A Note on Property III in Generalized Ordered Spaces

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Abstract

A topological space X has Property III provided there are subsets $U(k)$ and $D(k)$ of X for each $k \geq 1$ such that:

1) $U(k)$ is open in X and $D(k)$ is relatively closed in $U(k)$;
2) $D(k)$ is discrete in itself;
3) if p is a point of an open set G, then there is a $k \geq 1$ such that $p \in U(k)$ and $D(k) \cap G \neq \phi$.

We show that Property III is exactly what is needed, in a generalized ordered space, to transform a $\delta\theta$-base in the sense of Aull into a σ-disjoint base (equivalently, into a quasi-development). We also give an example of a linearly ordered topological space that has a $\delta\theta$-base, but not a point-countable base.

Key words and phrases: generalized ordered space, linearly ordered space, Property III, σ-disjoint base, quasi-developable space, point-countable base, $\delta\theta$-base, perfect space, σ-closed-discrete dense set.

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1.Introduction.

A topological space X has Property III provided there are subsets $U(k)$ and $D(k)$ of X for each $k \geq 1$ such that:

1) $U(k)$ is open in X and $D(k)$ is relatively closed in $U(k)$;
2) $D(k)$ is discrete in itself, i.e., when endowed with the relative topology from X, the space $D(k)$ is discrete;
3) if p is a point of an open set G, then there is a $k \geq 1$ such that $p \in U(k)$ and $D(k) \cap G \neq \phi$.

In [BL] we showed that Property III plays an important bridging role in the theory of generalized ordered spaces by proving that a generalized ordered space X has a σ-disjoint base if and only if it has a point-countable base and has Property III. In this paper we extend the main theorem of [BL] by showing that Property III plays an even larger bridging role in ordered space theory. We show that for generalized ordered spaces, Property III is the solution of the equation "σ-disjoint base = (?) + $\delta\theta$-base" by adding (e) to the list of equivalent conditions in the following theorem:

1.1) Theorem: *The following properties of a generalized ordered space X are equivalent:*

a) X is quasi-developable;
b) X has a σ-disjoint base;
c) X has a σ-point finite base;
d) X has a point countable base and has Property III;
e) X has a $\delta\theta$-base and has Property III.

Recall that a generalized ordered space or GO-space is a triple $(X, \mathcal{T}, <)$ such that $(X, <)$ is a linearly ordered set and where $\mathcal{T}$ is a Hausdorff topology on X that has a base of order-convex open sets. If it happens that $\mathcal{T}$ is the open interval topology of the ordering $<$, then X is a linearly ordered topological space or LOTS. The class of GO-spaces is exactly the class of subspaces of LOTS.

Quasi-developments, σ-point finite bases, and σ- disjoint bases are familiar objects: see [B1], [BL] and [L1] for general background material. The notion of a $\delta\theta$-base was intruduced by Aull in [Au]: A collection $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ is a $\delta\theta$-base for X provided that if G is open and $p \in G$, then for some n, some $B \in \mathcal{B}(n)$ has $p \in B \subset G$, and the order of p in $\mathcal{B}(n)$ is countable. (By the order of a point p in a collection $\mathcal{C}$ we mean the cardinality of the collection $\{C \in \mathcal{C} : p \in C\}$. We will denote the order of p in $\mathcal{C}$ by $\text{ord}(p, \mathcal{C})$.) Clearly, any point-countable base is a $\delta\theta$-base, but the converse is false, even among GO-spaces, as can be seen from the example constructed in Section 3.

Throughout this paper, we must carefully distinguish between the statements "E is a discrete subspace of X" and "E is a closed discrete subspace of X. We will do this, for example, by writing "Suppose X has a σ-closed discrete dense subset" even though that makes our terminology more cumbersome. For example, it is easy to see that the usual space of countable ordinals has a σ-discrete dense subspace but no dense subspace that is σ-closed-discrete.

2. Property III, $\delta\theta$-bases, and perfect GO spaces

We begin by recalling two technical results from [BL] and a characterization of hereditary paracompactness that follows immediately from [Fa].

2.1) Lemma:*Let X be a GO-space with Property III. Then:*

a) *Every subspace of X also has Property III;*

b) *X is hereditarily paracompact.*

2.2) Lemma: *Suppose $\{C(\alpha) : \alpha \in A\}$ is a pairwise disjoint collection of open subsets of a first-countable GO-space X. For each α, suppose that $E(\alpha)$ is a discrete-in-itself subspace of $C(\alpha)$. Then there is a σ-disjoint collection $\mathcal{B}$ of open subsets of X that contains a base at each point of $\bigcup\{E(\alpha) : \alpha \in A\}$.*

2.3)Lemma: *A GO-space X is hereditarily paracompact if and only if for each subspace $C \subset X$ there are sets E and E' such that:*

a) *E and E' are relatively closed subsets of C that are discrete;*

b) *E is well-ordered by the given ordering of X, and E' is reverse-well-ordered by the given ordering of X;*

c) *if $p \in C$ then $p \in [e, e']$ for some $e \in E$ and $e' \in E'$.*

2.4) Proposition: *Suppose that X is a GO-space with a $\delta\theta$-base. If X has Property III, then X has a σ- disjoint base.*

Proof: Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ be a $\delta\theta$-base for X. We may assume that members of $\mathcal{B}$ are convex subsets of X. Then X is certainly first countable. For each n let $X(n) = \{p \in X : \text{ord}(p, \mathcal{B}(n)) \leq \omega\}$. Because X is first countable, it is easy to show that the set $X(n)$ is a relatively closed subset of the open set $\bigcup \mathcal{B}(n)$. For $k \geq 1$, let $U(k)$ and $D(k)$ be the sets given by Property III. Let $O(n,k) = U(k) \cap (\bigcup \mathcal{B}(n))$. Then $O(n,k)$ is open in $U(k)$ and hence also in X, so that $O(n,k) - D(k)$ is open in X. Let $\{C(n,k,\alpha) : \alpha \in A(n,k)\}$ be the family of all convex components of $O(n,k) - D(k)$. In addition, notice that $X(n) \cap O(n,k)$ is a relatively closed subset of $O(n,k)$ so that for each $\alpha \in A(n,k)$, the set $X(n) \cap C(n,k,\alpha)$ is a relatively closed subset of $C(n,k,\alpha)$. Let $A^*(n,k) = \{\alpha \in A(n,k) :$

$X(n) \cap C(n,k,\alpha) \neq \phi\}$ and for each $\alpha \in A^*(n,k)$ use (2.3) to find discrete, relatively closed subsets $E(n,k,\alpha), E'(n,k,\alpha)$ of $X(n) \cap C(n,k,\alpha)$ as described in (2.3).

For each $\alpha \in A^*(n,k)$ let $\mathcal{H}(n,k,\alpha)$ be the collection of all convex components of the open set $C(n,k,\alpha) - E(n,k,\alpha)$ that have at least one endpoint in the set $E(n,k,\alpha)$. Define $\mathcal{H}'(n,k,\alpha)$ in an analogous way, using $E'(n,k,\alpha)$ in place of $E(n,k,\alpha)$. For each $H \in \mathcal{H}(n,k,\alpha)$, define $\mathcal{G}(n,k,\alpha,H)$ to be the collection of all sets $H \cap B$ where $B \in \mathcal{B}(n)$ and B contains at least one of the endpoints of H that belongs to $E(n,k,\alpha)$. Define collections $\mathcal{G}(n,k,\alpha,H')$ using the sets $E'(n,k,\alpha)$ in place of $E(n,k,\alpha)$ and sets $H' \in \mathcal{H}'(n,k,\alpha)$.

For a fixed (n,k,α,H), if $B \cap H \in \mathcal{G}(n,k,\alpha,H)$ then B contains at least one end point of H that belongs to $E(n,k,\alpha) \subset X(n)$, and $B \in \mathcal{B}(n)$. Because there are at most two such endpoints, the collection $\mathcal{G}(n,k,\alpha,H)$ is countable. Index $\mathcal{G}(n,k,\alpha,H)$ as $\{G(n,k,\alpha,H,j) : j \geq 1\}$. Similarly, index $\mathcal{G}'(n,k,\alpha,H')$ as $\{G'(n,k,\alpha,H',j) : j \geq 1\}$

Let $\mathcal{U}(n,k,j) = \{G(n,k,\alpha,H,j) : \alpha \in A^*(n,k) \textit{ and } H \in \mathcal{H}(n,k,\alpha)\}$. If α and β are distinct members of $A^*(n,k)$ then $G(n,k,\alpha,H_1,j)$ and $G(n,k,\beta,H_2,j)$ are, respectively, subsets of the disjoint convex components $C(n,k,\alpha)$ and $C(n,k,\beta)$, for any $H_1 \in \mathcal{H}(n,k,\alpha)$ and $H_2 \in \mathcal{H}(n,k,\beta)$. And if $H_1 \neq H_2$ belong to $\mathcal{H}(n,k,\alpha)$, then $G(n,k,\alpha,H_i,j) \subset H_i$ so that $G(n,k,\alpha,H_1,j) \cap G(n,k,\alpha,H_2,j) = \phi$. Therefore each $\mathcal{U}(n,k,j)$ is a pairwise disjoint collection. Analogously, $\mathcal{U}'(n,k,j)$, the collection of all sets $G'(n,k,\alpha,H',j)$ where $\alpha \in A^*(n,k)$ and $H' \in \mathcal{H}(n,k,\alpha)$, is a pairwise disjoint collection. Let $\mathcal{U} = \bigcup\{\mathcal{U}(n,k,j) : n,k,j \geq 1\}$ and define $\mathcal{U}' = \bigcup\{\mathcal{U}'(n,k,j) : n,k,j \geq 1\}$. Then $\mathcal{U}$ and $\mathcal{U}'$ are σ-disjoint collections of open subsets of X.

Let $Y = (\bigcup\{D(k) : k \geq 1\}) \cup (\bigcup\{E(n,k,\alpha) \cup E'(n,k,\alpha) : n,k \geq 1 \textit{ and } \alpha \in A^*(n,k)\}$ Repeated application of (2.2) yields a σ-disjoint collection $\mathcal{E}$ of open subsets of X that contains a base at each point $y \in Y$. Thus it will be enough to show that the collection $\mathcal{U} \cup \mathcal{U}'$ defined above contains a base at each point of $X - Y$.

To that end, suppose $p \in X - Y$ and $p \in G$, where G is open in X. Find n such that $\mathrm{ord}(p, \mathcal{B}(n))$ is countable, and such that for some $B \in \mathcal{B}(n)$, we have $p \in B \subset G$. Then $p \in X(n)$. Next find some k such that $p \in U(k)$ and some point $d \in D(k) \cap B$. Because $d \in Y$ while $p \notin Y$, we have $p \neq d$. There are two cases to consider. If $p < d$, then we will show that $\mathcal{U}$ contains a set that contains p and is a subset of $B \subset G$. If $d < p$ then an analogous argument (which we leave to the reader) shows that $\mathcal{U}'$ contains the required set.

Because $p \in U(k) \cap (\bigcup \mathcal{B}(n)) = O(n,k)$ and $p \notin D(k)$ there is a unique $\alpha \in A(n,k)$ such that $p \in C(n,k,\alpha)$. Because $p \in C(n,k,\alpha) \cap X(n)$, it follows that $\alpha \in A^*(n,k)$ so that

$E(n,k,\alpha)$ is defined. From $d \notin C(n,k,\alpha)$ it follows that either $C(n,k,\alpha) \subset\ ]\leftarrow, d[$ or else $C(n,k,\alpha) \subset\ ]d,\rightarrow[$. Because $p \in C(n,k,\alpha)$ and $p < d$, it follows that $C(n,k,\alpha) \subset\ ]\leftarrow, d[$. If p is the largest element of the set $X(n) \cap C(n,k,\alpha)$, then $\{p\} = E(n,k,\alpha) \subset Y$, contrary to $p \notin Y$. Therefore some $q \in E(n,k,\alpha)$ has $p < q$. We may assume that q is the first such point. Then q is the right endpoint of the unique convex component H of $C(n,k,\alpha) - E(n,k,\alpha)$ that contains p. Therefore $H \in \mathcal{H}(n,k,\alpha)$.

Because $B \in \mathcal{B}(n)$ is a convex set containing both p and d, and because $p < q$ and $q \in E(n,k,\alpha) \subset C(n,k,\alpha) \subset\ ]\leftarrow, d[$, we know that B contains q. Therefore, $B \cap H \in \mathcal{G}(n,k,\alpha,H)$ so that for some j we have $B \cap H \in \mathcal{U}(n,k,j) \subset \mathcal{U}$, and $p \in B \cap H \subset B \subset G$ as required. The case where $d < p$ being analogous, we now know that X has a σ- disjoint base. □

3. Examples

In this section we construct a LOTS Z with a $\delta\theta$-base that does not have a point-countable base. In the light of (2.4), Z cannot have Property III. The space Z is an extension of an example in [B] of a LOTS with a point-countable base, but not a σ-disjoint base.

3.1) Construction: Let P, Q, and R denote, respectively, the sets of irrational, rational, and real numbers. For each limit ordinal $\mu \leq \omega_1$, let $X(\mu)$ be the set of all functions $f : [0,\mu] \rightarrow R$ such that $f(\alpha) \in P$ whenever $\alpha < \mu$ and $f(\mu) \in Q$. Let $X = \bigcup\{X(\mu) : \mu < \omega_1\}$ and let $Y = X(\omega_1)$. Let $Z = X \cup Y$ and order Z lexicographically, i.e., for distinct $f, g \in Z$ define $f \prec g$ if $f(\alpha) < g(\alpha)$ where α is the first ordinal such that $f(\alpha) \neq g(\alpha)$. Let Z have the lexicographic order topology of $\prec$. For each $f \in Z$ let $\lambda(f)$ be the largest member of the domain of f. For each $f \in Z$ and each $n \geq 1$ define $B(f,n)$ to be the set

$$\{g \in Z : \lambda(g) \geq \lambda(f) \ and\ g(\alpha) = f(\alpha)\ for\ each\ \alpha < \lambda(f)\ and\ |g(\lambda(f)) - f(\lambda(f))| < 1/n\}.$$

3.2) Lemma: *With notation as above,*

a) *for each $f \in Z$, $\{B(f,n) : n \geq 1\}$ is a countable base of open neighborhoods for f in the space Z;*

b) *the collection $\mathcal{B} = \{B(f,n) : f \in X$ and $n \geq 1\}$ is point-countable at each point $g \in X$;*

c) *the collection $\mathcal{C} = \{B(f,n) : f \in Y$ and $n \geq 1\}$ is point-countable at each $g \in Z$;*

d) *Z has a $\delta\theta$-base.*

Proof: See [B2] for a proof of a) and b). As for c), suppose $g \in Z$. If $\lambda(g) < \omega_1$, then $g \notin \bigcup\mathcal{C}$. If $\lambda(g) = \omega_1$ and if $g \in B(f,n) \in \mathcal{C}$, then $f(\alpha) = g(\alpha)$ for each $\alpha < \omega_1$, and

$f(\omega_1)$ is one of the countably many numbers in Q. Hence g lies in at most countably many members of $\mathcal{C}$. Combining assertions a), b), and c) gives d). $\square$.

3.3) Lemma: *Suppose $f \in X$ and $n \geq 1$ are given. Then there is a function $g \in X$ such that:*

a) $\lambda(g) = \lambda(f) + \omega$

b) $B(g, 1)$ *is a proper subset of* $B(f, n)$.

Proof: Write $\lambda = \lambda(f)$ and choose an irrational number p such that $|f(\lambda) - p| < 1/n$. Define g on $[0, \lambda + \omega]$ by:

1) $g(\alpha) = f(\alpha)$ if $\alpha < \lambda$;
2) $g(\lambda) = p$;
3) $g(\alpha) = \pi$ if $\lambda < \alpha < \lambda + \omega$;
4) $g(\lambda + \omega) = 3.14$.

Then $B(g, 1) \subset B(f, n)$ as required, and the inclusion is proper because $f \in B(f, n) - B(g, 1)$. $\square$

3.4) Lemma: *The space Z does not have a point-countable base.*

Proof: For contradiction, suppose that $\mathcal{D}$ is a point- countable base of open sets for Z. Let $\{\mu_\alpha : \alpha < \omega_1\}$ be an increasing well-ordering of all limit ordinals less than ω_1. Define a function $f_0 : [0, \omega] \to R$ by the rule that $f_0(\alpha) = \pi$ for each $\alpha < \omega$ and $f(\omega) = 3.14$. Then $f_0 \in X$ has $\lambda(f_0) = \omega = \mu_0$. Choose any $D(0) \in \mathcal{D}$ with $f_0 \in D(0)$. Find n_0 such that $B(f_0, n_0) \subset D(0)$ and then use (3.3) to find a function $g_0 : [0, \lambda(f_0) + \omega] \to R$ such that $B(g_0, 1)$ is a proper subset of $B(f_0, n_0)$.

For induction hypothesis, suppose that $\alpha < \omega_1$ and that for each $\beta < \alpha$ we have defined functions $f_\beta,\ g_\beta$, an integer $m_\beta \geq 1$, and a set $D(\beta) \in \mathcal{D}$ such that:

1) $f_\beta,\ g_\beta \in X$ and $\lambda(g_\beta) = \lambda(f_\beta) + \omega > \lambda(f_\beta) \geq \mu_\beta$;
2) $N(g_\beta, 1) \subset N(f_\beta, m_\beta) \subset D(\beta)$, with each inclusion being proper;
3) if $\gamma < \beta < \alpha$ then $D(\beta) \subset N(g_\gamma, 1)$.

There are two cases to consider.

Case 1: Suppose that α is not a limit ordinal, say $\alpha = \beta + 1$. Then g_β is defined and we let $f_\alpha = g_\beta$. Then $f_\alpha \in N(g_\beta, 1)$ so that there is some $D(\alpha) \in \mathcal{D}$ such that $f_\alpha \in D(\alpha) \subset N(g_\beta, 1)$ with the last inclusion being proper. Choose m_α so that $N(f_\alpha, m_\alpha) \subset D(\alpha)$. Use Lemma (3.3) to find g_α with $\lambda(g_\alpha) = \lambda(f_\alpha) + \omega$ and $N(g_\alpha, 1) \subset N(f_\alpha, m_\alpha)$. Observe that $\lambda(g_\alpha) > \lambda(f_\alpha) = \lambda(g_\beta) = \mu_\beta + \omega = \mu_\alpha$. Thus, the induction continues across non-limit ordinals.

Case 2: Suppose α is a limit ordinal less than ω_1. Let $\lambda = \sup\{\lambda(f_\beta) : \beta < \alpha\}$. Then λ is a limit ordinal with $\lambda \geq \lambda(f_\beta) \geq \mu_\beta$ for each $\beta < \alpha$ so that $\lambda \geq \mu_\alpha$. Because $[0, \lambda] = \{\lambda\} \cup (\bigcup\{[0, \lambda(f_\beta)[: \beta < \alpha\})$, we may define $f_\alpha \in X$ by

a) $f_\alpha(\delta) = f_\beta(\delta)$ if $\delta < \lambda(f_\beta)$ for some $\beta < \alpha$;

b) $f_\alpha(\lambda) = 3.14$.

Observe that for each $\beta < \alpha$ we have $f_\alpha \in N(f_\alpha, 1) \subset N(g_\beta, 1) \subset D(\beta)$, with both inclusions being proper. Now find $D(\alpha) \in \mathcal{D}$ such that $f_\alpha \in D(\alpha) \subset N(f_\alpha, 1)$ and then an integer m_α such that $N(f_\alpha, m_\alpha) \subset D(\alpha)$. Use (3.3) to find $g_\alpha \in X$ such that $\lambda(g_\alpha) = \lambda(f_\alpha) + \omega$ and $N(g_\alpha, 1) \subset N(f_\alpha, m_\alpha) \subset D(\alpha)$. Thus, the induction continues across limit ordinals.

The above induction produces a subset $\{f_\alpha : \alpha < \omega_1\}$ of X and distinct sets $D(\alpha) \in \mathcal{D}$ for $\alpha < \omega_1$. Define $h : [0, \omega_1] \to R$ by

$g(\delta) = f_\alpha(\delta)$ whenever $\delta < \lambda(f_\alpha)$ and $\alpha < \omega_1$;

$g(\omega_1) = 3.14$.

Then $g \in Z$ and $g \in D(\alpha)$ for each $\alpha < \omega_1$. Because the sets $D(\alpha)$ are distinct members of $\mathcal{D}$, it follows that $\mathcal{D}$ cannot be point-countable in Z. $\square$

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