DUHAMEL'S PRINCIPLE FOR THE WAVE EQUATION
HEAT EQUATION WITH EXPONENTIAL GROWTH or DECAY
COOLING OF A SPHERE
DIFFUSION IN A DISK
SUMMARY of PDEs

MATH 4354
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Duhamel's Principle for the Wave Equation Takes the Source in the PDE and moves it to the Initial Velocity.

Suppose there is a force $f(x,t)$ in the PDE for the wave equation.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + f(x,t), \quad 0 < x < L, \ t > 0 \\ u(x,0) &= 0 = u_t(x,0), \quad 0 < x < L \\ u(0,t) &= 0 = u(L,t), \ t > 0. \end{aligned}$$

First, move the force to the initial velocity. The new IBVP is

$$\begin{aligned} w_{tt} &= c^2 w_{xx}, \quad 0 < x < L, \ t > 0 \\ w(x,0) &= 0, \quad w_t(x,0) = f(x,\tau), \quad 0 < x < L \\ w(0,t) &= 0 = w(L,t), \ t > 0. \end{aligned}$$

Second, the solution to $u(x,t)$ is given by

$$\boxed{u(x,t) = \int_0^t w(x, t-\tau)d\tau}$$

Note that u must have zero ICs:

$u(x,0) = 0$

$u_t(x,t) = w(x,0) + \int_0^t w_t(x, t-\tau)d\tau$

so that $u_t(x,0) = 0$.

An Example of Duhamel's Principle for the Wave Equation (Similar to Example 5).

Solve the wave equation with a source

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + \sin(3x), \quad 0 < x < \pi, \; t > 0 \\ u(x,0) &= 0 = u_t(x,0) \\ u(0,t) &= 0 = u(\pi,t) \end{aligned}$$

We put the source in the initial velocity

$$\begin{aligned} w_{tt} &= c^2 w_{xx}, \quad 0 < x < \pi, \; t > 0 \\ w(x,0) &= 0, \quad w_t(x,0) = \sin(3x), \quad 0 < x < \pi \\ w(0,t) &= 0 = w(\pi,t), \; t > 0. \end{aligned}$$

Separating variables $w(x,t) = y(x)g(t)$ we obtain

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0 = y(\pi).$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2 \quad \text{and} \quad y_n(x) = \sin(nx),$$

$n = 1, 2, \ldots$. The ODE for $g(t)$ is

$$g''(t) + c^2 \lambda_n g(t) = 0.$$

Hence, $g_n(t) = a_n \cos(cnt) + b_n \sin(cnt)$.

$$w(x,t) = \sum_{n=1}^{\infty} [a_n \cos(cnt) + b_n \sin(cnt)] \sin(nx)$$

$$w(x,t) = \sum_{n=1}^{\infty} [a_n \cos(cnt) + b_n \sin(cnt)] \sin(nx)$$

Applying the ICs:

$$w(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = 0.$$

For all n, $a_n = 0$.

$$w_t(x,0) = \sum_{n=1}^{\infty} b_n cn \sin(nx) = \sin(3x)$$

$b_3 c3 = 1$ **so** $b_3 = \dfrac{1}{3c}$. **All other** b_n **are zero.**

$$\boxed{w(x,t) = \frac{1}{3c} \sin(3ct) \sin(3x)}$$

Finally, the solution to $u(x,t)$ with the force in the PDE is

$$
\begin{aligned}
u(x,t) &= \int_0^t w(x, t-\tau)d\tau \\
&= \frac{1}{3c}\int_0^t \sin(3c(t-\tau))\sin(3x)d\tau \\
&= \frac{\sin(3x)}{3c}\int_0^t \sin(3c(t-\tau))d\tau \\
&= \frac{\sin(3x)}{3c}\frac{\cos(3c(t-\tau)}{3c}\Bigg|_0^t \\
u(x,t) &= \frac{\sin(3x)}{9c^2}[1-\cos(3ct)]
\end{aligned}
$$

The Heat Equation with Exponential Growth or Decay Can be Easily Solved by a Transformation.

Suppose there is exponential decay in the heat equation:

$$\boxed{u_t = k u_{xx} - r u}$$

Find the solution to $w(x,t)$, where $w_t = k w_{xx}$, then the answer to $u(x,t) = e^{-rt} w(x,t)$. Why? (See Chapter 1.3 # 4)

$$u_t = -r e^{-rt} w + e^{-rt} w_t = e^{-rt}[-rw + w_t].$$

$$u_{xx} = e^{-rt} w_{xx}$$

So

$$u_t - k u_{xx} + r u = e^{-rt}[-rw + w_t - k w_{xx} + rw] = 0$$

An Example with Exponential Decay (Similar to Example # 3)

Suppose the solution to the heat equation $u_t = ku_{xx}$ on $[0, L]$ is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-kn^2 t} \sin(nx).$$

Then the solution to $u_t = ku_{xx} - ru$ on $[0, L]$ is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-rt-kn^2 t} \sin(nx).$$

Cooling of A Sphere in the Special Case of Spherical Symmetry, $u(\rho, t)$, is Similar to Newton's Law of Cooling.

Suppose the solution to the heat equation on a sphere is spherically symmetric, $0 < \rho < \pi$,

$$u(\rho, \theta, \phi, t) = u(\rho, t)$$

Then $u_t = k\Delta u$ can be written as

$$u_t = k\left(u_{\rho\rho} + \frac{2}{\rho}u_\rho\right)$$

Suppose the temperature in the interior of the sphere is initially constant, IC:

$$u(\rho, 0) = T_0$$

and the surrounding temperature is kept at zero temperature for all time, BC:

$$u(\pi, t) = 0.$$

If $T_0 > 0$, we would expect the sphere to cool to the surrounding temperature, because heat is lost to the surroundings. The PDE with the preceding IC and BC can be solved by separation of variables $u(\rho, t) = y(\rho)g(t)$, where with some work (see page 145), we obtain

$$y_n(\rho) = \frac{\sin(n\rho)}{\rho} \quad \text{and} \quad g_n(t) = c_n e^{-kn^2 t}$$

Hence the solution to the Heat Equation on the Sphere is

$$u(\rho, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 kt} \frac{\sin(n\rho)}{\rho}$$

Now the IC, $u(\rho, 0) = T_0$ can be applied to find c_n:

$$u(\rho, 0) = \sum_{n=1}^{\infty} c_n \frac{\sin(n\rho)}{\rho} = T_0$$

or

$$T_0 \rho = \sum_{n=1}^{\infty} c_n \sin(n\rho)$$

Applying Fourier Sine Series, we obtain for c_n

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} T_0 \rho \sin(n\rho) d\rho \\ &= (-1)^{n+1} \frac{2T_0}{n}. \end{aligned}$$

Finally, see Equation (4.44):

$$u(\rho, t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2T_0}{n} e^{-kn^2 t} \frac{\sin(n\rho)}{\rho}$$

4.4 Exercise 2: What happens if $T_0 = 37$ degrees Celsius and $k = 5.58$ inches-squared per hour? Plot the temperature at the center of the sphere $\rho = 0$. There is a problem at $\rho = 0$. Take

$$\lim_{\rho \to 0} \frac{\sin(n\rho)}{\rho} = \lim_{\rho \to 0} \frac{n\cos(n\rho)}{1} = n$$

Hence, at the center of the sphere:

$$u(0, t) = \sum_{n=1}^{\infty} (-1)^{n+1} 2T_0 e^{-kn^2 t}$$

The Heat Equation or Diffusion Equation in a Disk is Not as Simple as the Previous Problem.

Suppose we have the heat equation in a disk in the special case of radial symmetry: $u(r, \theta, t) = u(r, t)$. Then $u_t = k\Delta u$ is given by

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0 \leq r < R,\ t > 0.$$

BC:

$$u(R, t) = 0, \quad t > 0$$

IC:

$$u(r, 0) = f(r),\ 0 \leq r < R.$$

Separate variables $u(r,t) = y(r)g(t)$ to obtain two ODEs:

$$\boxed{r^2 y''(r) + r y'(r) = -\lambda r^2 y(r), \quad y(R) = 0, \quad y(0) \text{ bounded}}$$

$$\boxed{g'(t) + \lambda k g(t) = 0}$$

The solution for g is straightforward:

$$g(t) = c e^{-\lambda k t}$$

The solution for the eigenvalue problem y is not so simple.

The eigenvalue problem in y is known as **Bessel's equation.**

Bessel's Equation has two Linearly Independent Solutions, One is Bounded at $r = 0$ and the Other is Not Bounded.

$$\boxed{r^2 y''(r) + r y'(r) = -\lambda r^2 y(r), \quad y(R) = 0, \quad y(0) \text{ bounded}}$$

There are two types of solutions because Bessel's equation is second order in r, Bessel's function of the first kind of order zero (bounded) and Bessel's function of the second kind of order zero (unbounded), denoted as $J_0(r\sqrt{\lambda})$ and $Y_0(r\sqrt{\lambda})$, respectively.

$$y(r) = c_1 J_0(r\sqrt{\lambda}) + c_2 Y_0(r\sqrt{\lambda}).$$

But since Y_0 is not bounded at $r = 0$,

$$y(r) = c_1 J_0(r\sqrt{\lambda})$$

The Eigenfunction is a Bessel Function of the First Kind, J_0 and the Eigenvalues are the Zeros of J_0.

The eigenfunctions are the zeros of this Bessel function, $J_0(R\sqrt{\lambda}) = 0$. Let $J_0(z_n) = 0$, where $z_n = R\sqrt{\lambda_n}$. The eigenvalues are

$$\lambda_n = \left(\frac{z_n}{R}\right)^2, \quad n = 1, 2, \ldots.$$

There are an infinite number of eigenvalues $\lambda_n > 0$ such that $\lambda_n \to \infty$ as $n \to \infty$.

Thus, the solution $u(r,t) = \sum_{n=1}^{\infty} g_n(t) y_n(r)$

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n t} J_0\left(r\sqrt{\lambda_n}\right) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n t} J_0\left(\frac{z_n}{R} r\right)}$$

In Practice, the Zeros of the Bessel Functions can be Approximated and the Bessel function for a finite number of Terms used to Plot the Solution to the Heat Equation on a Disk.

Bessel functions are orthogonal with weighting function r:

$$\int_0^R J_0(z_n r/R) J_0(z_m r/R) r \, dr = 0, \quad n \neq m$$

Thus,

$$c_n = \frac{\int_0^R f(r) J_0(z_n r/R) r \, dr}{\int_0^R [J_0(z_n r/R)]^2 r \, dr}$$

See Maple Program. Let $R = 1$, $f(r) = 5r^3(1 - r)$, and $k = 0.25$.

Summary of PDEs

PDEs arise from physical principles, such as the advection equation, heat or diffusion equation, wave equation and Laplace's equation. The three second order PDEs, heat equation, wave equation, and Laplace's equation represent the three distinct types of second order PDEs: parabolic, hyperbolic, and elliptic. These PDEs can be solved by various methods, depending on the spatial domain, whether it is infinite or finite, and its geometry, such as a line, disk, sphere, rectangle.

We have applied various techniques to solve these PDEs. The most useful techniques generally change the PDE into an ODE.

Methods:

Separation of Variables=applies to a homogeneous PDE and homogeneous BCs, reduces a PDE in n variables to n ODEs.

Integral Transforms= applies when the x variable is $-\infty < x < \infty$, Fourier transform-transforms x variable or when the x variable is $0 < x < \infty$, Laplace transform-transforms the t variable. The PDE in two variables is transformed to a ODE.

Change of Coordinates= This method changes the original PDE to an ODE or else another PDE (an easier one) by changing the coordinates, e.g., $u_t + cu_x = 0$. Let $\xi = x - ct$.

Transformation of the Dependent Variable= Changes the unknown of the PDE to a new unknown that is easier to find, $u_t = ku_{xx} + ru$, $u(x,t) = e^{rt}w(x,t)$, where $w_t = kw_{xx}$.

There are other methods that we have not discussed, e.g., numerical methods (finite difference methods and finite element methods), eigenfunction expansions.

There are many other PDEs that we have not discussed, e.g., nonlinear PDEs and higher order PDEs.

Diffusion and logistic growth is an example of a nonlinear PDE:

$$u_t = D\Delta u + ru\left(1 - \frac{u}{K}\right)$$

Exam # 3 Wednesday, December 14, 1:30-4 p.m.

Office Hours:

Monday, December 12, 10-12

Wednesday, December 14, 10-12.