Numerical Analysis of PDE I, Spring 2024, HW#4.

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Problem #1. Consider the Neumann problem:

$$-\Delta u = f \text{ in } \Omega, \text{ with } \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

(a) Assume that $f \in L^2(\Omega)$ and show that the condition

$$\int_\Omega f \mathrm{d}\boldsymbol{x} = 0$$

is necessary for the existence of a solution (because of the choice of boundary conditions).

(b) Note that if u solves the Neumann problem so does $w = u + c$ with c being an arbitrary constant. To obtain uniqueness we add the extra condition

$$\int_\Omega u \mathrm{d}\boldsymbol{x} = 0$$

requiring the mean value of u to be zero. Give this problem a variational formulation using the space

$$\mathbb{V} = \big\{ v \in H^1(\Omega) \,\big|\, \textstyle\int_\Omega v \mathrm{d}\boldsymbol{x} = 0 \big\}$$

(c) Show that if the weak solution $u \in \mathbb{V}$ is actually sufficiently regular then it solves

$$-\Delta u = f - \int_\Omega f \mathrm{d}\boldsymbol{x} \text{ in with } \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Problem #2. *Raviart-Thomas element of lowest order.* This problem illustrates how to design finite elements for the space $\mathbf{H}(\mathrm{div}, \Omega)$ where Ω is a polygonal domain in $\mathbb{R}^2$.

(a) $\mathbf{H}(\mathrm{div}, \Omega)$ is the Sobolev space of vector-valued functions $\boldsymbol{v}(\boldsymbol{x}) \in \mathbb{R}^d$ satisfying the following definition:

$$\mathbf{H}(\mathrm{div}, \Omega) = \big\{ \boldsymbol{v} \in [L^2(\Omega)]^d \,\big|\, \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega) \text{ and } \textstyle\int_\Omega \boldsymbol{v} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x} = -\int_\Omega \mathrm{div}\, \boldsymbol{v} \varphi \,\mathrm{d}\boldsymbol{x} \text{ for all } \varphi \in \mathcal{C}_0^\infty(\Omega) \big\} \quad (*)$$

In other words: $\mathbf{H}(\mathrm{div}, \Omega)$ is the space of L^2-integrable vector-valued function with a well-defined weak divergence. Given $\Gamma \subset \Omega$, a surface of discontinuity dividing Ω in two pieces; i.e. $\Omega_1 \cup \Omega_2 = \Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma$; show that functions $\boldsymbol{v} \in H(\mathrm{div}, \Omega)$ cannot have normal jumps $[\![\boldsymbol{v}]\!] \cdot \boldsymbol{n} = 0$ across Γ since this would fundamentally contradict definition **(∗)**.

(b) Consider the following space $\mathbb{P}$ of vector valued polynomials over a triangle $K \subset \Omega$:

$$\mathbb{P} = [\mathbb{P}_0(K)]^2 + \boldsymbol{x}\mathbb{P}_0(K)$$

Hence a function $\boldsymbol{v} \in \mathbb{P}$ is of the form $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x}$ where $\boldsymbol{a} = [a_1, a_2]^\top \in \mathbb{R}^2$, $b \in \mathbb{R}$, and $\boldsymbol{x} = [x_1, x_2]^\top$ is the usual vector of spatial coordinates. Note that the polynomial space $\mathbb{P}$ has a total of three degrees of freedom: a_1, a_2 and b. Consider the alternative choice degrees of freedom $\{\sigma_1, \sigma_2, \sigma_3\}$ of the vector-valued polynomial space $\mathbb{P}$:

$$\sigma_i(\boldsymbol{v}) = \int_{F_i} \boldsymbol{v} \cdot \boldsymbol{n} \,\mathrm{d}\boldsymbol{s} \text{ for each } i \in \{1, 2, 3\} \quad (**)$$

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where $F_i \subset \partial K$ represents a face of the element K. Note that $\sigma_i(\boldsymbol{v}) : \mathbb{P} \to \mathbb{R}$ is *not* a nodal value. Prove that the set of degrees of freedom $\{p_1, p_2, p_3\}$ is *unisolvent*. Start by showing that $\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \equiv$ *const* at each face.

(c) Prove that all functions $\boldsymbol{v}_h$ in the finite element space resulting from pasting together affine equivalent triangles are in $\mathbf{H}(\text{div}, \Omega)$. Note however that $\boldsymbol{v}_h$ is not continuous since its tangent component may jump at every interface between two elements.
Hint: show that the normal components of discrete vector fields $\boldsymbol{v}_h$ are continuous across interelement boundaries, meaning $[\![\boldsymbol{v}_h]\!] \cdot \boldsymbol{n}_F \equiv 0$ for each face F in the skeleton of the mesh and that this implies the existence of a weak divergence.

Problem #3. *Relationship between norms and nodal values.* Let $\mathcal{T}_h$ be a family of quasi-uniform shape-regular triangulation of the polygonal domain Ω. Let $\mathbb{V}_h$ be a finite element space defined on such mesh $\mathcal{T}_h$. Let $\{\phi_i\}_{i=1}^N$ be the canonical Lagrange basis function on $\mathbb{V}_h$, and let $v_h(\boldsymbol{x}) = \sum_{i=1}^N V_i \phi_i(\boldsymbol{x}) \in \mathbb{V}_h$. Prove the following local estimates:

$$c_1 \|v\|_{L^2(K)}^2 \leq h_K^d \sum_{\boldsymbol{x}_i \in K} V_i^2 \leq c_2 \|v_h\|_{L^2(K)}^2$$

$$\|v_h\|_{H^1(K)}^2 \leq c_1 h_K^{d-2} \|v_h\|_{L^\infty(K)}^2 \leq c_2 h_K^{d-2} \sum_{\boldsymbol{x}_i \in K} V_i^2$$

Problem #4. *Applications of Bramble-Hilbert lemma I.* Let $\mathcal{T}_h$ be a family of quasi-uniform shape-regular triangulations of the polygonal domain Ω. By this we mean that:

Quasi-uniformity: $\exists\, c_1, c_2 > 0$ such that $c_1 h_{K'} \leq h_K \leq c_2 h_{K'}$ for all $K, K' \in \mathcal{T}_h$

Shape regularity: $\exists\, \sigma > 0$ such that $\frac{h_K}{\rho_K} \leq \sigma$ for all $K \in \mathcal{T}_h$

In colloquial terms: quasi-uniformity means that all the element sizes are comparable, while shape regularity means that the elements are not "too flat" (assuming that $\sigma = \mathcal{O}(1)$). For each $\mathcal{T}_h$ let

$$\mathbb{V}_h = \left\{ v_h \in L^2(\Omega) \,\middle|\, v_h|_K \in \mathbb{P}_0 \ \forall K \in \mathcal{T}_h \right\}$$

Define the piecewise averaging operator $(\Pi_h v)(x) = \sum_{K \in \mathcal{T}_h} \overline{v}_K \mathbb{I}_K(\boldsymbol{x})$ where $\mathbb{I}_K(\boldsymbol{x})$ is the indicator function at the element K and the "nodal value" $\overline{v}_K$ is defined as

$$\overline{v}_K = \tfrac{1}{|K|} \textstyle\int_K v \, \mathrm{d}\boldsymbol{x}$$

The operator Π_h is nothing else than the $L^2(\Omega)$-projector onto the $\mathbb{V}_h$ space. Prove the error bounds

$$\|v - \Pi_h v\|_{L^2(\Omega)} \leq ch|u|_{H^1(\Omega)} \quad \text{where} \quad h = \max_{K \in \mathcal{T}_h} h_K$$

$$\left| \textstyle\int_K u(v - \Pi_h v) \, \mathrm{d}\boldsymbol{x} \right| \leq c h_K^2 |u|_{H^1(K)} |v|_{H^1(K)}$$

Note: The proof of the first inequality requires defining a (possibly nonlinear) functional $F(u)$ in the reference element, that is non-negative, subadditive, satisfying $F(p) \equiv 0$ for all $p \in \mathbb{P}_k$, then invoke the Bramble-Hilbert lemma. The estimate follows by using a scaling argument.

Problem #5. *Applications of Bramble-Hilbert lemma II.* Let $\mathcal{T}_h = \{x_i\}_{i=0}^N$ be a partition of $\Omega = (0, 1)$. Let $Q(w) = \sum_{i=1}^N Q_i(w)$ be the trapezoidal quadrature rule, where:

$$Q_i(w) = \frac{h_i}{2}(w(x_{i-1}) + w(x_i))$$

and $h_i = x_i - x_{i-1}$ is the local meshsize. Show that for all $w \in W_1^2(\Omega)$ the following error estimate holds:

$$\left|Q(w) - \int_0^1 w(x)\mathrm{d}x\right| \leq const \cdot \sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} |w''|\mathrm{d}x$$

Hint: use the fact that $Q_i(w)$ is exact if $w \in \mathbb{P}^1$ in the unit interval, then use Bramble-Hilbert lemma, and finish by using scaling arguments. You will receive zero credits if you solve this problem using Taylor series expansions or related arguments.
