# Formulas for Chapter 4

## Table of Laplace Transforms

| $f(t)$ for $t \geq 0$ | $F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$ |
| $t^a$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$ |
| $\sin bt$ | $\frac{b}{s^2+b^2}$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}$ |
| $f'(t)$ | $sF(s) - f(0)$ |
| $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ |
| $f^{(n)}(t)$ | $s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0)$ |
| $t^n f(t)$ | $(-1)^n \frac{d^nF}{ds^n}(s)$ |
| $e^{at} f(t)$ | $F(s-a)$ |
| $u(t-a) = \begin{cases} 0 & t \leq a \\ 1 & t > a \end{cases}$ | $\frac{e^{-as}}{s}$ |
| $u(t-a)f(t-a)$ | $e^{-as}F(s)$ |
| $u(t-a)g(t)$ | $e^{-as}\mathcal{L}(g(t+a))$ |
| $\delta(t-a)$ | $e^{-as}$ |
| $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau$ | $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$ |
| If $f(t+T) = f(t)$ for all $t$ | $\mathcal{L}(f) = \frac{\left(\int_0^T e^{-s\tau} f(\tau)\, d\tau\right)}{\left(1 - e^{-Ts}\right)}$ |
| $\int_0^t f(\tau)\, d\tau$ | $\frac{1}{s}F(s)$ |

# Partial Fractions

These notes are concerned with decomposing rational functions

$$\frac{P(s)}{Q(s)} = \frac{a_M s^M + a_{M-1}s^{M-1} + \cdots + a_1 s + a_0}{s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}$$

Note: we can (without loss of generality assume that the coefficient of $s^N$ in the denominator is 1.

I. **Degree $\boldsymbol{P}(s) >$ Degree $\boldsymbol{Q}(s)$** In this case first carry out long division to obtain

$$\frac{P(s)}{Q(s)} = P_1(s) + \frac{P_2(s)}{Q(s)}$$

where Degree($P_2$) < Degree($Q$).

## II. Nonrepeated factors

If $Q(s) = (s - r_1)(s - r_2)\cdots(s - r_n)$ and $r_i \neq r_j$ for $i \neq j$

$$\frac{P(s)}{Q(s)} = \frac{A_1}{(s - r_1)} + \frac{A_2}{(s - r_2)} + \cdots + \frac{A_n}{(s - r_n)}$$

## III. Repeated Linear Factors

If $Q(s)$ contains a factor of the form $(s-r)^m$ then you must have the following terms

$$\frac{A_1}{(s-r)} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_m}{(s-r)^m}$$

## IV. A Nonrepeated Quadratic Factor

If $Q(s)$ contains a factor of the form $(s^2 - 2\alpha s + \alpha^2 + \beta^2) = (s-\alpha)^2 + \beta^2$ then you must have the following term

$$\frac{A_1 s + B_1}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)}$$

## V. Repeated Quadratic Factors

If $Q(s)$ contains a factor of the form $(s^2 - 2\alpha s + \alpha^2 + \beta^2)^m$ then you must have the following terms

$$\frac{A_1 s + B_1}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)} + \frac{A_2 s + B_2}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)^2} + \cdots + \frac{A_m s + B_m}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)^m}$$