Numerical Analysis of PDE I, Spring 2024, HW#6.

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Problem #1. *Discrete maximum principle.* Let Ω be a polygonal domain in $\mathbb{R}^2$ and let $\mathcal{T}_h$ be an affine simplicial mesh of Ω. Assume that we use an approximation consisting of piecewise linear finite $\mathbb{P}_1$ elements. Let $\{\phi_i\}_{i\in\{1,N\}}$ be the global shape functions and let $\boldsymbol{K}$ be the stiffness matrix associated to the laplace operator, i.e. $\boldsymbol{K}_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j$ for $0 \leq i, j \leq N$. If we assume that all the angles of the triangles in $\mathcal{T}_h$ are acute, then, it can be proven that $\boldsymbol{K}_{ij} < 0$ for all $i \neq j$.

(a) Show that $\boldsymbol{K}$ is an M-matrix, i.e. all its off-diagonal entries are non-positive and its row-wise sums are non-negative. Show that $\sum_{j=1}^{N} \boldsymbol{K}_{ij} = 0$

(b) Prove the following discrete maximum principle: if $f \in L^2(\Omega)$ is such that $f \leq 0$ in Ω, the finite element solution u_h to the homogeneous Dirichlet problem with right hand side f is such that $u_h \leq 0$ in Ω.
Hint: you will have to use the fact that $\sum_{j=1}^{N} \boldsymbol{K}_{ij} = 0$ in order to develop a contradiction argument.

Note: preservation of maximum and minimum principles are not "variational" properties of the scheme. By this I mean that they cannot be deduced by "staring" at the bilinear form $a(u,v) = f(u)$, using coercivity properties, using galerkin orthogonality, or any other Hilbert-space concept. Strictly speaking, max/min principles are purely algebraic property of the method. You have to inspect the matrix and vector that result from the scheme in order to prove them.

Problem #2. Let $\mathbb{V} = \big\{ v \in H^2(0,1) \;:\; v(1) = 0 \big\}$. Consider the following variational (weak) formulation: Find $u \in V$ s. t.

$$a(u,\varphi) = l(\varphi) \quad \forall \varphi \in \mathbb{V}, \qquad \text{where}$$

$$a(u,\varphi) \;=\; \int_0^1 (u''\varphi'' + u\varphi)\,\mathrm{d}x + u'(1)v'(1), \qquad l(\varphi) \;=\; \int_0^1 f\varphi\,\mathrm{d}x \;+\; \beta_0\varphi(0)$$

(a) What is the correct norm for the function space $H^2(0,1)$?

(b) Derive the strong form of this problem

(c) Show that $a(u,\varphi)$ is a bounded and elliptic symmetric bilinear form on $\mathbb{V}$.

(d) Show that $l(\varphi)$ is a bounded linear function on $\mathbb{V}$.

(e) With the results of (c) and (d) give a short argument that the weak formulation of the problem admits a unique solution.

(d) Is it possible to use continuous (only) finite element basis for this problem? If not, what is the main obstacle?

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Problem #3. *Nitsche-style implementation of Dirichlet boundary conditions.* Consider the discretization of the Dirichlet problem

$$-\Delta u = f \ \in \ \Omega \text{ and } u = g \in H^{\frac{1}{2}}(\partial\Omega) \tag{1}$$

As discussed in Homework #3 Problem #5, after the assembly of the matrix and right hand side vector corresponding to problem (1) we have to carry out some algebraic manipulations in order enforce Dirichlet boundary conditions for this and related problems. This is somewhat undesirable. Nitsche (1971) proposed the following bilinear form:

$$(\nabla u, \nabla v)_\Omega - (\nabla u \cdot \boldsymbol{n}, v)_{\partial\Omega} - (\nabla v \cdot \boldsymbol{n}, u)_{\partial\Omega} + \eta(u, v)_{\partial\Omega} = (f, v)_\Omega - (\nabla v \cdot \boldsymbol{n}, g)_{\partial\Omega} + \eta(v, g)_{\partial\Omega} \tag{2}$$

where $\eta > 0$ is a large parameter. The variational formulation (2) can be proven to be formally consistent with problem (1). The bilinear form associated to (2) is clearly symmetric, but it's not coercive in the infinite dimensional setting. However, if we consider its finite dimensional counterpart

$$\begin{aligned} &(\nabla u_h, \nabla v_h)_\Omega - (\nabla u_h \cdot \boldsymbol{n}, v_h)_{\partial\Omega} - (\nabla v_h \cdot \boldsymbol{n}, u_h)_{\partial\Omega} + \eta(u_h, v_h)_{\partial\Omega} = \\ &\quad (f, v_h)_\Omega - (\nabla v_h \cdot \boldsymbol{n}, g)_{\partial\Omega} + \eta(v_h, g)_{\partial\Omega} \end{aligned} \tag{3}$$

the corresponding bilinear form can be proven to be coercive if $\eta = \frac{\kappa}{h}$ with $\kappa = \mathcal{O}(1)$ sufficiently large. Prove that indeed the bilinear form of (3) can be made coercive.

Hint: Start by setting $v_h = u_h$ in (3): then you get that

$$a(u_h, u_h) = \|\nabla u_h\|^2_{L^2(\Omega)} - 2(\nabla u_h \cdot \boldsymbol{n}, u_h)_{\partial\Omega} + \eta\|u_h\|^2_{L^2(\partial\Omega)}$$

Clearly the term $-(\nabla u_h \cdot \boldsymbol{n}, u_h)_{\partial\Omega}$ is unsigned, thefore it has to be absorbed by the other two terms which are guaranteed to be positive. Decompose this integral on the boundary as contributions from all the faces $F \subset \partial\Omega$. For each one of these contributions you will have to use C-S inequality $|-(\nabla u_h \cdot \boldsymbol{n}, u_h)_F| \geq -\|\nabla u_h \cdot \boldsymbol{n}\|_{L^2(F)}\|u_h\|_{L^2(F)}$. One of the tricks that saves the day is using trace inequalities of the form $h_K^{\frac{1}{2}}\|w_h\|_{L^2(F)} \leq c_K\|w_h\|_{L^2(K)}$. Do not prove this trace inequality: just use it.

Note: the Nitsche implementation of Dirichlet boundary conditions is widely used in practice. It is a standard technique usually quite frequently used in the context of interior penalty methods.
