Numerical Analysis of PDE I, Spring 2024, HW#2.

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Problem #1. *Continuous and discrete Maximum principles.* Let u be the solution of the mixed boundary value problem

$$Lu = -u'' + u = f(x) \text{ for all } x \in (0,1) \text{ with } u(0) = u_0 \text{ and } u'(1) = \beta$$

with $u_0, \beta \in \mathbb{R}$.

(a) Write a finite difference method on a uniform partition $\mathcal{T}_h = \{x_i\}_{i=0}^N$ with $0 = x_0 < x_1 < ... < x_N = 1$ and meshsize h. Write 2 discretizations of the Neumann condition $u'(1) = \beta$ that introduce error $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$.

(b) Let $U \in \mathbb{R}^N$ be the discrete solution, $F \in \mathbb{R}^N$ be the right hand side, and $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ be the corresponding matrix so that $\boldsymbol{K}U = F$. Show the discrete maximum principle: if $F_j \leq 0$ for all $j \in \{1, N\}$ then $U_j \leq 0$ for all for all $j \in \{1, N\}$. Conclude that $\boldsymbol{K}$ is non-singular.

(c) Show the continuous maximum principle for the operator L, meaning: show that $Lw \leq 0$ in $(0,1)$ and $w(0) \leq 0$, $w'(1) < 0$, then

$$\max_{0\leq x\leq 1} w(x) \leq 0$$

To make a concrete case assume that $Lw = -1$ and that $w(0) = 0$ and $w'(1) = -1$.

(d) Show the discrete maximum principle. Using the information from (c), let $W \in \mathbb{R}^N$ be given by $W_j = w(x_j)$: show that $(\boldsymbol{K}W)_i \leq -\frac{1}{2}$ for all $1 < i < N$.

(e) Make use of the discrete maximum principle in order to derive a bound for $\|U\|_{\ell^\infty(\mathbb{R}^N)}$ in terms of u_0, β and f.

(f) Prove the error estimate of the form

$$\max_{1\leq i\leq N} |u(x_i) - U_i| \leq \mathit{const} \cdot h^\gamma$$

where $\mathit{const} > 0$ will depend on the regularity of the exact solution, $\gamma = 1$ or 2 depending on the discretization of the Neumann boundary condition. Make the required regularity of u explicit in each case.

Problem #2. *Higher order finite difference formula.* Derive the following 4th order finite difference formula.

$$u''(x_i) = \tfrac{1}{12h^2}(-u(x_{i-2}) + 16u(x_{i-1}) - 30u(x_i) + 16u(x_{i+1}) - u(x_{i+2})) + \mathcal{O}(h^4)$$

upon combining 2nd order centered differences for uniform mesh spacing h and $2h$ at $x = x_i$. To this end write the Taylor expansions for these two formulas and eliminate the leading error term, making explicit the dependence on the regularity of u in the error term $\mathcal{O}(h^4)$ above.

Problem #3. *MATLAB/PYTHON project.* Consider the two-point boundary value problem with parameter $b \in \mathbb{R}$:

$$-u'' + bu' + u = 2x \text{ in } (0,1)\ ,\ u(0) = u(1) = 0 \tag{1}$$

(a) Find the exact solution of $u(x)$ in terms of the parameter b.

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(b) Write the finite difference approximation of (1) using centered finite differences and upwind difference on a uniform mesh with meshsize h. Write the matrix of the system and examine how the equations would change if $u(1) = \alpha \neq 0$.

(c) Implement the finite difference method using MATLAB or PYTHON. If you use Matlab: use the command `diag` to construct the corresponding tridiagonal matrix (type `help diag` to learn about this command). Use the backslash command \ to find the solution to `Ax=b` as `x` = `A\b`.

(d) Solve the linear system for $h = \frac{1}{5}2^{-k}$ for $0 < k < 5$ and $b = 0, 100$. Compute the maximum norm error at the nodes and plot it vs h in a log-log plot. Explain your findings.

(e) Plot the exact solution $u(x)$ and the computed solution as piecewise linear function over the corresponding grid for $h = \frac{1}{20}, \frac{1}{80}$ and $b = 0, 100$. Draw conclusions.

Problem #4. *Transmission conditions.* Consider two domains Ω_1 and Ω_2 with common boundary $\gamma \in \mathcal{C}^1$ and $\Gamma_i = \partial\Omega_i \backslash \gamma$ for $i = 1, 2$. Let $a(\boldsymbol{x}), c(\boldsymbol{x}) \in \mathcal{C}^0(\overline{\Omega})$ for $i = 1, 2$ but discontinuous across γ. Let $u \in \mathcal{C}^2(\overline{\Omega}_i) \cap \mathcal{C}^0(\overline{\Omega})$ for $i = 1, 2$ and $u = g$ on $\partial\Omega$ be the solution of the weak (or variational) problem:

$$\int_\Omega a\nabla u \cdot \nabla v + cuv \, \mathrm{d}\boldsymbol{x} = \int_\Omega fv \, \mathrm{d}\boldsymbol{x} \quad \forall v \in \mathcal{C}_0^\infty(\Omega) \tag{2}$$

(a) Derive the strong form of the PDE including the conditions satisfied by u and ∇u across γ

(b) Obtain an energy $E(u, \nabla u)$ and compute its first variation, which gives the Euler-Lagrange equation (2).

Note: What do we mean by energy? It can be proven that $u \in \mathcal{C}^2 \cap C^0(\Omega)$ solves the Poisson problem $-\Delta u = f$ if and only if u is a minimizer of the functional $E(u, \nabla u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - fu \, \mathrm{d}\boldsymbol{x}$. This property is called Dirichlet principle. If you are unfamiliar with this concept read the book of Evans p.41-43 in order to get familiar with this concept and the usual proof devices. Note that we expect that the "energy" associated to this problem will not be exactly the same as that one of the Poisson problem.

Problem #5. Given the strong form of the boundary value problem

$$-\mathrm{div}\,(a\nabla u) = f \text{ in } \Omega \tag{3}$$

$$a\partial_n u + h(u - g) = k \text{ on } \partial\Omega \tag{4}$$

with $a(\boldsymbol{x})$, $h(\boldsymbol{x})$, and $k(\boldsymbol{x})$ smooth bounded coefficients and

$$a(\boldsymbol{x}) \geq a_0 > 0\,, h(\boldsymbol{x}) \geq h_0 > 0$$

(a) Derive the corresponding weak (or variational) formulation: identify the bilinear form and the right hand side functional.

(b) Show that the resulting bilinear form is coercive: you will have to use the Friedrich's inequality proved in the previous homework.

(c) Show that the resulting right hand side functional $F(v)$ is indeed bounded in $H^1(\Omega)$: you will have to invoke a trace inequality: see for instance the book of Evans p .272, or Theorem A.4 in the appendix of Larsson-Thomee's book.

(d) Show that the solution satisfies a stability estimate of the form

$$\|u\|_{H^1(\Omega)} \leq c(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} + \|k\|_{L^2(\partial\Omega)})$$

(e) Find an energy $E(u, \nabla u)$ whose first variation gives rise to (3).