Numerical Analysis of PDE I, Spring 2024, HW#1.

Assignment day: January 15th 2024

Ignacio Tomas[1], Department of Mathematics and Statistics, Texas Tech University.

Problem #1. *1d Poincaré's inequality and norm-equivalence.* Given a Banach space $\mathbb{B}$: while there is usually one "natural" choice of norm $\|u\|_{\mathbb{B}}$, it is perfectly possible to consider other norms. More precisely, we say that the norms $\|u\|_{\mathbb{A}}$ and $\|u\|_{\mathbb{B}}$ are equivalent if there exists $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|u\|_{\mathbb{A}} \leq \|u\|_{\mathbb{B}} \leq c_2\|u\|_{\mathbb{A}} \text{ for all } u \in \mathbb{B} \tag{1}$$

with c_1 and c_2 independent of u.

For the specific case of the $H^1(\Omega)$ space with $\Omega = (a, b) \subset \mathbb{R}$ the natural norm is clearly $\|u\|_{H^1(\Omega)} = (\int_a^b u^2 + |u'|^2 \mathrm{d}x)^{\frac{1}{2}}$. Prove that for any continuously differentiable $u : \Omega \to \mathbb{R}$, $\Omega = [a, b]$, with satisfying $u(a) = 0$ there holds,

$$\int_\Omega u^2(x)\,\mathrm{d}x \leq (b-a)^2 \int_\Omega |u'(x)|^2\,\mathrm{d}x.$$

While the proof requires $C^1(a, b)$ regularity, the inequality actually holds true for H^1 functions which, strictly speaking, may not be differentiable. Here $c_p = |b - a|$ is the so-called Poincaré's constant.

Conclude that, for the case of functions $u \in H^1_0(\Omega)$ with $H^1_0(\Omega)$ defined as

$$H^1_0(\Omega) = \{u \in H^1(\Omega) | u(a) = u(b) = 0\}$$

the semi-norm $|u|_{H^1(\Omega)} = (\int_a^b |u'|^2\,\mathrm{d}\boldsymbol{x})^{\frac{1}{2}}$ is actually a norm since it is equivalent to $\|u\|_{H^1(\Omega)}$. Provide the expressions of c_1 and c_2 in terms of the Poincaré's constant.

Hint: Start with the identity

$$u(x) = \underbrace{u(a)}_{=0} + \int_a^x u'(s)\,\mathrm{d}s$$

and observe that $\int_a^x u'(s)\,\mathrm{d}s = \int_a^x 1\, u'(s)\,\mathrm{d}s$. You will have to use some the elementary inequalities we have learned in class.

Problem #2. *Multi-d Poincaré's inequality and norm-equivalence.* Let $u \in \mathcal{C}^1_0(\Omega)$. Here $\mathcal{C}^1_0(\Omega)$ is the space of differentiable functions with compact support in Ω. The following Poincaré's inequality holds true

$$\|u\|_{L^2(\Omega)} \leq c_p \|\nabla u\|_{L^2(\Omega)} \text{ for all } u \in \mathcal{C}^1_0(\Omega)$$

with c_p independent of u, but depending on geometric properties of Ω. We will not prove it here, but this Poincaré's inequality holds true for functions in $u \in H^1_0(\Omega)$ as well. Consider the following:

(a) *A first crude proof.* Prove that Poincaré's inequality holds true: start by considering the fact that

$$\int_\Omega \mathrm{div}\,(\boldsymbol{x}u^2)\,\mathrm{d}\boldsymbol{x} = 0 \;\text{ for all } u \in \mathcal{C}^1_0(\Omega)$$

because of the divergence theorem and the compact support of u. Then proceed using elementary inequalities learned in class. Provide an explicit expression for the Poincaré's constant c_P.

[1] `https://www.math.ttu.edu/~igtomas/`, igtomas@ttu.edu

(b) *A better proof.* The proof provided in the previous step is somewhat defective since the Poincaré's constant will depend on the value of $M = \max_{x\in\Omega} |\boldsymbol{x}|_{\ell^2(\mathbb{R}^d)}$. Technically speaking, the proof of step (a) is not incorrect, but the resulting Poincaré's constant could be exaggeratedly large. Using technical jargon: the Poincaré's constant should be translation invariant. In particular, it should not depend on the distance of Ω with respect to the origin. We have to fix this!

A better proof should lead to a Poincaré's constant depending on the diameter of the domain defined as

$$\mathrm{diam}(\Omega) = \max_{\boldsymbol{x},\boldsymbol{y}\in\overline{\Omega}} |\boldsymbol{x}-\boldsymbol{y}|_{\ell^2(\mathbb{R}^d)}$$

Re-do the proof of the part (a): start by thinking how to modify the proof of part (a) in order make the dependence on the diameter of the domain appear. Provide an improved expression for the Poincaré's constant.

Conclude that the $H^1(\Omega)$-norm and the seminorm $\|\nabla u\|_{L^2(\Omega)}$ are equivalent in the space $H^1_0(\Omega)$: provide an expression for the equivalence constants in terms of the Poincaré's constant c_P.

Problem #3. *Weak derivatives.* Compute the (weak) derivative $\partial_{x_i} v(\boldsymbol{x})$ with $v(x) = \log|\log\frac{|\boldsymbol{x}|}{2}|$ in the unit ball of $\mathbb{R}^d$. Prove that $v(x) \in W^{1,d}(\Omega)$. In particular, this implies that $H^1(\Omega)$ functions may not be bounded in dimension $d \geq 2$.
Hint: the expectation is that you should start by showing that $v(x) \in L^d(\Omega)$, and the proceed to show that $\partial_{x_i} v(\boldsymbol{x}) \in L^d(\Omega)$ as well.

Problem #4. Assuming that solutions of the following boundary value problem

$$-(a(x)u'(x))' + b(x)\,u'(x) + c(x)u(x) = f(x), \quad \text{with} \quad u(a) = u(b) = 0,$$

with

$$a(x) \geq a_0 > 0\,,\ c(x) \geq c_0 > 0\,,\ c(x) - \tfrac{b'(x)}{2} \geq 0 \tag{2}$$

exists, prove that it admits the a-priori estimate

$$\|u\|_{L^2(a,b)} \leq c\|f\|_{L^2(a,b)}$$

Provide an expression for the constant in terms of the bounds satisfied by the cofficients $a(x)$, $b(x)$, and $c(x)$.
Hint: This is just an integration by parts exercise. In addition, it will require using some of the inequalities learned in class.

Problem #5. Prove the Friedrich's inequality.

$$\|u\|_{L^2(\Omega)} \leq c_F\big(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\big)^{\frac{1}{2}} \quad \text{for all } u \in \mathcal{C}^1(\overline{\Omega})\,,$$

where Ω is a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$. Though it's not necessary in practice, it is perfectly possible to obtain an explicit expression for the constant c_F. Even better: it is possible to verify (directly) that c_F does NOT depend on u but only on geometric properties of the domain Ω.
Hint: Integrate by parts the identity $\int_\Omega u^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega u^2 \Delta\phi\,\mathrm{d}\boldsymbol{x}$ where $\phi(\boldsymbol{x}) = \frac{1}{2d}|\boldsymbol{x}|^2$.
Note: we will see later in the class that Friedrich's inequality allows us to prove a specialized forms of norm-equivalence. This is useful to prove the requirements of Lax-Milgram theorem

---

Problem #6. Consider a scalar valued-function $u(x) : \mathbb{R} \to \mathbb{R}$ with $x \in \mathbb{R}$. Consider the definitions:

$$x_i := ih \text{ and } u_i := u(x_i)\,,$$

with i integer and $h > 0$ a real number. For simplicity let's assume that $h = 1/N$ with N a positive integer. Given the finite difference formulas

$$\delta u_j = \frac{u_{j+1} - u_{j-1}}{2h}\,, \quad \text{and} \quad \delta^2 u_j = \frac{u_{j+1} + 2u_j - u_{j-1}}{h^2}\,,$$

prove the following estimates

$$|\delta^2 U_j - \tfrac{d^2u}{dx^2}(x_j)| \le Ch^2 \|u\|_{C^4(\Omega)} \text{ for all } x_j \in \Omega\,,$$
$$|\delta U_j - \tfrac{du}{dx}(x_j)| \le Ch^2 \|u\|_{C^3(\Omega)} \text{ for all } x_j \in \Omega\,,$$

where $\Omega = (0, 1)$.

Hint: You will have to use Taylor's polynomial. You might want to consider using the remainder/truncation formula in integral form:

$$\mathcal{R}_k = \int_a^x \frac{u^{k+1}(s)}{k!}(x - s)^k ds$$

instead of the usual remainder in Cauchy's form. This remainder formula usually helps get sharper constants. The power of h of the error estimate usually pop-out from the term $(x - s)^k$ inside the remainder.

Problem #7. Consider the nonlinear boundary value problem:

$$-u'' + u = e^u \text{ in } \Omega = (0, 1) \text{ with } u(0) = u(1) = 0$$

Assuming that solutions to this problem exist: use the maximum principle to show that all solutions are nonnegative i.e. $u(x) \ge 0$ for all $x \in \overline{\Omega}$. Use the strong version of the maximum principle to show that all solutions are positive, i.e. $u(x) > 0$ for all Ω.

Note: a quick and short outline of maximum/minimum principles techniques can be found in Chapter 2 of the book of Larsson-Thomee (it's less than a handful of pages). Study the main ideas/steps and adapt the proofs to this problem.