Numerical Analysis of PDE I, Spring 2024, HW#5.

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Problem #1. *Applications of Deny-Lions.* Prove the following Poincare-like inequalities invoking the Deny-Lions lemma:

$$\|u\|_{H^1(\Omega)} \leq c_p\big(\|\nabla u\|_{L^2(\Omega)} + |\textstyle\int_{\partial\Omega} u \,\mathrm{d}\boldsymbol{s}|\big)$$
$$\|u\|_{H^1(\Omega)} \leq c_p\big(\|\nabla u\|_{L^2(\Omega)} + |\textstyle\int_{\Omega} u \,\mathrm{d}\boldsymbol{x}|\big)$$
$$\|u\|_{H^1(\Omega)} \leq c_p\big(\|\nabla u\|_{L^2(\Omega)} + |\textstyle\int_{\Omega_0} u \,\mathrm{d}\boldsymbol{x}|\big)$$

where Ω_0 is any subset of Ω of positive measure.

Problem #2. *Proper mapping for $H(div,\Omega)$ functions.* Consider the scalar-valued function $v(\boldsymbol{x})$ that is related to the function $\widehat{v}(\widehat{\boldsymbol{x}})$ by the relationship $v(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})) = \widehat{v}(\widehat{\boldsymbol{x}})$, where $\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) : \widehat{K} \to K$. Assume that the mapping is affine, that is $\boldsymbol{T}_K(\widehat{\boldsymbol{x}})) = A_K\widehat{\boldsymbol{x}} + b_K$ and that $\det A_K > 0$. Then we have that $\boldsymbol{x} = \boldsymbol{T}_K(\widehat{\boldsymbol{x}})$ and $\nabla_{\widehat{\boldsymbol{x}}}\boldsymbol{x} = A_K$. The relationship $v(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})) = \widehat{v}(\widehat{\boldsymbol{x}})$ is often called the pullback map, and preserves a few important properties. In particular, if $q(\boldsymbol{x})$ vanishes on the boundary of K, then $\widehat{q}(\widehat{\boldsymbol{x}}) := q(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))$ vanishes on the boundary of $\widehat{K}$ (and converse). However, we might be interested in mappings that preserve other important properties. Let $\boldsymbol{v}(\boldsymbol{x}) : \mathbb{R}^d \to \mathbb{R}^d$ be a vector-valued function. Consider the mapping for vector-valued functions defined by $\boldsymbol{v}(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})) = \frac{1}{\det A_K} A_K \widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})$, which is known as the contravariant Piola transform. Then:

1. Using the chain rule show that $\nabla_{\boldsymbol{x}} v = A_K^{-\top}\nabla_{\widehat{\boldsymbol{x}}}\widehat{v}$.
2. Show that $\operatorname{div}\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{\det A_K}\widehat{\operatorname{div}}\,\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})$, where $\widehat{\operatorname{div}}$ is the divergence with respect to $\widehat{\boldsymbol{x}}$.
   Hint: do not try to prove this identity as a sheer brute-force chain-rule rule exercise. Instead consider proving that $\int_K \operatorname{div}\boldsymbol{v}(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\widehat{K}} \widehat{\operatorname{div}}\,\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})q(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\,\mathrm{d}\widehat{\boldsymbol{x}}$ for all $q(\boldsymbol{x}) \in \mathcal{C}_0^\infty(K)$. Note that $\boldsymbol{v}(\boldsymbol{x})$ is mapped using the contravariant map, while $q(\boldsymbol{x})$ is mapped using the pullback transform. You will have to use the result of Part 1.
3. Multiply both sides of the identity $\operatorname{div}\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{\det A_K}\widehat{\operatorname{div}}\,\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})$ by the measure of volume $\mathrm{d}\boldsymbol{x}$ and integrate in K: What do you get?

Note. The contravariant Piola-transform is important for the implementation of $H(\operatorname{div},\Omega)$ finite elements. The contravariant transform does the right job even if $\boldsymbol{T}_K(\widehat{\boldsymbol{x}})$ is non-affine. On the other hand, the pullback map cannot be used to map div-conformig elements since it does not preserve the divergence or normal components of $\boldsymbol{v}(\boldsymbol{x})$.

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Problem #3. *Condition numbers.* Let $\mathbf{A} \in \mathbb{R}^{N\times N}$ be nonsingular symmetric matrix. The condition number of $\mathbf{A}$ in the 2-norm is $\kappa(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2$. This number dictates the performance iterative solvers. Show that:

(a) If $\mathbf{A}$ is symmetric positive definite, then $\kappa(\mathbf{A}) = \frac{\lambda_{\max}}{\lambda_{\min}}$, where $\lambda_{\max}$ (resp $\lambda_{\min}$) is the largest (resp smallest) eigenvalue of $\mathbf{A}$.

(b) The condition number of the mass matrix $\mathbf{M} \in \mathbb{R}^{N\times N}$ with entries given by $\mathbf{M}_{ij} = \int_\Omega \phi_i(\boldsymbol{x})\phi_j(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ over a quasi-uniform triangulation $\mathcal{T}_h$ of size h satisfies $\kappa(\mathbf{M}) = \mathcal{O}(1)$.
Hint: use the results of Problem #3 from Homework #4.

(c) The condition number of the stiffness matrix $\mathbf{K}$ with entries $\mathbf{K}_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \mathrm{d}\boldsymbol{x}$ for the case of vanishing Dirichlet boundary conditions over a quasi-uniform triangulation $\mathcal{T}_h$ of size h satisfies $\kappa(\mathbf{K}) = \mathcal{O}(h^{-2}) = \mathcal{O}(N^{\frac{2}{d}})$ where N is the total number of degree of freedom.
Hint: use the results of Problem #3 from Homework #4.

Problem #4. *Consistency of quadrature under lumping.* Let $a(u, v) = (c\nabla u, \nabla v)_{L^2(\Omega)}$, where $c(\boldsymbol{x}) \in \mathcal{C}^2(\Omega)$ and $F(v) = (f, v)_{L^2(\Omega)}$ denote the "exact" bilinear form and right hand side functional. Consider the three-point quadrature rule

$$q_K(g) = \tfrac{1}{3}|K| \sum_{i=1}^{3} g(\boldsymbol{x}_i)$$

where $g \in \mathcal{C}^0(K)$ and $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ are the vertex coordinates of the triangle K, in order to approximate the integral $\int_K g \,\mathrm{d}\boldsymbol{x}$. Indeed, this quadrature formula is a generalization of the trapezoidal rule. Using similar techniques to those of problem #5 in homework #4 we can easily prove that:

$$|q_K(g) - \textstyle\int_K g \,\mathrm{d}\boldsymbol{x}| \leq ch^2 |g|_{W^{2,1}(K)} \qquad \textbf{(*)}$$

Consider the functionals $a_{h,K}(u, v)$ and $F_{h,K}(v)$ defined in each element as

$$a_{h,K}(u, v) := q_K(uv) \text{ and } F_{h,K}(v) := q_K(fv)$$

such that $a_h(u, v) = \sum_{K\in\mathcal{T}_h} a_{h,K}(u, v)$ and $F_h(v) = \sum_{K\in\mathcal{T}_h} F_{h,K}(v)$. Let $\mathbb{V}_h$ be defined by

$$\mathbb{V}_h = \big\{ v \in \mathcal{C}(\Omega) \,|\, v|_K \in \mathbb{P}_1(K) \; \forall K \in \mathcal{T}_h \big\}$$

Prove that:

$$
\begin{aligned}
&c_1 |v_h|_{H^1(\Omega)} \leq a_h(v_h, v_h) \leq c_2 |v_h|_{H^1(\Omega)} \\
&|a_h(v_h, w_h) - a(v_h, w_h)| \leq ch^2 \|a\|_{\mathcal{C}^2(\Omega)} |v_h|_{H^1(\Omega)} |w_h|_{H^1(\Omega)} \\
&|F_h(v_h) - F(v_h)| \leq ch^2 \|f\|_{H^2(\Omega)} |v_h|_{H^1(\Omega)}
\end{aligned}
$$

for all $v_h, w_h \in \mathbb{V}_h$, where the constants are independent or v_h and w_h.
Hint. First important observation is that the functional $q_K(g)$ is exact for all $g \in \mathbb{P}_1(K)$. The second important observation is that if $v_h|_K \in \mathbb{P}_1(K)$ therefore $\nabla v_h|_K \in [\mathbb{P}_0(K)]^d$. The proofs of the second and third inequalites will require using the error estimate ***(*)****. At some point of the proof you will have to use the fact that the elementwise hessian of functions in the space $\mathbb{V}_h$ is zero.*
