Section 12.6: Non-homogeneous Problems

1 Introduction

Up to this point all the problems we have considered are we what we call homogeneous problems. This means that for an interval $0 < x < \ell$ the problems were of the form

$$\boxed{\begin{aligned} &u_t(x,t) = ku_{xx}(x,t), \\ &\mathcal{B}_0(u) = 0, \quad \mathcal{B}_1(u) = 0 \\ &u(x,0) = f(x) \end{aligned}}$$

In contrast, in Section we are concerned with some non-homogeneous cases:

$$\boxed{\begin{aligned} &u_t(x,t) = ku_{xx}(x,t) + R(x), \\ &\mathcal{B}_0(u) = \gamma_0, \quad \mathcal{B}_1(u) = \gamma_1 \\ &u(x,0) = f(x) \end{aligned}}$$

where γ_j are constants and R is a function of x but not t. I would like to do the case where it is a function of t but the book does not and there is no time.

Here we have used the notation $\mathcal{B}_j(u)$ to indicate our usual boundary conditions

$$\mathcal{B}_0(u) = \alpha_0 u_x(0,t) + \alpha_1 u(0,t), \quad \mathcal{B}_1(u) = \beta_0 u_x(\ell,t) + \beta_1 u(\ell,t).$$

Specifically then for Dirichlet boundary conditions we have $\mathcal{B}_0(u) = u(0,t)$, $\mathcal{B}_1(u) = u(\ell,t)$ and for Neumann conditions we have $\mathcal{B}_0(u) = u_x(0,t)$, $\mathcal{B}_1(u) = u_x(\ell,t)$.

1.1 Non-Homogeneous Equation, Homogeneous Dirichlet BCs

We first show how to solve a non-homogeneous heat problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned} &u_t(x,t) = ku_{xx}(x,t) + R(x), \quad 0 < x < \ell, \quad t > 0 \qquad (1) \\ &u(0,t) = 0, \quad u(\ell,t) = 0 \\ &u(x,0) = f(x) \end{aligned}$$

Let us recall from all our examples involving Dirichlet BC the Sturm-Liouville problem gives

$$\lambda_n = -\mu_n^2, \quad \mu_n = \frac{n\pi}{\ell}, \quad \varphi_n(x) = \sqrt{\frac{2}{\ell}}\sin(\mu_n x).$$

To solve this problem we look for a function $\psi(x)$ so that the change of dependent variables $u(x,t) = v(x,t) + \psi(x)$ transforms the non-homogeneous problem into a homogeneous problem. In particular we have

$$R = u_t - ku_{xx} = (v+\psi)_t - k(v+\psi)_{xx} = v_t - kv_{xx} - k\psi''.$$

So if we want $v_t - kv_{xx} = 0$ then we need

$$\psi'' = -\frac{1}{k}R.$$

In addition we want ψ to satisfy the BCs so we look for ψ satisfying:

$$\begin{aligned}\psi''(x) &= -\frac{1}{k}f(x), \quad 0 < x < \ell,\\ \psi(0) &= 0, \quad \psi(\ell) = 0.\end{aligned}$$

Notice this is a non-homogeneous second order constant coefficient boundary value problem.

At $t = 0$ we have $v(x,0) = u(x,0) - \psi(x) = f(x) - \psi(x)$ so the heat problem for v has a different initial condition.

Thus, in order to solve (1) we need to solve the following:

$$\begin{aligned}\psi''(x) &= -\frac{1}{k}f(x), \quad 0 < x < \ell,\\ \psi(0) &= 0, \quad \psi(\ell) = 0.\end{aligned}$$

and then

$$\begin{aligned}v_t(x,t) &= kv_{xx}(x,t),\\ v(x,0) &= 0, \quad v(\ell,t) = 0\\ v(x,0) &= f(x) - \psi(x) \equiv v_0(x)\end{aligned}$$

to obtain

$$u(x,t) = v(x,t) + \psi(x).$$

We already know how to solve the heat problem:

$$\begin{aligned}v(x,t) &= \sum_{n=1}^{\infty} c_n e^{k\lambda_n t}\varphi_n(x),\\ c_n &= \int_0^{\ell} v_0(x)\varphi_n(x)\,dx.\end{aligned}$$

Before we turn to finding ψ let us make a very important remark.

Remark 1.1. The solution v approaches zero as t goes to infinity which means that u converges to a non-constant *steady state*, i.e., the solution consists of two parts which are often referred to as the *transient* and the *steady state.*

$$u(x,t) = v(x,t) + \psi(x) \to \psi(x) \quad \text{as} \quad t \to \infty.$$

This is a very important property of the heat equation. Solutions tend to a equilibrium solution., i.e., a solution for which $u_t(x,t) = 0$ which means a solution of

$$0 = ku_{xx} + R(x), \quad u(x,t) = 0, \quad u(\ell,t) = 0.$$

Example 1.1. Find the steady state solution for the heat problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t) - 6x, \quad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \quad u(1,t) = 0 \\ &u(x,0) = \varphi(x) \end{aligned}$$

As described in the remark the steady state problem is obtained by setting $u_t = 0$ and solving the non-homogeneous BVP

$$\begin{aligned} &\psi''(x) = 6x, \quad 0 < x < 1, \\ &\psi(0) = 0, \quad \psi(1) = 0 \end{aligned}$$

For this problem we apply the techniques from an elementary ODE class. Namely, we know that the general solution is the sum of the general solution of the homogenous problem ψ_h and any particular solution ψ_p. The general solution of the homogeneous problem $\psi''(x) = 0$ is $\psi_h(x) = c_1 x + c_2$ and it is clear that $\psi_p(x) = x^3$ is a particular solution. *N.B. Remember we learned two methods to find a particular solution: Undetermined Coefficients, Variation of Parameters.*

So we have $\psi(x) = \psi_h(x) + \psi_p(x) = c_1 x + c_2 + x^3$. Now we try to find c_1 and c_2 so that the boundary conditions are satisfied. We need

$$0 = \psi(0) = c_2, \quad \text{and} \quad 0 = \psi(1) = c_1 + 1^3$$

which implies $c_1 = -1$ and

$$\psi(x) = x^3 - x.$$

Thus for every initial condition $\varphi(x)$ the solution $u(x,t)$ to this forced heat problem satisfies

$$\lim_{t\to\infty} u(x,t) = \psi(x).$$

More generally we can give a formula for the solution of the steady state problem

$$\psi'' = -\frac{1}{k}R, \quad \psi(0) = 0, \quad \psi(\ell) = 0$$

using the method of variation of parameters.

First the general solution of the homogeneous problem $\psi'' = 0$ is $\psi_h = c_1 + c_2 x$. So we next need to find a particular solution y_p and then the general solution is $y = y_h + y_p$.

The method of Variation of Parameters is used to give a particular solution y_p for a problem $y'' + P(x)y' + Q(x)y = R(x)$. It requires having a pair of linearly independent solutions of the homogeneous problem, y_1 and y_2. The formula is

$$\boxed{y_p(x) = -y_1(x) \int^t \frac{y_2(s)R(s)}{W(s)}\, ds + y_2(x) \int^t \frac{y_1(s)R(s)}{W(s)}\, ds, \qquad W(s) = \det \begin{bmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{bmatrix}}$$

In our case we take $y_1 = 1$ and $y_2 = x$ and notice that the right hand side has $9 - 1/k)r(x)$ so we have

$$W(s) = \det \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = 1.$$

$$y_p(x) = \frac{1}{k} \left(\int_0^x sR(s)\, ds - x \int_0^x R(s)\, ds \right).$$

Then the general solution is

$$\psi(x) = \frac{1}{k} \left(\int_0^x sR(s)\, ds - x \int_0^x R(s)\, ds \right) + c_1 + c_2 x.$$

Next we apply the BCs to find c_1 and c_2.

$$0 = \psi(0) = c_1 \quad c_1 \Rightarrow 0.$$

Next

$$0 = \psi(\ell) = \frac{1}{k} \left(\int_0^\ell sR(s)\, ds - \ell \int_0^x R(s)\, ds \right) + c_2 \ell$$

This implies

$$c_2 = \frac{1}{k\ell} \left(-\int_0^\ell sR(s)\, ds + \ell \int_0^\ell R(s)\, ds \right)$$

so we have

$$\psi(x) = \frac{1}{k} \left(\int_0^x sR(s)\, ds - x \int_0^x R(s)\, ds + \frac{x}{\ell} \left(-\int_0^\ell sR(s)\, ds + \ell \int_0^\ell R(s)\, ds \right) \right).$$

This can be considerably simplified by factoring out an ℓ to obtain

$$\psi(x) = \frac{1}{k\ell} \left(\ell \int_0^x sR(s)\, ds - x\ell \int_0^x R(s)\, ds - x \int_0^\ell sR(s)\, ds + x\ell \int_0^\ell R(s)\, ds \right).$$

in the denominator then combining the first and third integrals and the second and fourth integrals. We have

$$\ell \int_0^x sR(s)\, ds - x \int_0^\ell sR(s)\, ds = (\ell - x) \int_0^x sR(s)\, ds + x \int_x^\ell sR(s)\, ds$$

and

$$-x\ell \int_0^x R(s)\,ds + x\ell \int_0^\ell R(s)\,ds = x\ell \int_x^\ell R(s)\,ds.$$

Combining these results we have

$$\psi(x) = \frac{1}{k\ell}\left((\ell - x)\int_0^x s\,R(s)\,ds + x\int_x^\ell (\ell - s)R(s)\,ds\right).$$

Collecting these results we have the following results.

$$\psi''(x) = -\frac{1}{k}R(x), \quad 0 < x < \ell,$$
$$\psi(0) = 0, \quad \psi(\ell) = 0.$$

has solution

$$\psi(x) = \frac{1}{k\ell}\left((\ell - x)\int_0^x s\,R(s)\,ds + x\int_x^\ell (\ell - s)R(s)\,ds\right).$$

1.2 Non-homogeneous Dirichlet Boundary Conditions

In this section we consider forcing through Dirichlet boundary conditions

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= \gamma_0, \quad u(\ell,t) = \gamma_1 \\ u(x,0) &= f(x) \end{aligned} \tag{2}$$

In order to obtain a continuous solution we also need to impose the compatibility conditions

$$f(0) = \gamma_0, \quad f(\ell) = \gamma_1.$$

Our method to solve this problem is to transform it to a homogeneous problem like we did in the previous section. In order to do this we introduce the function

$$h(x) = \gamma_0 + \frac{x}{\ell}(\gamma_1 - \gamma_0).$$

Then we introduce a new function $v(x,t)$ by

$$w(x,t) = u(x,t) - h(x).$$

Our goal is to see what problem $w(x,t)$ satisfies. To this end we note that

$$h_{xx}(x) = 0 \quad h_t(x) = 0$$

We see that

$$w_t - kw_{xx} = (u(x,t) - h(x))_t - k\,(u(x,t) - h(x))_{xx} = 0$$

and

$$w(0,t) = u(0,t) - h(0) = \gamma_0 - \gamma_0 = 0, \qquad w(\ell,t) = u(\ell,t) - h(\ell) = \gamma_1 - \gamma_1 = 0.$$

$$w(x,0) = u(x,0) - h(x) = f(x) - \left(\gamma_0 + \frac{x}{\ell}(\gamma_1 - \gamma_0)\right) \equiv w_0(x). \tag{3}$$

Collecting this information we find that $w(x,t)$ satisfies

$$\begin{aligned} &w_t(x,t) = kw_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ &w(0,t) = 0, \quad w(\ell,t) = 0 \\ &w(x,0) = w_0(x) \end{aligned} \tag{4}$$

So we can apply our earlier results to obtain a formula for $w(x,t)$.

Once we do this (see below) we can obtain the desired solution from

$$u(x,t) = w(x,t) + h(x).$$

Then for the initial condition we compute

$$w_0(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x). \tag{5}$$

where

$$c_n = \int_0^{\ell} \varphi_n(x) w_0(x)\,dx. \tag{6}$$

Combining these results we obtain

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{k\lambda_n t} \varphi_n(x) \tag{7}$$

and finally

$$u(x,t) = w(x,t) + h(x).$$

Notice in this case, just as in the case of the non-homogeneous equation, that as $t \to \infty$ all the exponential terms in the sum tend to zero and we have

$$\lim_{t\to\infty} u(x,t) = h(x).$$

This represents a nonzero and non constant steady state temperature profile.

1.3 More General Non-homogeneous Boundary Conditions

In this section we consider forcing through Neumann boundary conditions.

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= \gamma_0, \quad u_x(\ell,t) + k_1 u(\ell,t) = \gamma_1 \\ u(x,0) &= f(x) \end{aligned} \tag{8}$$

The primary difference between this problem and that considered in the previous section (i.e., (2)) is that we need a different function h.

In order to find an appropriate function h let us examine the properties we desire. We want a function that satisfy the following conditions:

$$h_t = 0, \quad h_{xx} = 0, \quad h(0) = \gamma_0, \quad h_x(\ell) + k_1 h(\ell) = \gamma_1.$$

We see that this first suggests h not depend on t and since $h_{xx} = 0$ we would have $h(x) = c_1 x + c_2$. Applying the boundary conditions we would have

$$\gamma_0 = h(0) = c_2, \quad \gamma_1 = h'(\ell) + k_1 h(\ell) = c_1(1 + k_1 \ell) + k_1 c_2.$$

This implies

$$c_2 = \gamma_0, \quad c_1 = \frac{(\gamma_1 - k_1\gamma_0)}{(1 + k_1\ell)}.$$

So we have

$$h(x) = \frac{(\gamma_1 - k_1\gamma_0)}{(1 + k_1\ell)} x + \gamma_0.$$

Next we make a change of variables $u(x,t) = w(x,t) + h(x)$ and we find

$$\begin{aligned} w_t(x,t) &= k w_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ w_x(0,t) &= 0, \quad w_x(\ell,t) + k_1 w(\ell,t) = 0 \\ w(x,0) &= w_0(x) = f(x) - h(x). \end{aligned} \tag{9}$$

Notice once again that the initial condition w_0 is not just the original initial condition and we obtain

$$u(x,t) = w(x,t) + h(x).$$

The only thing that remains is to solve for w and we do this using an eigenfunction expansion. In this case, due to the boundary conditions, we have Sturm-Liouville problem

$$\varphi'' = \lambda\varphi, \quad \varphi(0) = 0, \quad \varphi'(\ell) + k_1\varphi(\ell) = 0.$$

We can readily show that the eigenvalues are negative since

$$\lambda\|\varphi\|^2 = \int_0^\ell \varphi''(x)\varphi(x)\,dx = -\|\varphi'\|^2 + \varphi'(x)\varphi(x)\Big|_0^\ell = -\|\varphi'\|^2 - k_1\varphi(\ell)^2.$$

So $\lambda = -\mu^2$ and $\varphi(x) = a\cos(\mu x) + b\sin(\mu x)$. To satisfy the first boundary condition we need

$$0 = \varphi(0) = a \quad \Rightarrow \quad a = 0, \quad \Rightarrow \quad \varphi(x) = b\sin(\mu x).$$

Then we need

$$0 = vp'(\ell) + k_1\varphi(\ell) = b\mu\cos(\mu\ell) + k_1 b\sin(\mu\ell),$$

which implies

$$\tan(\mu\ell) = -\frac{\mu}{k_1}.$$

This equation has infinitely solutions which diverge to infinity but we cannot solve for them in closed form. We denote them, the associated eigenvalues and the normalized eigenfunctions by

$$\mu_n, \quad \lambda_n = -\mu_n^2, \quad \varphi_n(x) = \kappa_n \sin(\mu_n x).$$

Then the solution of our heat problem for w is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{k\lambda_n t}\varphi_n(x), \qquad \text{where} \quad c_n = \int_0^\ell w_0(x)\varphi_n(x)\,dx.$$

Now, once again, since the $\lambda_n < 0$ (and tend to minus infinity) we have

$$\sup_{x\in[0,\ell]} |w(x,t)| \xrightarrow{t\to\infty} 0.$$

Therefore we have

$$u(x,t) = w(x,t) + h(x) \xrightarrow{t\to\infty} h(x).$$

1.4 Heat Equation with Conduction and Convection

Another variation on the heat equation is to add extra terms that correspond to heat conduction and convection.

$$\begin{aligned}
u_t(x,t) &= k\big(u_{xx}(x,t) - 2au(x,t)_x + bu(x,t)\big), \quad 0 < x < \ell, \quad t > 0 && (10)\\
u(0,t) &= 0, && (11)\\
u(\ell,t) &= 0, && (12)\\
u(x,0) &= \varphi(x). && (13)
\end{aligned}$$

There are many different ways to approach this problem. One such method would be to apply separation of variable directly. The dissadvantange to this is that one gets a more complicated ode for $\varphi(x)$ and there is a more difficult analysis of the eigenvalues and eigenvectors.

We will take a different approach which allows us to use our earlier work after a change of dependent variables. So to this end let us define $v(x,t)$ via

$$u(x,t) = e^{ax+\beta t}v(x,t), \qquad \beta = k(b - a^2). \tag{14}$$

Thus we have

$$v(x,t) = e^{-(ax+\beta t)}u(x,t)$$

and we can compute

$$\begin{aligned}
v_t - kv_{xx} &= e^{-(ax+\beta t)}\left(-\beta u + u_t\right) - k\left[e^{-(ax+\beta t)}\left(-au+u_x\right)\right]_x \\
&= e^{-(ax+\beta t)}\left\{(-\beta u + u_t) - k\left[-a(-au+u_x) + (-au_x + u_{xx})\right]\right\} \\
&= e^{-(ax+\beta t)}\left[u_t - k(u_{xx} - 2au_x + a^2 u) + \beta u\right] \\
&= e^{-(ax+\beta t)}\left[u_t - k(u_{xx} - 2au_x + a^2 u + (b-a^2)u)\right] \\
&= e^{-(ax+\beta t)}\left[u_t - k(u_{xx} - 2au_x + bu)\right] = 0.
\end{aligned}$$

Furthermore

$$v(0,t) = e^{-\beta t}u(0,t) = 0, \qquad v(\ell,t) = e^{-(a\ell+\beta t)}u(\ell,t) = 0$$

and

$$v(x,0) = e^{-ax}u(x,0) = e^{-ax}\varphi(x).$$

Therefore, $v(x,t)$ is the solution of

$$\begin{aligned}
v_t(x,t) &= kv_{xx}(x,t) \\
v(0,t) &= 0, \qquad v(\ell,t) = 0 \\
v(x,0) &= e^{-ax}\varphi(x).
\end{aligned}$$

We have eigenvalues and eigenfunctions

$$\mu_n = \left(\frac{n\pi}{\ell}\right), \quad \lambda_n = -\mu_n^2, \quad \varphi_n(x) = \sqrt{\frac{2}{\ell}}\sin\left(\mu_n x\right)$$

and we obtain the solution to this problem as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right) \quad \text{with} \;\; b_n = \int_0^\ell e^{-ax}\varphi(x)\varphi_n(x)\,dx.$$

Finally our solution to (10)-(13) can be written as

$$u(x,t) = e^{ax+\beta t}\sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right).$$