$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

1. **Hyperbolic** if $B^2 - 4AC > 0$
2. **Parabolic** if $B^2 - 4AC = 0$
3. **Elliptic** if $B^2 - 4AC < 0$

Let $f(x)$ be a 2ℓ-periodic function for which $\int_{-\ell}^{\ell} |f(x)|\,dx$ exists. Then the Fourier series of $f(x)$ satisfies

$$\frac{(f(x^-) + f(x^+))}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(\mu_n x) + b_n \sin(\mu_n x)\big), \quad \mu_n = \frac{n\pi}{\ell},$$
$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\mu_n x)\,dx, \quad b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\mu_n x)\,dx \quad n = 0, 1, 2, \ldots$$

Fourier Cosine: Given f on $[0, \ell]$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx, a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi}{\ell}x\right)dx$$

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right)$$

Fourier Sine: Given f on $[0, \ell]$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right)$$

$\varphi'' = \lambda\varphi, \ 0 < x < \ell, \varphi(0) = 0, \ \varphi(\ell) = 0 \quad \Rightarrow$

$\mu_n = \frac{n\pi}{\ell}, \quad \lambda_n = -\mu_n^2, \quad \varphi_n(x) = \sqrt{2/\ell}\sin(\mu_n x)$

$\varphi'' = \lambda\varphi, \ 0 < x < \ell, \varphi'(0) = 0, \ \varphi'(\ell) = 0 \quad \Rightarrow$

$\lambda_0 = 0, \ \varphi_0 = 1/\sqrt{\ell}, \ \mu_n = \frac{n\pi}{\ell},$

$\lambda_n = -\mu_n^2, \quad \varphi_n(x) = \sqrt{2/\ell}\cos(\mu_n x)$

$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0$

$u(0,t) = 0, \quad u(\ell,t) = 0$

$u(x,0) = f(x)$

$\mu_n = \frac{n\pi}{\ell}, \quad \lambda_n = -\mu_n^2,$

$\varphi_n(x) = \sqrt{2/\ell}\sin(\mu_n x)$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{k\lambda_n t}\varphi_n(x)$$

$$c_n = \int_0^{\ell} f(x)\varphi_n(x)\,dx.$$

$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0$

$u_x(0,t) = 0, \quad u_x(\ell,t) = 0$

$u(x,0) = \varphi(x)$

$\lambda_0 = 0, \quad \varphi_0(x) = 1/\sqrt{\ell},$

$\mu_n = \frac{n\pi}{\ell}, \quad \lambda_n = -\mu_n^2,$

$\varphi_n(x) = \sqrt{2/\ell}\cos(\mu_n x)$

$$u(x,t) = c_0\varphi_0(x) + \sum_{n=1}^{\infty} c_n e^{k\lambda_n t}\varphi_n(x)$$

$$c_n = \int_0^{\ell} \varphi(x)\varphi_n(x)\,dx.$$

$$u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0$$

$$u(0,t) = 0, \quad u_x(\ell,t) = 0$$

$$u(x,0) = \varphi(x)$$

$$\mu_n = \frac{(2n-1)\pi}{2\ell}, \quad \lambda_n = -\mu_n^2, \quad \varphi_n(x) = \sqrt{2/\ell}\,\sin(\mu_n x)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{k\lambda_n t} \varphi_n(x) \quad c_n = \int_0^\ell \varphi(x)\varphi_n(x)\,dx.$$

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0$$

$$u(0,t) = 0, \quad u(\ell,t) = 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\mu_n\, ct\right) + b_n \sin\left(\mu_n\, ct\right)\right) \varphi_n(x),$$

$$\mu_n = \frac{n\pi}{\ell}, \quad \lambda_n = -\mu_n^2, \quad \varphi_n(x) = \sqrt{\frac{2}{\ell}}\,\sin(\mu_n x)$$

$$a_n = \int_0^\ell f(x)\varphi_n(x)\,dx, \quad b_n = \frac{1}{\mu_n c}\int_0^\ell g(x)\varphi_n(x)\,dx.$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right), \qquad \sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right),$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right).$$

With $\mu_n = \frac{n\pi}{\ell}$ we have

$$\int_{-\ell}^{\ell} \cos^2(\mu_n x)\,dx = \ell$$

$$\int_{-\ell}^{\ell} \sin^2(\mu_n x)\,dx = \ell$$

$$\int_{-\ell}^{\ell} \cos(\mu_n x)\cos(\mu_m x)\,dx = \frac{1}{2}\int_{-\ell}^{\ell} [\cos((\mu_n+\mu_m)x) + \cos((\mu_n-\mu_m)x])\ dx = 0$$

$$\int_{-\ell}^{\ell} \sin(\mu_n x)\sin(\mu_m x)\,dx = \frac{1}{2}\int_{-\ell}^{\ell} [\cos((\mu_n-\mu_m)x) - \cos((\mu_n-\mu_m)x)]\ dx = 0$$

$$\int_{-\ell}^{\ell} \sin(\mu_n x)\cos(\mu_m x)\,dx = \frac{1}{2}\int_{-\ell}^{\ell} [\sin((\mu_n+\mu_n)x) + \sin((\mu_n-\mu_n)x)]\ dx = 0$$