Some Comments on Sturm-Liouville Problems

Consider the following Sturm-Liouville Boundary Eigenvalue Problem on a finite interval $a < x < b$

$$\begin{aligned}
&\varphi''(x) = \lambda\varphi(x) \\
&\varphi'(a) - k_0\varphi(a) = 0,\ \ k_0 > 0 \\
&\varphi'(b) + k_1\varphi(b) = 0,\ \ k_1 > 0
\end{aligned}$$

1. **Eigenvalues Less Than or Equal Zero** For this regular Sturm-Liouville eigenvalue problem we show that the eigenvalues λ must satisfy $\lambda \le 0$.

   Indeed we have the following calculation. Multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

   $$\begin{aligned}
   \lambda\|\varphi\|^2 = \lambda\int_0^\ell \varphi(x)^2\,dx &= \int_0^\ell \varphi(x)\varphi''(x)\,dx = -\int_0^\ell \varphi'(x)^2\,dx + \varphi(x)\varphi'(x)\Big|_0^\ell \\
   &= \|\varphi'\|^2 - k_1\varphi(\ell)^2 - k_0\varphi(0)^2 \text{ which is } \begin{cases} < 0 & \text{if } k_0, k_1 > 0 \\ \le 0 & \text{if } k_0, k_1 = 0 \end{cases}
   \end{aligned}$$

   So λ is strictly negative unless both k_0 and $k_1 = 0$ (i.e., Neumann BCs) and in this case again the right hand side is negative unless $\|\varphi'\| = 0$. Notice that if

   $$\|\varphi'\|^2 = \int_0^\ell \varphi'(x)^2\,dx = 0$$

   then $\varphi'(x) = 0$ for all x if φ' is continuous. This implies $\varphi = C$. So we would have $\lambda_0 = 0$ and the normalized eigenfunction would be $\varphi_0 = 1/\sqrt{\ell}$.

2. **Eigenfunctions for Distinct Eigenvalues are Orthogonal** Assume that $\varphi_j'' = \lambda\varphi_j$ for $j = 1, 2$ with $\lambda_1 \ne \lambda_2$. We want to show that

   $$\langle\varphi_1, \varphi_2\rangle = \int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0.$$

   To this end we first show that $(\varphi_1'(x)\varphi_2(x) - \varphi_1(x)\varphi_2'(x))\Big|_{x=a}^{x=b} = 0$. We have

   $$\begin{aligned}
   [\varphi_1'(x)\varphi_2(x) - \varphi_1(x)\varphi_2'(x)]\Big|_{x=a}^{x=b} &= [\varphi_1'(b)\varphi_2(b) - \varphi_1(b)\varphi_2'(b)] - [\varphi_1'(a)\varphi_2(a) - \varphi_1(a)\varphi_2'(a)] \\
   &= [-k_1\varphi_1(b)\varphi_2(b) + k_1\varphi_1(b)\varphi_2(b)] - [k_0\varphi_1(a)\varphi_2(a) - k_0\varphi_1(a)\varphi_2(a)] = 0
   \end{aligned}$$

   Now we have

   $$\begin{aligned}
   \lambda_1\langle\varphi_1, \varphi_2\rangle = \langle\lambda_1\varphi_1, \varphi_2\rangle = \langle\varphi_1'', \varphi_2\rangle &= -\langle\varphi_1', \varphi_2'\rangle + \varphi_1(x)\varphi_2(x)\Big|_{x=a}^{x=b} \\
   &= \langle\varphi_1, \varphi_2''\rangle + [\varphi_1'(x)\varphi_2(x) - \varphi_1(x)\varphi_2'(x)]\Big|_{x=a}^{x=b} \\
   &= \langle\varphi_1, \varphi_2''\rangle = \langle\varphi_1, \lambda_2\varphi_2\rangle = \lambda_2\langle\varphi_1, \varphi_2\rangle.
   \end{aligned}$$

   So we have $\lambda_1\langle\varphi_1, \varphi_2\rangle = \lambda_2\langle\varphi_1, \varphi_2\rangle$ or $(\lambda_2 - \lambda_1)\lambda_2\langle\varphi_1, \varphi_2\rangle = 0 \Rightarrow \lambda_2\langle\varphi_1, \varphi_2\rangle = 0$.