A Metric Space of A.H. Stone and an Example Concerning σ-Minimal Bases

by

Harold R. Bennett, Texas Tech University, Lubbock, TX

and

David J. Lutzer, College of William and Mary, Williamsburg, VA

Abstract

In this paper we use a metric space Y due to A. H. Stone and one of its completions X to construct a linearly ordered topological space $E = E(Y, X)$ that is Čech complete, has a σ-closed-discrete dense subset, is perfect, hereditarily paracompact, first-countable, and has the property that each of its subspaces has a σ-minimal base for its relative topology. However, E is not metrizable and is not quasi- developable. The construction of $E(Y, X)$ is a point-splitting process that is familiar in ordered spaces, and an orderability theorem of Herrlich is the link between Stone's metric space and our construction.

Key words and phrases: linearly ordered space, generalized ordered space, Cech complete, paracompact, perfect space, σ-minimal base, metrization theory.

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1. Introduction.

A collection $\mathcal{A}$ of subsets of a space X is minimal (or irreducible) if each member of $\mathcal{A}$ contains a point that belongs to no other member of $\mathcal{A}$. Spaces with σ-minimal bases were introduced and studied by Aull in [A1, A2], and the basic examples clarifying the relation of σ-minimal bases to other topological properties were constructed in [BB]. However, there remains one context in which the role of Aull's σ-minimal bases is not yet clear, and that is the theory of linearly ordered and generalized ordered spaces.

Among generalized ordered spaces, quasi-developability, the existence of a σ-disjoint base, and the existence of a σ- point-finite base are mutually equivalent properties ([B],[L]). Among generalized ordered spaces, each implies the existence of a σ-minimal base, but the converse is obviously false – the lexicographic square is a compact linearly ordered space that has a σ-minimal base but is not quasi-developable. Indeed, there are linearly ordered spaces that have σ-minimal bases that are not even first countable. However, all such spaces are hereditarily paracompact [BL1] even though the existence of a σ-minimal base is not itself a hereditary property.

Familiar examples suggest that generalized ordered spaces with σ-minimal bases are either quasi-developable or else contain a particular type of pathological subspace that does not have a σ-minimal base for its relative topology, and that suggests asking ([BL2],[L2]) whether a compact linearly ordered space must be metrizable if each of its subspaces has a σ-minimal base. That question is still open. (See "Added in Proof.") A weaker version of the question (originally mis-posed in [BL2]) asks whether a linearly ordered, or generalized ordered, space X must be quasi-developable if every subspace of X has a σ- minimal base for its relative topology. That is the question that we resolve negatively in this paper by constructing a non-metrizable, perfect linearly ordered space X such that every subspace of X has a σ-minimal base for its topology. The space X has many other valuable features: it is hereditarily paracompact, first-countable, Čech-complete, and has a dense subspace that is σ-closed-discrete in X. But the space X is not quasi-developable, as can be seen from the fact that X is perfect but not metrizable.

The two key components of our construction are an elegant metric space constructed by A.H. Stone in [St] (and used as the basis for many other topological examples, e.g., [P1,P2]) together with an orderability theorem of H. Herrlich [H].

Recall that a linearly ordered topological space (or LOTS) is a triple $(X, <, \mathcal{T})$ where $<$ is a linear ordering of X and where $\mathcal{T}$ is the usual open interval topology of $<$. Unfortunately, a subspace of a LOTS may fail to be a LOTS, and that leads to the study of generalized ordered spaces (or GO-spaces), i.e., spaces that can be embedded in some LOTS. An internal characterization of GO-spaces is that they are triples $(X, <, \mathcal{T})$ where $<$ is a linear ordering of X and where $\mathcal{T}$ is a Hausdorff topology on X that has an open base consisting of order-convex sets.

In our paper we must carefully distinguish between subsets of a space X that are the union of a countable collection of closed, discrete subspaces of X (such subspaces will be called σ-closed-discrete) and subspaces that are countable unions of discrete, but not necessarily closed, subspaces (such spaces are said to be σ-discrete-in-themselves). However, as Stone pointed out in [St], the two notions are equivalent in any metric space, and indeed in any perfect space, where a space X is perfect if every closed subset of X is a G_δ in X.

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2. Stone's metric space and preliminary properties of E(Y,X)

In [St], A.H. Stone constructed a special metric space Y that has been used as the starting point of many important examples in topology; see, for example, [P1,P2]. The crucial properties of Stone's space Y are:

S-1) Y is a subspace of D^ω where D is a discrete space of cardinality ω_1;

S-2) the cardinality of Y is ω_1;

S-3) Y is not the union of countably many subspaces, each discrete-in-itself;

S-4) if C is a countable subset of Y, then the closure of C in Y is also countable.

Throughout this paper, X will be the closure of Y in the metric space D^ω and $\mathcal{M}$ will denote the topology that X inherits from D^ω. Let d be a metric on X that induces the topology $\mathcal{M}$ and, for $x \in X$ and $\epsilon > 0$ let $B(d, x, \epsilon)$ denote the open d-ball of radius ϵ centered at x.

2.1 **Proposition**: *With X as above, $(X, \mathcal{M})$ is a completely metrizable space and there is a linear ordering of the set X that induces $\mathcal{M}$ as its open interval topology.*

Proof: Being closed in D^ω, $(X, \mathcal{M})$ is completely metrizable. That some linear ordering of X makes $(X, \mathcal{M})$ into a LOTS follows from a result of Herrlich ([H]; see [E, Problem 6.3.2] for another proof) because X is strongly zero-dimensional. $\square$

2.2 **Proposition**: *Let $Z \subset X$. Define $J(1) = \{z \in Z :$ for some $x \in X,\ z < x$ and $]z, x[\ \cap Z = \emptyset\}$ and $J(-1) = \{z \in Z :$ for some $x \in X,\ z > x$ and $]x, z[\ \cap Z = \emptyset\}$. Then $J = J(-1) \cup J(1)$ is a σ-closed-discrete subset of $(X, \mathcal{M})$.*

Proof: Because X is metrizable, it will be enough to show that J is the union of countably many subsets, each of which is discrete-in-itself. Consider $J(1)$, the argument for $J(-1)$ being analogous. For each $z \in J(1)$ let z^+ denote an element of X such that $z < z^+$ and $]z, z^+[\ \cap Z = \emptyset$. Then there is an integer $n = n(z) \geq 1$ such that $B(d, z, \frac{1}{n}) \subset\ ]\leftarrow, z^+[$. Let $J(1, m) = \{z \in J(1) : n(z) = m\}$. If $p < q$ are points of $J(1, m)$, then $p^+ \leq q$ so that we have $B(d, p, \frac{1}{m}) \subset\ ]\leftarrow, p^+[$ showing that $q \notin B(d, p, \frac{1}{m})$ so that $d(p, q) \geq \frac{1}{m}$ as required. $\square$

For any $Z \subset X$, there is a second natural topology on Z, namely the Sorgenfrey topology $\mathcal{S}_Z$ that has sets of the form $Z \cap [z, b[$ as a base, where $z \in Z$ and $b \in X$ has $z < b$. Clearly $\mathcal{M}_Z \subset \mathcal{S}_Z$, where $\mathcal{M}_Z$ is the topology that Z inherits from the metrizable space $(X, \mathcal{M})$.

2.3 **Corollary**: *For any $Z \subset X$, the space $(Z, \mathcal{S}_Z)$ has a dense set D that is the union of countably many closed, discrete subsets of $(Z, \mathcal{S}_Z)$.*

Proof: The metrizable space $(Z, \mathcal{M}_Z)$ has a dense set E that is σ-closed-discrete in $(Z, \mathcal{M}_Z)$ and hence is also σ-closed-discrete in $(Z, \mathcal{S}_Z)$. From (2.2) the set $J(1) = \{z \in Z :$ for some

$x \in X$ with $x > z$, $[z, x[\, \cap Z = \{z\}\}$ is also σ-closed-discrete in $(Z, \mathcal{M}_Z)$ and therefore also in $(Z, \mathcal{S}_Z)$. Because $J(1) \cup E$ is dense in $(Z, \mathcal{S}_Z)$, the lemma is proved. $\square$

2.4 **Construction of** $E(Y, X)$. Let Y be Stone's metric space and X its closure in D^ω. Let $\mathcal{M}$ be the metrizable topology that X inherits from the product space. Use (2.1) to choose a linear ordering of X that induces $\mathcal{M}$ as its open interval topology. Let

$$E(Y, X) = (Y \times \{-1, 1\}) \cup ((X - Y) \times \{0\}).$$

Order $E(Y, X)$ lexicographically and let $E(Y, X)$ carry the open interval topology of that ordering. Let $\pi : E(Y, X) \to X$ be the function $\pi(x, i) = x$. One can think of $E(Y, X)$ as being the result of splitting each point of Y into two consecutive points.

2.5 **Proposition**: *The mapping π is a perfect mapping from $E(Y, X)$ onto $(X, \mathcal{M})$. Hence $E(Y, X)$:*

a) is Čech-complete;

b) has a dense subset that is σ-closed-discrete;

c) is perfect, hereditarily paracompact, and first countable.

Proof: It is easy to see that π is continuous and closed so that, because $|\pi^{-1}(x)| \leq 2$ for each $x \in X$, π is perfect. Hence a) holds.

Let D be a dense σ-closed-discrete subset of the metric space $(X, \mathcal{M})$. Then $E_1 = \pi^{-1}[D]$ is a σ-closed-discrete subset of $E(Y, X)$. Consider the sets $J(-1) = \{z \in X :$ for some $x < z$, $]x, z[\, = \emptyset\}$ and $J(1) = \{z \in X :$ for some $x > z$, $]z, x[\, = \emptyset\}$. According to (2.2), each is σ-closed-discrete in $(X, \mathcal{M})$. Hence $E_2 = \pi^{-1}[J(-1) \cup J(1)]$ is σ-closed-discrete in $E(Y, X)$. It is easy to verify that $E_1 \cup E_2$ is dense in $E(Y, X)$ so that (b) holds. Now (c) follows from the general theory of ordered spaces (see [L,BLP,Fa,vW]). $\square$

2.6 **Proposition**: $E(Y, X)$ *is not metrizable.*

Proof: For contradiction, suppose $E(Y, X)$ is metrizable. Then so is its subspace $Y \times \{1\}$, which is homeomorphic to $(Y, \mathcal{S}_Y)$, where $\mathcal{S}_Y$ is the Sorgenfrey topology on Y described just before (2.3). Let d be a metric on Y that induces the topology $\mathcal{M}_Y$ and suppose ρ is a metric on Y that induces the topology $\mathcal{S}_Y$. We will denote the open balls with respect to d and ρ by $B(d, x, \epsilon)$ and $B(\rho, x, \epsilon)$, respectively.

For each $y \in Y$ there is an integer $n = n(y) \geq 1$ such that $B(\rho, y, \frac{1}{n}) \subset [y, \to [$. Let $Y(m) = \{y \in Y : n(y) = m\}$ and observe that

(*) if $p < q$ are points of $Y(m)$, then $B(\rho, q, \frac{1}{m}) \subset [q, \to [$ so that $\rho(p, q) \geq \frac{1}{m}$.

Thus each $Y(m)$ is closed and discrete in $(Y, \mathcal{S}_Y)$.

For each $y \in Y(m)$ there is a $b > y$ such that $Y \cap [y, b[\subset B(\rho, y, \frac{1}{m})$. Because $] \leftarrow, b[$ is a neighborhood of y in $(Y, \mathcal{M}_Y)$, there is a $k = k(y, m) \geq 1$ such that $B(d, y, \frac{1}{k}) \subset] \leftarrow, b[$. Hence $B(d, y, \frac{1}{k}) \cap [y, \rightarrow [\subset [y, b[\subset B(\rho, y, \frac{1}{m})$. Let $Y(m, j) = \{y \in Y(m) : k(y, m) = j\}$ for each $m, j \geq 1$. Because $Y = \bigcup\{Y(m, j) : m, j \geq 1\}$, it follows from property (S-3) of Y that for some $m, j \geq 1$, the set $Y(m, j)$ is not discrete-in-itself when Y is topologized using $\mathcal{M}_Y$. Hence we may choose a strictly monotonic sequence $\{y_k : k \geq 1\}$ in $Y(m, j)$ that converges to a point $y_0 \in Y(m, j)$. Because $Y(m, j) \subset Y(m)$ and $Y(m)$ is closed and discrete in $(Y, \mathcal{S}_Y)$, it must be the case that $y_1 < y_2 < y_3 < ....$ Choose integers $k < i$ such that $\{y_k, y_i\} \subset B(d, y_0, \frac{1}{3j})$. The triangle inequality yields $d(y_k, y_i) < \frac{1}{j}$ so that $y_i \in B(d, y_k, \frac{1}{j}) \cap [y_k, \rightarrow [$. Because $y_k \in Y(m, j)$ we have $B(d, y_k, \frac{1}{j}) \cap [y_k, \rightarrow [\subset B(\rho, y_k, \frac{1}{m})$ so that $\rho(y_k, y_i) < \frac{1}{m}$, and that contradicts (*) above, because $y_k, y_i \in Y(m, j) \subset Y(m)$. That contradiction shows that $(Y, \mathcal{S}_Y)$ cannot be metrizable and completes the proof. $\square$

3. σ-minimal bases

3.1) **Lemma**: *Let Z be a subspace of a first-countable GO-space Y. Suppose that Z, in its relative topology, has a σ-minimal base. Then there is a σ-minimal collection of open subsets of Y that contains a neighborhood base at each point of Z.*

Proof: Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ be a σ-minimal base of relatively open subsets of Z. For each $B \in \mathcal{B}$, let B^* be the convex hull of B in Y. The set B^* might not be open, but because Y is first countable, there is a sequence $B^*(k)$ of convex open subsets of Y such that if G is a convex open subset of Y that contains B^*, then for some k, $B^*(k) \subset G$.

For each $B \in \mathcal{B}(n)$, choose an open set $C(B)$ in Y such that $B = C(B) \cap Z$, and define $D(B, n, k) = B^*(k) \cap C(B)$. Each $D(B, n, k)$ is open in Y and the collection $\mathcal{D}(n, k) = \{D(B, n, k) : B \in \mathcal{B}(n)\}$ is irreducible. Finally, $\mathcal{D} = \bigcup\{\mathcal{D}(n, k) : n, k \geq 1\}$ contains a local base at each $y \in Y$. For suppose G is a convex open subset of Y and that $z \in G \cap Z$. Then there is an index n and a set $B \in \mathcal{B}(n)$ such that $z \in B \subset G \cap Z$. Because G is convex, $B^* \subset G$. As noted above, there is an index $k \geq 1$ such that $B^*(k) \subset G$. Then $z \in D(B, n, k) = C(B) \cap B^*(k) \subset G$, as required. $\square$

3.2 **Corollary**: *Suppose Z is a first-countable GO space and $Z = \bigcup\{Z(n) : n \geq 1\}$ where each subspace $Z(n)$ has a σ-minimal base for its relative topology. Then Z has a σ-minimal base.* $\square$

3.3 **Lemma**: *With $(Y, \mathcal{S}_Y)$ as in Section 2, each subspace of $(Y, \mathcal{S}_Y)$ has a σ-minimal base for its relative topology.*

Proof: Let $Z \subset Y$. According to (2.3) there is a dense subset $D = \bigcup\{D(n) : n \geq 1\}$ of $(Z, \mathcal{S}_Z)$ where each $D(n)$ is a closed, discrete subspace of $(Z, \mathcal{S}_Z)$. Because $(Z, \mathcal{S}_Z)$ is first countable and hereditarily collectionwise normal, there is a σ-disjoint collection $\mathcal{D}$ of open subsets of Z that contains a neighborhood base at each point of D.

Let $Z_0 = \{z \in Z : \text{some } \mathcal{S}_Z\text{-neighborhood of } z \text{ is countable }\}$. Then Z_0 is an open subspace of Z and, each countable GO-space being metrizable, Z_0 is locally metrizable. The existence of the σ-closed-discrete, dense set D makes Z perfect and hence hereditarily paracompact, so that Z_0 is a metrizable subspace of Z. Hence there is a σ-disjoint collection $\mathcal{E}$ of $\mathcal{S}_Z$-open sets that contains a neighborhood base at each point of Z_0.

Suppose $z \in Z-(D \cup Z_0)$. Then each $\mathcal{S}_Z$ neighborhood of z is uncountable. We claim that for each $b \in X$ with $b > z$, the set $C = [z, b[\ \cap D$ is uncountable. For if not, then property (S-4) of Stone's space $(Y, \mathcal{M}_Y)$ forces the $\mathcal{M}_Y$-closure of C to be countable. But then $[z, b[\ \cap Z$ is a subset of the $\mathcal{S}_Y$-closure of C which is a subset of the $\mathcal{M}_Y$-closure of C which is countable, so that $z \in Z_0$, contradicting $z \in Z-(D \cup Z_0)$.

Provided $z \in Z-(D \cup Z_0)$, each $B(d, z, \frac{1}{k}) \cap [z, \rightarrow[$ is an $\mathcal{S}_Z$-neighborhood of z. Hence for some $n \geq 1$, $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\ \cap D(n)| > \omega$. Let $A(n, k)$ be the set of all $z \in Z-(D \cup Z_0)$ such that $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\ \cap D(n)| > \omega$. Because $|Y| = \omega_1$ according to property (S-2), there is a 1-1 function $\phi_{n,k} : A(n, k) \rightarrow D(n)$ such that $\phi_{n,k}(z) \in B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\ \cap D(n)$ for each $z \in A(n, k)$. For $z \in A(n, k)$ define

$$C(z, n, k) = (Z \cap [z, \rightarrow[\ \cap B(d, z, \frac{1}{k})) - (D(n) - \{\phi_{n,k}(z)\})$$

and let $\mathcal{C}(n, k) = \{C(z, n, k) : z \in A(n, k)\}$. Each collection $\mathcal{C}(n, k)$ consists of $\mathcal{S}_Z$-open sets and is a minimal collection because if z, w are distinct points of $A(n, k)$, then $\phi_{n,k}(z) \in C(z, n, k) - C(w, n, k)$.

We claim that $\mathcal{C} = \bigcup\{\mathcal{C}(n, k) : n, k \geq 1\}$ contains an $\mathcal{S}_Z$-neighborhood base at each point $z \in Z-(D \cup Z_0)$. For suppose $z < b$ with $b \in X$, and consider $Z \cap [z, b[$. Because $]\leftarrow, b[$ is an $\mathcal{M}_Y$-neighborhood of z, there is a $k \geq 1$ with $B(d, z, \frac{1}{k}) \subset\ ]\leftarrow, b[$. Then $B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\ \subset [z, b[$. Choose n so that $|B(d, z, \frac{1}{k}) \cap [z, \rightarrow[\ \cap D(n)| > \omega$. Then $C(z, n, k) \in \mathcal{C}(n, k)$ and $z \in C(z, n, k) \subset [z, b[\ \cap Z$ as required. $\square$

3.4 **Corollary**: *Every subspace of $E(Y, X)$ has a σ-minimal base for its topology.*

Proof: Let $Z \subset E(Y, X)$ and define $Z(i) = (Y \times \{i\}) \cap Z$ if $i \in \{-1, 1\}$ and $Z(0) = Z \cap ((X-Y) \times \{0\}$. In the light of (3.2) it will be enough to show that each $Z(i)$ has a σ-minimal base for its relative topology.

The subspace $Z(0)$ is metrizable and therefore has a base of the required kind. The subspace $Z(1)$ is homeomorphic to a subspace of $(Y, \mathcal{S}_Y)$ so that it has a σ-minimal base in the light of (3.3). The argument to show that $Z(-1)$ has a σ-minimal base for its relative topology is entirely analogous to the argument for $Z(1)$, and that completes the proof. $\square$

Added in Proof: In a paper that will apprear in *Proceedings of the American Mathematical Society*, Wei-Xue Shi constructed a non- mertrizable compact LOTS X, every subspace of which has a σ-minimal base. Shi's result completely settles the question posed in [BL2] and [L2].

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