & \xrightarrow{\Sigma \phi \ 0 \ -\Sigma \delta^1} & \Sigma N^1 \\
\end{array}
\]

is an isomorphism from the mapping cone candidate triangle of \((\tilde{\varphi}, \tilde{\psi}, \tilde{\chi})\) to the mapping cone candidate triangle of \((\varphi, \psi, \chi)\).


34. MacLane, S.: Categories for the working mathematician. Springer-Verlag, New York (1971).

35. Massey, W.S.: A basic course in algebraic topology, Graduate Texts in Mathematics, vol. 5


Glossary

Here we recapitulate, briefly, the definitions of several key notions. For other standard terms that we use but do not define, we refer to the following textbooks.

- “Categories for the Working Mathematician” [35] by MacLane for notions in category theory,
- “Lectures on Modules and Rings” [32] and “A First Course in Noncommutative Rings” [33] by Lam for notions in general ring theory, and

**Abelian category.** An additive category in which every morphism has a kernel and a cokernel, every monomorphism is the kernel of a morphism, and every epimorphism is the cokernel of a morphism. See also [35, VIII.3].

In the Abelian categories $\mathcal{M}(R)$ and $\mathcal{C}(R)$, a kernel is identified with its source and a cokernel is identified with its target.

**Divisible module over a domain.** An $R$-module $M$ with $rM = M$ for all $r \neq 0$ in the domain $R$. See also [32, §3C].

**Division ring.** A unital ring in which every element has a multiplicative inverse; see [33, §13].

**Filtered set.** A preordered set $(U, \leq)$ with the property that for any two elements $u$ and $v$ in $U$ there is a $w \in U$ with $u \leq w$ and $v \leq w$; see [35, IX.1].

**(Left) hereditary ring.** A ring whose every left ideal is projective; see [32, §2E]. A ring $R$ is called hereditary if it is both left and right hereditary; that is, $R$ and $R^\text{op}$ are both left hereditary.

**Invariant basis number (IBN).** A left/right symmetric property: $R$ has IBN if finitely generated free $R$-modules are isomorphic only if their bases have the same number of elements; see [32, §1].

**Indecomposable module.** A module $N$ with no other direct summands than 0 and $N$; see [33, §7].

**Jacobson radical** of a ring. The intersection of all maximal left ideals or, equivalently, all maximal right ideals. In particular, the Jacobson radical is an ideal; see [33, §4].

**Local ring.** A ring with a unique maximal left ideal or, equivalently, a unique (the same) maximal right ideal; see [33, §19].

**Middle $R$-linear map.** A map $\varphi : M \times N \to X$, where $M$ is an $R^\text{op}$-module, $N$ is an $R$-module, and $X$ is a $k$-module, such that $\varphi(mr, n) = \varphi(m, rn)$ holds for all $m \in M$, $n \in N$, and $r \in R$.
Noetherian ring. A ring \( R \) that is both left and right Noetherian; that is, \( R \) and \( R^o \) are both left Noetherian. A ring \( R \) is left Noetherian if it satisfies the ascending chain condition on left ideals, equivalently, every submodule of a finitely generated \( R \)-module is finitely generated; see [33, §1].

Preordered set. A set endowed with a reflexive and transitive binary relation ‘\( \leq \)’; see [35, I.2].

Semi-simple module. A module whose every submodule is a direct summand; see [33, §2].

Semi-simple ring. Every \( R \)-module (equivalently: every \( R^o \)-module, the \( R \)-module \( R \), or the \( R^o \)-module \( R \)) is semisimple. A cyclic module over a semi-simple ring is isomorphic to a direct sum of simple ideals generated by idempotents. See also [33, §§2–3].

Simple module. A module \( M \neq 0 \) with no other submodules than \( 0 \) and \( M \); see [33, §2].

Simple ring. A non-zero ring \( R \) with no other ideals than \((0)\) and \( R \); see [33, §1].

Torsion. An element \( m \) of a \( \mathbb{k} \)-module \( M \) is torsion if one has \( xm = 0 \) for some non-zero divisor \( x \) in \( \mathbb{k} \). The torsion elements in \( M \) form a submodule \( M_T \) of \( M \). If one has \( M_T = M \), then \( M \) is torsion, and \( M \) is torsion free if \( M_T \) is the zero-module. See also [32, §4B].

von Neumann regular ring. A ring \( R \) such that for every \( x \in R \) there is an \( r \in R \) with \( x = xrx \); equivalently, every finitely generated left ideal (equivalently every finitely generated right ideal) in \( R \) is generated by an idempotent; see [33, thm. (4.23)].
List of Symbols

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≪ sufficiently small, xxiii
≫ sufficiently large, xxiii
∼ homotopy, 54
≃ quasi-isomorphism,
in category of complexes, 119
→ injective map, xxiii
↠ surjective map, xxiii
⊂ proper subset, xxiii
⊆ subset, xxiii
\ difference of sets, xxiii
⊔ disjoint union of sets, xxiii
⊕ biproduct, 5
direct sum,
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∧ wedge product, 37
Π coproduct,
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∏ product,
in category of modules, 6
in category of complexes, 80, 81
⊔ pushout, 92
⊓ pullback, 105
∧ exterior algebra, 37
\text{R}(\cdot) module generated by a set, 17, 36
\lfloor \cdot \rfloor degree of an element, 35
\lceil \cdot \rceil homology class, 51
\lfloor \cdot \rfloor_N coset w.r.t. submodule N, 4
\lfloor \cdot \rfloor^1 underlying graded module, 38
\lfloor \cdot \rfloor^o opposite ring, xxiii
\lfloor \cdot \rfloor^{op} opposite category/ functor, xxiii
\lfloor \cdot \rfloor^r torsion submodule, 436
\lfloor U \rfloor U\text{-fold coproduct}, 7
\lfloor U \rfloor^U U\text{-fold product}, 6
\lfloor \cdot \rfloor^p p\text{th tensor power}, 36
\lfloor \cdot \rfloor^c hard truncation above, 74
\lfloor \cdot \rfloor^c hard truncation below, 74
\lfloor \cdot \rfloor^c soft truncation above, 74
\lfloor \cdot \rfloor^c soft truncation below, 74
∇ superdiagonal, 87
\partial^M differential, 38
\text{1}^M identity morphism, 5
Σ shift, 47
\delta^M_X biduality,
\text{in category of modules}, 32
\text{in category of complexes}, 140
\rho^X^M homomorphism evaluation,
\text{in category of modules}, 31
\text{in category of complexes}, 136
\rho^{MXN} tensor evaluation,
\text{in category of modules}, 29
\text{in category of complexes}, 132
\sigma^{MN} commutativity,
\text{in category of modules}, 13
\text{in category of complexes}, 127
\iota^{X^M} \text{adjunction},
\text{in category of modules}, 15
\text{in category of complexes}, 129
\varsigma^M degree shift, 48
\varsigma^M_X swap,
\text{in category of modules}, 15
\text{in category of complexes}, 130
List of Symbols

\[ \tau^M_{\mathcal{C}} \] truncation morphisms, 74
\[ \omega^{\mathcal{N}}_{\mathcal{C}} \] associativity,
  in category of modules, 14
  in category of complexes, 128

\[ \mathbb{C} \] complex numbers, xxiii
\[ \mathbb{E} \] faithfully injective module, 25
\[ \mathbb{F} \] finite field, xxiii
\[ \mathbb{N} \] natural numbers, xxiii
\[ \mathbb{Q} \] rational numbers, xxiii
\[ \mathbb{R} \] real numbers, xxiii
\[ \mathbb{Z} \] integers, xxiii

\[ \mathfrak{k} \] commutative ground ring, xxiii

\[ \mathcal{C} \] category of complexes, 42
\[ \mathcal{M} \] category of modules, 3
\[ \mathcal{M}_{\text{gr}} \] category of graded modules, 36
\[ \mathcal{U}(\cdot,\cdot) \] hom-set in category \( \mathcal{U} \), 5

\[ \text{B}(\cdot) \] boundary complex, 48
\[ \text{C}(\cdot) \] cokernel complex, 48
\text{Coker} \[ \text{Coker}(\cdot) \] cokernel, 42
\text{Cone} \[ \text{Cone}(\cdot) \] mapping cone, 113
\text{H}(\cdot) \[ \text{H}(\cdot) \] homology complex, 49
\text{Hom} \[ \text{Hom}(\cdot,\cdot) \] Homomorphism functor,
  in category of modules, 4
  in category of complexes, 56
\text{Id} \[ \text{Id}(\cdot) \] identity functor, xxiii
\text{Im} \[ \text{Im}(\cdot) \] image, 42
\text{Ker} \[ \text{Ker}(\cdot) \] kernel, 42
\text{M}_{m\times n} \[ \text{M}_{m\times n}(\cdot) \] set of \( m \times n \) matrices, 8
\text{Z}(\cdot) \[ \text{Z}(\cdot) \] cycle complex, 48

\[ \text{amp} \] amplitude, 69
\text{colim} \[ \text{colim}(\cdot) \] colimit, 88
\text{inf} \[ \text{inf}(\cdot) \] infimum, 69
\text{lim} \[ \text{lim}(\cdot) \] limit, 100
\text{rank} \[ \text{rank}(\cdot) \] rank of free module, 18
\text{sup} \[ \text{sup}(\cdot) \] supremum, 69
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