Some Review Problems

1. Find the position vector $\vec{R}(t)$ and acceleration vector $\vec{A}(t)$ given the velocity of a moving particle $\vec{V}(t) = \vec{i} + 4\pi\cos(2\pi t)\vec{k}$ and initial position $\vec{R}(2) = \vec{j}$.

2. Find the unit tangent vector $\vec{T}(t)$ and principal unit normal vector $\vec{N}(t)$ for the curve given by $\vec{R}(t) = 3\sin(4t)\vec{i} - 2\vec{j} - 3\cos(4t)\vec{k}$.

3. (a) Find the curvature and radius of curvature of the plane curve given by the equation $y = x + e^{x-2}$ at the point $P_0(x_0, y_0)$ on this curve where $x_0 = 2$.

   (b) Use the cross derivative formula to find the curvature of the curve $\vec{R}(t) = (1-t)\vec{i} - t^2\vec{j} + t\vec{k}$.

4. Find the tangential component of acceleration A_T and normal component of acceleration A_N of an object's acceleration $\vec{A}(t)$ if the position vector is $\vec{R}(t) = e^{-t}\vec{i} - e^{t}\vec{k}$. Evaluate A_T and A_N at $t = 0$.

5. Find the limit: $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^4 - 16y^4}{x^2 - 4y^2}$. Show the limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$.

6. (a) Determine the first order partial derivatives f_x and f_y and the second order partial derivatives f_{xx}, f_{xy}, and f_{yy} of the function $f(x,y) = e^{xy}\cos(y)$.

   (b) Find the total differential of this function $f(x,y)$ at the point (1,0).

   (c) Approximate the increment $\Delta f(1.01, -0.02)$.

7. Write the equation for the tangent plane to the surface given by the function $z = f(x,y) = \dfrac{4}{x^2 + y^2 - 6}$ at the point $P_0(x_0, y_0, z_0)$ on this surface with $x_0 = 1, y_0 = -1$.

8. (a) Apply the chain rule for two independent parameters to find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$, where

$$z = \tan\left(\frac{y}{x}\right) \quad \text{and} \quad x = uv, \quad y = \frac{u}{v}$$

   (b) Assume z is an implicit function of x and y, $z = f(x,y)$. If $x^2 - 2yz + z^3 = 10$ use implicit differentiation and the chain rule to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answers:

1. $\vec{R}'(t) = (t-2)\vec{i} + \vec{j} + 2\sin(2\pi t)\vec{k}$, $\vec{A}(t) = -8\pi^2\sin(2\pi t)\vec{k}$
2. $\vec{T}(t) = \cos(4t)\vec{i} + \sin(4t)\vec{k}$, $\vec{N}(t) = -\sin(4t)\vec{i} + \cos(4t)\vec{k}$
3. (a) $\kappa = 1/5^{3/2}$

   (b) $\kappa = \dfrac{\sqrt{8}}{(2+4t^2)^{3/2}}$
4. $A_T = \dfrac{-e^{-2t} + e^{2t}}{\sqrt{e^{-2t} + e^{2t}}}$, $A_N = \dfrac{2}{\sqrt{e^{-2t} + e^{2t}}}$ At $t=0$, $A_T = 0$, $A_N = \sqrt{2}$
5. Limit $= 8$, Limit does not exist since for $y = mx$ $L = \dfrac{3m}{1+m^2}$, for different m, the Limit is not unique.
6. (a) $f_x = ye^{xy}\cos(y)$, $f_y = xe^{xy}\cos(y) - e^{xy}\sin(y)$, $f_{xx} = y^2e^{xy}\cos(y)$, $f_{xy} = e^{xy}(\cos(y) + xy\cos(y) + y\sin(y))$

   (b) $df = ye^{xy}\cos(y)dx + [xe^{xy}\cos(y) - e^{xy}\sin(y)]dy \Big|_{\substack{x=1\\y=0}} = dy$ $\qquad f_{yy} = e^{xy}(x^2\cos(y) - 2x\sin(y) - \cos(y))$

   (c) $\Delta f \approx df = f_x\Delta x + f_y\Delta y = \Delta y = -0.02$
7. $z + 1 = -\frac{1}{2}(x-1) + \frac{1}{2}(y+1)$
8. (a) $z_u = -\frac{y}{x^2}\sec^2\left(\frac{y}{x}\right)v + \frac{1}{x}\sec^2\left(\frac{y}{x}\right)\frac{1}{v}$, $z_v = -\frac{y}{x^2}\sec^2\left(\frac{y}{x}\right)u - \frac{1}{x}\sec^2\left(\frac{y}{x}\right)\left(\frac{u}{v^2}\right)$

   (b) $z_x = \dfrac{2x}{2y - 3z^2}$, $z_y = \dfrac{2z}{3z^2 - 2y}$

Review for Exam # 1, Math 2350-H02
Chapters 10.1-10.5, 11.1-11.5

Review: Homework and Webwork Problems.
Topics: Vector-valued functions and curves in space: $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$. Dot product is a scalar: $\vec{F}(t)\cdot\vec{G}(t)$. Cross product is a vector: $\vec{F}(t)\times\vec{G}(t)$, $\lim_{t\to t_0}\vec{F}(t)$, $\lim_{t\to t_0}\vec{F}(t)\times\vec{G}(t)$, other limits.

$$\text{Position vector}: \ \vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}. \tag{1}$$

$$\text{Velocity vector}: \ \vec{V}(t) = \vec{R}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.$$

$$\text{Acceleration vector}: \vec{A}(t) = \vec{R}''(t) = x''(t)\vec{i} + y''(t)\vec{j} + z''(t)\vec{k}.$$

Speed: $\|\vec{V}(t)\| = \dfrac{ds}{dt}$, Direction of motion: $\dfrac{V(t)}{\|V(t)\|}$. If $\vec{V}(t_0) \neq \vec{0}$, then $\vec{V}(t_0)$ is tangent to the graph of $\vec{R}(t)$ at $t = t_0$.

$$\int \vec{F}(t)\,dt = \left(\int f_1(t)\,dt + c_1\right)\vec{i} + \left(\int f_2(t)\,dt + c_2\right)\vec{j} + \left(\int f_3(t)\,dt + c_3\right)\vec{k}.$$

Motion of a projectile: $x(t) = (v_0\cos(\alpha))t$, $y(t) = -\dfrac{1}{2}gt^2 + (v_0\sin(\alpha))t + s_0$, where v_0 = initial speed, α = angle of elevation, s_0 = initial height, and g = acceleration due to gravity, e.g., 32 f/s^2 or 9.8 m/s^2.
Time of flight: T_f when $y = 0$. Range of flight: $R_f = v_0\cos(\alpha)T_f$.
Unit Tangent vector: $\vec{T}(t) = \dfrac{\vec{R}'(t)}{\|\vec{R}'(t)\|}$. Principal Unit Normal vector: $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$.
Given the position vector $\vec{R}(t)$ in (1), the arclength from t_0 to t is $s(t) = \displaystyle\int_{t_0}^{t}\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,dt$.
Curvature: $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\| = \dfrac{\|\vec{T}'(t)\|}{\|\vec{R}'(t)\|} = \dfrac{\|\vec{R}'(t)\times\vec{R}''(t)\|}{\|\vec{R}'(t)\|^3}$. Plane curve: $\kappa = \dfrac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$.
Radius of curvature, Center of curvature. What is the curvature of a circle of radius r? What type of curve has a curvature $\kappa = 0$?
Tangent and Normal components of Acceleration: $\vec{A}(t) = A_T\vec{T} + A_N\vec{N}$,

$$A_T = \frac{d^2s}{dt^2} = \frac{\vec{R}'(t)\cdot\vec{R}''(t)}{\|\vec{R}'(t)\|}, \quad A_N = \kappa\left(\frac{ds}{dt}\right)^2 = \frac{\|\vec{R}'(t)\times\vec{R}''(t)\|}{\|\vec{R}'(t)\|}$$

$$\|\vec{A}(t)\|^2 = A_T^2 + A_N^2$$

Give an example of a curve with $A_T = 0$. An example with $A_N = 0$.
Functions of several variables: $z = f(x,y)$ = surface in space $\mathbf{R}^3$; $w = g(x,y,z)$ =hypersurface in $\mathbf{R}^4$. $f(x,y)$ = constant = level curve in the x-y plane. $g(x,y,z)$ = constant = level surface in space. Sketch level curves and level surfaces, Domain and Range of functions. Limits: $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$. Show a limit does not exist. Determine values (x,y) in the plane where $f(x,y)$ is continuous.
Partial differentiation: $\dfrac{\partial f}{\partial x} = f_x$, $\dfrac{\partial f}{\partial y} = f_y$, Higher order derivatives, $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, $f_{xy} = \dfrac{\partial^2 f}{\partial y\partial x}$, $f_{yx} = \dfrac{\partial^2 f}{\partial x\partial y}$, etc.
Tangent plane to $z = f(x,y)$ at a point $P_0(x_0,y_0,z_0)$: $z - z_0 = f_x(x-x_0) + f_y(y-y_0)$, where the partials are evaluated at (x_0,y_0); Incremental Approximation at $P(x_0,y_0)$: $\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y) \approx f_x\Delta x + f_y\Delta y$, where the partials are evaluated at (x_0,y_0); Total differential for $f(x,y,z)$: $df = f_xdx + f_ydy + f_zdz$; Differentiability of a function f, differentiability implies continuity.
Chain rule for $z = f(x,y)$, $x \equiv x(t)$, $y \equiv y(t)$ $z \equiv f(x(t),y(t))$: $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$.
For $z = f(x,y)$, $x \equiv x(u,v)$, $y \equiv y(u,v)$, $z = f(x(u,v),y(u,v))$: $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$, $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$.
Implicit differentiation: $F(x,y(x)) = c : F_x + F_y\dfrac{dy}{dx} = 0$ which means $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.
Higher order differentiation with implicit differentiation and the chain rule.

Additional Problems: Chapter 10: Practice Problems p. 680 # 24-30, Supplementary Problems pp. 680-682, # 3,11,17, 25,37,41,56, 61. Chapter 11: Practice Problems pp. 770-771 # 31, 32, 34, 35, 36. Supplementary Problems p. 771-773 # 1, 5, 6, 11, 15, 17, 19, 21, 23, 26, 31.