1 Math 4350 Lecture 5 -2/4/2020 Limit Theorems

We establish several basic facts about limits that help us greatly both in understanding the theory and in being able to prove news theorems and attack individual concrete limit problems more efficiently.

Online resources:

https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-series-optional/v/proving-a-sequence-converges

https://www.youtube.com/watch?v=RyWen5Zss7o

(We put the definition of a limit of a sequence on the board again for review)

Recall we have already seen one handy theorem:

Theorem 1 *(In the book 3.1.10) Suppose we have a sequence* (x_n) *and let* $x \in \mathbb{R}$ *(our potential limiting value). If there is a sequence of postive numbers* (a_n) *with* $\lim_{n\to\infty} a_n$ *and a* $C > 0$ *such that*

$$|x_n - x| \leq Ca_n$$

then we have $\lim_{n\to\infty} x_n = x$.

Example 2 *Since* $\left|\frac{1}{1+2n} - 0\right| = \frac{1}{1+2n} < \left(\frac{1}{2}\right)\frac{1}{n}$ *and* $\frac{1}{n} \to 0$ *we see that* $\lim_{n\to\infty} \frac{1}{1+2n} = 0$

Example 3 *In the book we use a similar but trickier argument to show that* $\lim_{n\to\infty} b^n = 0$ *for* $0 < b < 1$*. Check it out. Can you follow it? (It is important to get used to using the book!)*

Another very basic and important fact is that all convergent sequences are bounded:

Theorem 4 *If* (x_n) *converges then there is an* $M \in \mathbb{R}$ *such that* $|x_n| \leq M$ *for all* n

Can you find a "bound" M for the sequence $x_n = \frac{n^2}{n^2-1}$?

An idea familiar from basic calculus is that sequences can be added, subtracted, multiples and divided. We can also just scale a sequence by a fixed number.

1. $(x_n) \pm (y_n) = (x_n \pm y_n)$
2. $(x_n)(y_n) = (x_n y_n)$
3. $(x_n)/(y_n) = (x_n/y_n)$ as long as $y_n \neq 0$ for all $n \in \mathbb{N}$ (natural numbers)

4. $c(x_n) = (cx_n)$

For example, we can add the sequences $x_n = \frac{n^2}{n^2-1}$ and $y_n = (1/2)^n$ to get a new sequence $z_n = \frac{n^2}{n^2-1} + (1/2)^n$. Very obvious and simple. We have the following expected theorem:

Theorem 5 *If (x_n) and (y_n) converge to x and y respectively then $(x_n) \pm (y_n)$ converges to $x \pm y$ and $(x_n)(y_n) = (x_n y_n)$ converges to xy while also $c(x_n)$ converges to cx. Furthermore, if $(x_n)/(y_n) = (x_n/y_n)$ is defined (no zero denominators) then it converges to x/y as long as $y \neq 0$. Finally, $|x_n|$ converges to $|x|$.*

We can write this roughly and in more familiar form as

1. $\lim(x_n + y_n) = \lim x_n + \lim y_n$ as long as 2 of the 3 limits exist
2. $\lim(x_n - y_n) = \lim x_n - \lim y_n$ as long as 2 of the 3 limits exist
3. $\lim(x_n y_n) = (\lim x_n)(\lim y_n)$ as long as 2 of the 3 limits exist
4. $\lim(x_n/y_n) = \lim x_n / \lim y_n$ assuming no zero denominators and $\lim x_n$ and $\lim y_n$ both exist.
5. $\lim(cx_n) = c \lim x_n$ for any $c \in \mathbb{R}$ (limits respect scaling)
6. $\lim |x_n| = |x|$

Example 6 *Since $\lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2$ and $\lim_{n\to\infty} n^{1/n} = 1$ we have $\lim_{n\to\infty} \left(\frac{2n^2}{n^2-1} + n^{1/n}\right) = 2 + 1$, $\lim_{n\to\infty} \left(\frac{2n^2}{n^2-1} n^{1/n}\right) = 2 * 1$ etc.*

Note: Very important for our class is to know how these are proved! We will prove as an example the hardest case

Proof. We will prove that $\lim(x_n/y_n) = \lim x_n / \lim y_n$ assuming no zero denominators and $\lim x_n$ and $\lim y_n$ both exist. We will assume we have proven the other limit theorem whose proofs are easier (you should try). Let $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. In order to prove the result that $\lim(x_n/y_n) = x/y$ we will need to look at $|x_n/y_n - x/y| = \left|\frac{yx_n - xy_n}{yy_n}\right| = \left|\frac{yx_n - xy + xy - xy_n}{yy_n}\right| \leq \frac{1}{|yy_n|}(|yx_n - xy| + |xy - xy_n|)$. We need a bound on $\frac{1}{|yy_n|}$ Since we know by (5) above that $\lim_{n\to\infty} yy_n = y^2 > 0$ and hence $\lim_{n\to\infty} |yy_n| = y^2 > 0$ by (6) we can find K_1 such that for all $n > K_1$ we have

$$\left|y^2 - y_n y\right| < \frac{1}{2} y^2$$

which forces $\frac{1}{|yy_n|} < \frac{2}{y^2}$. To see this use the triangle inequality: $y^2 - |y_n y| \leq \left|y^2 - y_n y\right| < \frac{1}{2} y^2$ to get $\frac{1}{2} y^2 < |y_n y|$. So we have

$$\begin{aligned} |x_n/y_n - x/y| &\leq \frac{1}{|yy_n|}(|yx_n - xy| + |xy - xy_n|) \\ &\leq \frac{2}{y^2}(|yx_n - xy| + |xy - xy_n|) \end{aligned}$$

Now $yx_n \to xy$ and $xy \to xy_n$ so we can choose N_1 and N_2 big enough that for all $n > \max\{N_1, N_2\}$ we have

$$|yx_n - xy| < \frac{\epsilon y^2}{4} \text{ and } |xy - xy_n| < \frac{\epsilon y^2}{4}$$

which gives $|x_n/y_n - x/y| \leq \frac{2}{y^2}(|yx_n - xy| + |xy - xy_n|) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n > \max\{K_1, N_1, N_2\}$. Do you see why we need to include K_1? ■

Exercise 7 *Without looking at the book or any other resource, try to prove that if (x_n) and (y_n) converge to x and y respectively then $(x_n) + (y_n)$ converges to $x + y$.*

Exercise 8 *Prove that if (x_n) and (y_n) converge to x and y respectively then $(x_n y_n)$ converges to $x + y$. That is, $\lim(x_n y_n) = \lim x_n \lim y_n$.*

We proved this in class and you can find the proof in the book as well. Can you do it without looking?

In class we will prove the following seemingly obvious result. Can you prove it on your own.

Theorem 9 *If (x_n) and (y_n) converge to x and y respectively and for all n sufficiently large $x_n, y_n \in [a, b]$ then $x \in [a, b]$ and $y \in [a, b]$.*

We prove this in class and you can find the proof in the book as well. Can you do it without looking?

A very useful theorem that you may remember from basic calculus class is the **Squeeze Theorem**:

Theorem 10 *Suppose we have sequences $(x_n), (y_n)$ and (z_n) and that*

$$x_n \leq y_n \leq z_n \text{ for all } n \in \mathbb{N}$$

Then if $\lim x_n = \lim z_n$ then

$$\lim x_n = \lim y_n = \lim z_n$$

Proof. We prove this on the board in class. Can you do it? ■

Example 11 *For example since $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$ we easily get $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$*

Example 12 *Notice that $\lim(a + b/n) = \lim a + \lim b/n = 2 + 0$ so for example $\lim_{n\to\infty} \frac{2n+1}{n+5} = \lim_{n\to\infty} \frac{2+1/n}{1+5/n} = \frac{\lim(2+1/n)}{\lim(1+5/n)} = \frac{2}{1} = 2$*

Example 13 *A slightly trickier than the example above is the following:* $\lim_{n\to\infty}\left(\frac{2n}{n^2+1}\right) =$?
Notice that although $\frac{2n}{n^2+1} = \frac{2n}{n+1/n}$ *this doesn't do us any good. Instead we try*

$$\frac{2n}{n^2+1} = \frac{2/n}{1+1/n^2}$$

then since $\lim 2/n = 0$ *and* $\lim\left(1+1/n^2\right) = 0$ *(why?) we have* $\lim_{n\to\infty}\left(\frac{2n}{n^2+1}\right) = \lim_{n\to\infty}\frac{2/n}{1+1/n^2} = \frac{\lim_{n\to\infty}2/n}{\lim_{n\to\infty}(1+1/n^2)} = 0/1 = 0$

Example 14 *In class we show that if* $\lim x_n = x$ *and* $x_n \geq 0$ *for all* n *then* $\lim\sqrt{x_n} = \sqrt{x}$

Try this on your own. ***This may be on an exam!***