1 Math 4350 Lecture 6 -2/6/2020 Limit Theorems Continued

.

Theorem 1 *If (x_n) converge to x and for all n sufficiently large we have $x_n \geq 0$ then* $\lim x_n = x \geq 0$.

Proof. Suppose that $x < 0$. Then let $\epsilon = -x > 0$ and so the is an N such that for all $n \geq N$ we have $x_n \geq 0$ and

$$x - \epsilon < x_n < x + \epsilon = x + (-x) = 0$$

In particular, $x_N < 0$ which is a contradiction. ■

Corollary 2 *If (x_n) and (y_n) converge to x and y respectively and for all n sufficiently large $x_n \leq y_n$ then $x \leq y$*

Proof. We do this in class but it is pretty obvious. Just consider $z_n = y_n - x_n$ and use the previous theorem. ■

Remark 3 *What happens if we try to replace the inequalities above by strict inequalities?*

Theorem 4 *If (x_n) converges to x and for all n sufficiently large $x_n \in [a, b]$ then $x \in [a, b]$*

We prove this in class and you can find the proof in the book as well. Can you do it without looking?

A very useful theorem that you may remember from basic calculus class is the **Squeeze Theorem**:

Theorem 5 *Suppose we have sequences $(x_n), (y_n)$ and (z_n) and that*

$$x_n \leq y_n \leq z_n \text{ for all } n \in \mathbb{N}$$

Then if $\lim x_n = \lim z_n$ *we also have that* $\lim_{n\to\infty} y_n$ *exist and*

$$\lim x_n = \lim y_n = \lim z_n$$

Proof. Sketch: Let $w = \lim x_n = \lim y_n$.For large enough n we have

$$\begin{aligned} -\epsilon &< x_n - w < \epsilon \\ &\text{and} \\ -\epsilon &< x_n - w < \epsilon \end{aligned}$$

Now subtract w through the inequalities $x_n \leq y_n \leq z_n$ to get

$$\begin{aligned} x_n - w &\leq y_n - w \leq z_n - w \\ &\text{and so} \\ -\epsilon &< y_n - w < \epsilon \end{aligned}$$

Which is $|y_n - w| < \epsilon$ for all sufficiently large n. ■

Example 6 *For example since* $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$ *we easily get* $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Example 7 *Notice that* $\lim(a + b/n) = \lim a + \lim b/n = a + 0$ *so for example* $\lim_{n\to\infty} \frac{2n+1}{n+5} = 2$ *since* $\frac{2}{1} = \frac{\lim(2+1/n)}{\lim(1+5/n)} = \lim_{n\to\infty} \frac{2+1/n}{1+5/n} = \lim_{n\to\infty} \frac{2n+1}{n+5}$ *(we used the quotient theorem)*

Example 8 *A slightly trickier than the example above is the following:* $\lim_{n\to\infty} \left(\frac{2n}{n^2+1}\right) = ?$
Notice that although $\frac{2n}{n^2+1} = \frac{2n}{n+1/n}$ *this doesn't do us any good. Instead we try*

$$\frac{2n}{n^2+1} = \frac{2/n}{1+1/n^2}$$

then since $\lim 2/n = 0$ *and* $\lim\left(1 + 1/n^2\right) = 0$ *(why?) we have* $\lim_{n\to\infty} \left(\frac{2n}{n^2+1}\right) = \lim_{n\to\infty} \frac{2/n}{1+1/n^2} = \frac{\lim_{n\to\infty} 2/n}{\lim_{n\to\infty}(1+1/n^2)} = 0/1 = 0$
These inequalities give a more logical flow written in reverse order. That was we don't write equalities before we no the limits exist.

Example 9 *Later we show that if* $\lim x_n = x$ *and* $x_n \geq 0$ *for all* n *then* $\lim \sqrt{x_n} = \sqrt{x}$

Try this on your own. ***This may be on an exam!***

Exercise 10 *Use* $||x_n| - |x|| \leq |x_n - x|$ *to prove that if* $\lim_{n\to\infty} x_n = x$ *then* $\lim_{n\to\infty} |x_n| = |x|$ *(we assumed this result last lecture).*

Recall that we mentioned the following in class and just above. Did you try to prove it?

Theorem 11 *If* $\lim_{n\to\infty} x_n = x$ *and* $x_n \geq 0$ *for all (sufficiently large)* n *then* $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$

Proof. The case $x = 0$ is special. In this case we use the fact that for all sufficiently large n we have

$$0 \leq x_n = x_n - 0 < \epsilon^2$$

to conclude that for all such n we also have $0 \leq \sqrt{x_n} < \epsilon$ where we have used the proven fact that $\sqrt{}$ preserves order. Thus for all sufficiently large n we have $0 \leq \left|\sqrt{x_n} - 0\right| = \sqrt{x_n} < \epsilon$.

Now for the case $x > 0$ (why is < 0 impossible?) we have

$$\sqrt{x_n} - \sqrt{x} = \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}(\sqrt{x_n} - \sqrt{x}) = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

and $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$ so that $\left(\sqrt{x_n} + \sqrt{x}\right)^{-1} \leq (\sqrt{x})^{-1}$ giving

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{1}{\sqrt{x}}|x_n - x|$$

and the result follows easily (from 3.1.10 for example) ■

Theorem 12 *If (x_n) is a sequence of positive real numbers and $\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = L < 1$ then $\lim_{n\to\infty}(x_n) = 0$*

Proof. We know by now that $L \geq 0$. Let $\epsilon = r - L$ where $0 < r < L$. Now there is a $K > 0$ such that whenever $n \geq K$ we have

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon = r - L$$

and so $\frac{x_{n+1}}{x_n} < L + \epsilon = L + (r - L) = r$. So for $n \geq K$ we have

$$0 < x_{n+1} < x_n r < x_{n-1}r^2 < \cdots < x_K r^{n-K+1} \text{ (think about this!)}$$

If we set $C = x_K/r^K$ then $0 < x_{n+1} < Cr^{n+1}$. But since $0 < r < 1$ we know that $\lim_{n\to\infty} r^{n+1} = 0$ and the result follows by squeezing. ■

Example 13 *For example try $x_n = n/2^n$.*

2 Monotone Sequences

Definition 14 *A sequence of real numbers (a_n) is said to be **Increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Similarly, a sequence of real numbers (a_n) is said to be **Decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is said to be **Monotone** or **Monotonic** if it is either increasing or decreasing*

Theorem 15 (Monotone Convergence Theorem) *A monotone sequence of real numbers is convergent if and only if it is bounded. If (x_n) is bounded and increasing then $\lim_{n\to\infty} x_n = \sup\{x_n\}$ and if it is bounded and decreasing then $\lim_{n\to\infty} x_n = \inf\{x_n\}$.*

Consider the case of a increasing sequence (x_n). By assumption, $\{x_n\}$ is non-empty and bounded above. By the least-upper-bound property or completeness property of real numbers, $c = \sup\{x_n\}$ exists and is finite. Now, for every $\epsilon > 0$, there exists N such that $N > c - \epsilon$, since otherwise $c - \epsilon$ would be an upper bound of which contradicts to the definition of c. Then since (x_n) is increasing, and c is its upper bound, for every $n > N$, we have $|c - x_n| \leq |c - x_N| < \epsilon$

Thus $c = \sup\{x_n\}$ is the limit of the sequence.

The decreasing case is similar.