Eigenvalues & Eigenvectors for Periodic Boundary Conditions

The main idea of these notes is to give you a hand doing one of the homework problems.

Consider the eigenvalue problem

$$y''(x) = \lambda y(x), \qquad y(0) = y(\ell), \quad y'(0) = y'(\ell)$$

1. You show that $\lambda = 0$ is an eigenvalue and find the normalized eigenfunction.
2. For $\lambda = \mu^2 < 0$, our ODE becomes $y'' - \mu^2 y = 0$ which implies $y = c_1 e^{\mu x} + c_2 e^{-\mu x}$. Now we try to satisfy the BCs.

   $$y(0) = y(\ell) \quad \Rightarrow \quad c_1 + c_2 = c_1 e^{\mu\ell} + c_2 e^{-\mu\ell}$$

   or

   $$(1 - e^{\mu\ell})c_1 + (1 - e^{-\mu\ell})c_2 = 0.$$

   Then

   $$y'(0) = y'(\ell) \quad \Rightarrow \quad \mu c_1 - \mu c_2 = c_1 \mu e^{\mu\ell} - \mu c_2 e^{-\mu\ell}$$

   or, dividing by μ,

   $$(1 - e^{\mu\ell})c_1 + (1 + e^{-\mu\ell})c_2 = 0.$$

   For this system of equations to have a non-zero solution we would need the determinant of the coefficient matrix would need to be zero. That is, we would need to find a μ so that this determinant is zero.

   $$\begin{vmatrix} (1 - e^{\mu\ell}) & (1 - e^{-\mu\ell}) \\ (1 - e^{\mu\ell}) & (1 + e^{-\mu\ell}) \end{vmatrix} = e^{-\mu\ell}(1 - e^{\mu\ell})$$

   and this is only zero when $\mu = 0$. So $\lambda = \mu^2 > 0$ cannot be an eigenvalue for any μ.
3. For $\lambda = -\mu^2 < 0$, I will help you show that there is a infinite set of numbers $\{\mu_n\}_{n=1}^{\infty}$ giving eigenvalues $\lambda_n = -\mu_n^2$. The main difference with this case and the Regular Sturm-Liouville case is that these eigenvalues have multiplicity two.

   The means that for each eigenvalue λ_n there are two (normalized) linearly independent eigenfunctions $y_n^1(x)$ and $y_n^1(x)$ satisfying

   $$(y_n^j)''(x) = \lambda_n y_n^j(x), \qquad y_n^j(0) = y_n^j(\ell), ; (y_n^j)'(0) = (y_n^j)'(\ell), \quad j = 1, 2.$$

   In fact these eigenfunctions are orthonormal, i.e.,

   $$\int_0^\lambda y_n(x) y_m(x)\, dx = \delta_{n,m}.$$

   To see this we note that the ODE for this case is $y'' + \mu^2 y = 0$ and the general solution is

   $$y(x) = a\sin(\mu x) + b\cos(\mu x).$$

We see that

$$y(0) - y(\ell) = 0 \quad \Rightarrow \quad b - (a\sin(\mu\ell) + b\cos(\mu\ell)) = 0$$

and

$$y'(0) - y'(\ell) = 0 \quad \Rightarrow \quad a\mu - (a\mu\cos(\mu\ell) - b\mu\sin(\mu\ell)) = 0$$

or

$$\begin{pmatrix} -\sin(\mu\ell) & 1-\cos(\mu\ell) \\ \mu(1-\cos(\mu\ell)) & \mu\sin(\mu\ell) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We obtain a nontrivial solution if

$$\begin{vmatrix} -\sin(\mu\ell) & 1-\cos(\mu\ell) \\ \mu(1-\cos(\mu\ell)) & \mu\sin(\mu\ell) \end{vmatrix} = -\mu\sin^2(\mu\ell) - \mu(1-\cos(\mu\ell))^2 = 0.$$

Expanding the second term and simplifying we arrive at $\cos(\mu\ell) = 1$ and so

$$\mu = \frac{2n\pi}{\ell}, \quad n = 1, 2, \cdots$$

In this case a, b are arbitrary and so to each eigenvalue

$$\lambda_n = \left(\frac{2n\pi}{\ell}\right) \quad n = \pm 1, \pm 2, \cdots$$

we have

$$y_n(x) = a_n \sin\left(\frac{2n\pi}{\ell}x\right) + b_n \cos\left(\frac{2n\pi}{\ell}x\right)$$

where a and b are arbitrary. So, for example we could take $a = C$ and $b = 0$ to get eigenfunctions $C\sin\left(\frac{2n\pi}{\ell}x\right)$ or we could take $a = 0$ and $b = C$ to get eigenfunctions $C\cos\left(\frac{2n\pi}{\ell}x\right)$. From this we conclude that for each eigenvalue there are two linearly independent eigenfunctions.