Math 5311 – A short introduction to function spaces

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For this course, the important thing to take away from these notes is the language: L^2 is the space of square-integrable functions, H^1 is the space of functions such that $u_x^2 + u^2$ is integrable, and so on. You need to know that L^2 and H^1 are vector spaces, and you should be able to verify when a function is, or is not, a member of L^2 and H^1. The discussion of Riemann and Lebesgue integrability theory is background, as is the discussion of the Lebesgue Dominated Convergence Theorem.

1 Continuity class

The simplest classification of functions is by continuity of their derivatives. The set $C^n(\Omega)$ contains those functions whose n-th derivatives are continuous everywhere on the set Ω. Right away, we can conclude that $C^n \subset C^{n-1}$: Any function whose n-th derivative is continuous must necessarily have a continuous $(n-1)$-th derivative, so any member of C^n is necessarily a member of C^{n-1}. Consequently, C^n is a subset of C^{n-1}.

2 Integrability class

Given a function $f(x)$ on an interval $\Omega \in \mathbb{R}$, does its definite integral $\int_\Omega f(x)\,dx$ exist?

First, let's clear up right away that this is *not* the same as asking whether we can actually *compute* the integral by finding an antiderivative (indefinite integral). There are many functions, for instance, e^{-x^2}, whose antiderivative can't be computed in a finite number of "elementary" operations such as exponentials, cosines, cube roots, and so on. It's not just that nobody's been smart enough to find the antiderivative of e^{-x^2}, it's that it can't be done, at least not as a finite number of elementary operations (it *is* possible to find a convergent infinite series for the integral of e^{-x^2}, but then that's not a finite number of operations!). Theorems on the impossibility of such calculations are part of Galois theory in abstract algebra, well beyond the scope of this course. In any event, that's not what we mean when wondering about existence: the *existence* of an integral and the *computability* of an integral are two different questions. As an analogy to arithmetic, $\sqrt{2}$ exists as a real number but it cannot be computed exactly in a finite number of operations on rational numbers.

What we mean by existence is this: is there a finite number, to be denoted by $I = \int_\Omega f(x)\,dx$, such that I measures the area under the curve represented by $f(x)$? Can we then define the integral $\int_\Omega f(x)\,dx$ as the limit of some sequence of operations on f so that it converges to the area under the curve? If so, then we say the integral $\int_\Omega f(x)\,dx$ exists; there are two requirements behind that statement: first, that the area is finite, and second, that we have some limiting process that, in principle, could compute it. In calculus, you learned that a definite integral can be defined by chopping up the interval into a bunch of subintervals, putting rectangles on each subinterval, and adding up the areas; take the limit as the subintervals become smaller, and you get the area. The rigorous treatment of this intuitive idea is called the Riemann integral.

In calculus courses, what you learned to think of as "integration" is really a method for short-cutting the limit process implied by the Riemann integral. The fundamental theorem of calculus says that the Riemann integral can be computed as a difference of antiderivatives, so integration becomes the game of finding antiderivatives for functions. The antiderivative game is often difficult, occasionally impossible, but it is not the same game as wondering whether the definite integral exists in the first place.

2.1 Riemann integrability

The space $\mathcal{R}(\Omega)$ is the set of all functions whose Riemann integrals exist on Ω. All functions that are continuous on Ω, or piecewise continuous with a finite number of bounded "jumps" on Ω are Riemann integrable on Ω. What functions aren't Riemann integrable? Functions that have a non-integrable singularity within Ω, for example,

$$\int_0^1 \frac{dx}{x}$$

are not Riemann integrable. Perverse, wildly discontinuous functions such as

$$\mu(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

are not Riemann integrable.

If most functions we meet are Riemann integrable, then why don't we just use the Riemann integral? The problem with the Riemann integral is subtle: a limit of a sequence of Riemann integrable functions is not necessarily Riemann integrable. The important consequence of that deficiency is that it is difficult to prove convergence of approximating sequences based on Riemann integrals.

2.2 Lebesgue integrability

In the early 20th century Henri Lebesgue developed a formulation of the integral that overcame the limitations of the Riemann theory of integration. The details of the Lebesgue theory aren't important to us at this point; what matters are the consequences.

So how do the integrals mentioned above fare in the Lebesgue theory? First of all, any Riemann integrable function is also Lebesgue integrable. Integrals such as $\int_0^1 dx/x$ also diverge in the Lebesgue theory. Many perverse functions such as $\mu(x)$ above can be integrated (the integral of $\mu(x)$ turns out to be zero). For "ordinary" functions like $\sin x$ and x^{-2} the Riemann and Lebesgue integrals are, for practical purposes, identical. So why do we care?

The importance of the Lebesgue integral is that it allows proof of the Dominated Convergence Theorem: a theorem establishing that, under very weak conditions, we can exchange the order of limits and integrals:

$$\lim_{n\to\infty} \int u_n \, dx = \int \lim_{n\to\infty} u_n \, dx.$$

As an example of why this matters, recall that the solution to a BVP can be computed as an integral involving the Green's function

$$u(x) = \int G(x, y) f(y) \, dy.$$

Now, suppose we have a sequence of approximations to the Green's function, $G_n(x, y)$. Suppose that G_n converges to G, i.e.,

$$\lim_{n\to\infty} G_n(x, y) = G(x, y).$$

Using G_n, we can form a sequence of functions

$$u_n(x) = \int G_n(x, y) f(y) \, dy,$$

but do we know that $\lim_{n\to\infty} u_n(x) = u(x)$? With the Riemann integral, no, we don't know. With the Lebesgue integral, yes, provided that $f(y)G_n(x, y)$ satisfies very simple, very weak conditions (which it nearly always will). If so, we invoke the Lebesgue DCT to show that

$$\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \int G_n(x, y) f(y) \, dy = \int \lim_{n\to\infty} G_n(x, y) f(y) \, dy = \int G(x, y) f(y) \, dy = u(x).$$

The Lebesgue theory lets us put convergence of approximations like this on a sound footing. Mathematicians used approximating sequences for centuries without being able to prove convergence, and most scientists and engineers today use them without knowing why they work. We don't need to actually use the Lebesgue theory in this course, but we will need to know that if we restrict ourselves to Lebesgue-integrable functions we can "usually" exchange limits and integrals, giving us guaranteed convergence of certain approximating sequences.

2.3 L^2 integrability

The space $L^2(\Omega)$ consists of all functions whose squares are Lebesgue integrable:

$$L^2(\Omega) = \left\{ f : \int_\Omega f(x)^2 \, dx < \infty \right\}.$$

I'll write this as L^2 when the discussion is independent of the region of integration, or when the region of integration can be inferred from a problem. You should check that L^2 is indeed a vector space.

Note that there are examples of Riemann integrable functions that are not in L^2, for example, $f(x) = x^{-1/2}$ is integrable on $[0, 1]$ but is not in $L^2([0, 1])$ because

$$\int_0^1 x^{-1} \, dx$$

does not exist. Conversely, there are functions in $L^2(\Omega)$ that are not Riemann integrable on Ω, for example, $f(x) = x^{-1}$: the integral

$$\int_1^\infty f(x)^2 \, dx$$

converges but

$$\int_1^\infty f(x) \, dx$$

does not. Neither $\mathcal{R}(\Omega) \subset L^2(\Omega)$ nor $L^2(\Omega) \subset \mathcal{R}(\Omega)$ is true.

3 Sobolev spaces: differentiability and integrability

Sobolev spaces are central to the advanced theory of PDE and the optimization of functionals. Problems such as

$$\min_u \int_\Omega (\nabla u)^2 + u \, dx$$

are naturally set in a Sobolev space.

A Sobolev space imposes both differentiability and integrability requirements on its members: to be a member of a Sobolev space, certain derivatives of a function must be integrable. In this course we'll work with $H^1(\Omega)$, defined as

$$H^1(\Omega) = \left\{ f : \int_\Omega \left[f_x^2 + f^2 \right] \, dx < \infty \right\}.$$

For the second term in the integral to exist, f must be in L^2, therefore, we know right away that $H^1 \subset L^2$.

3.1 Weak differentiability and Sobolev spaces

Is $|x| \in H^1([-1,1])$? At first glance the answer is no, because $|x|$ is not differentiable at zero. However, the mixture of differentiation and integration in the definition of H^1 has an interesting effect: it lets us work with functions whose derivatives exist only weakly. Recall that the derivative of $|x|$ is *weakly* equal to $\text{sgn}(x)$. If we use this weak derivative in the integral defining H^1, we get

$$\int_{-1}^{1} (|x|')^2 + |x|^2 \, dx = \int_{-1}^{1} \text{sgn}(x)^2 + x^2 \, dx = 2 + \frac{2}{3}.$$

The failure of $|x|'$ to exist at a single point doesn't harm its integrability. So surprisingly, it is *not* true that $H^1 \subset C^1$: there are members of H^1 whose derivatives are discontinuous.

This property of Sobolev spaces is essential in the theory of finite element methods, because it lets us work with functions that are only piecewise differentiable.

4 Exercises

1. Prove that $L^2(a,b)$ is a vector space
2. Prove that $H^1(a,b)$ is a vector space
3. Let $S = \left\{ f : f \in H^1(0,1) \wedge f(0) = 1 \wedge f(1) = 0 \right\}$. Prove that S is not a vector space.
4. Prove that $H^1(a,b) \subset C^0(a,b)$, in other words, all members of $H^1(a,b)$ are continuous on (a,b).
5. For each of the following functions, determine whether they are members of each space: H^1, L^2, C^0, C^1, $\mathcal{R}$:
   (a) $f(x) = \sigma(x) \sin x$ on $[-1,1]$
   (b) $f(x) = x^{-1}$ on $[1,\infty)$
   (c) $f(x) = e^x \delta(x)$ on $[-1,1]$
   (d) $f(x) = e^{-|x|}$ on $(-\infty,\infty)$
   (e) $f(x) = \begin{cases} x & x < 1 \\ \sin \pi x & x \geq 1 \end{cases}$ on $[0,2]$
6. Prove that $H_0^1(a,b)$ is a vector space
7. Give an example of a sequence of functions $f_n(x)$ for which $\lim_{n\to\infty} \int f_n(x)\,dx \neq \int \lim_{n\to\infty} f_n(x)\,dx$.
8. The L^2 inner product $\langle f,g \rangle_{L^2}$ on $[a,b]$ is defined by $\langle f,g \rangle_{L^2} = \int_a^b f(x)g(x)\,dx$. Prove that $\langle f,g \rangle_{L^2}$ necessarily exists when both f and g are in $L^2(a,b)$.