Solving PDEs using Laplace Transforms

Given a function $u(x,t)$ defined for all $t>0$ and assumed to be bounded. we can apply the Laplace transform in t considering x as a parameter.

$$L(u(x,t)) = \int_0^\infty e^{-st}u(x,t)\,dt \equiv U(x,s)$$

In applications to PDEs we need the following:

$$L(u_t(x,t) = \int_0^\infty e^{-st}u_t(x,t)\,dt = e^{-st}u(x,t)\big|_0^\infty + s\int_0^\infty e^{-st}u(x,t)\,dt = sU(x,s)-u(x,0)$$

so we have

$$L(u_t(x,t) = sU(x,s)-u(x,0)$$

In exactly the same way we obtain

$$L(u_{tt}(x,t) = s^2U(x,s)-su(x,0)-u_t(x,0).$$

We also need the corresponding transforms of the x derivatives:

$$L(u_x(x,t)) = \int_0^\infty e^{-st}u_x(x,t)\,dt = U_x(x,s)$$

$$L(u_{xx}(x,t)) = \int_0^\infty e^{-st}u_{xx}(x,t)\,dt = U_{xx}(x,s)$$

Consider the following examples.

Example 1.

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t} = x, \quad x>0, \quad t>0,$$

with boundary and initial condition

$$u(0,t)=0 \quad t>0, \qquad \text{and} \quad u(x,0)=0, \quad x>0.$$

As above we use the notation $U(x,s)=L(u(x,t))(s)$ for the Laplace transform of u.

Then applying the Laplace transform to this equation we have

$$\frac{dU}{dx}(x,s)+sU(x,s)-u(x,0)=\frac{x}{s} \quad \Rightarrow \quad \frac{dU}{dx}(x,s)+sU(x,s)=\frac{x}{s}.$$

This is a constant coefficient first order ODE. We solve it by finding the integrating factor

$$\mu = e^{\int s\,dx} = e^{sx}$$

Thus we have

$$\frac{d}{dx}\left[e^{sx}U(x,s)\right] = e^{sx}\frac{x}{s}.$$

We integrate both sides to get

$$U(x,s) = \frac{e^{-sx}}{s}\left(\int e^{sr}r\,dr\right)+Ce^{-sx}.$$

We can use integration by parts to evaluate the integral:

$$\begin{aligned}\int e^{sx} x\, dx &= \int \left(\frac{e^{sx}}{s}\right)' x\, dx \\ &= \frac{xe^{sx}}{s} - \int \left(\frac{e^{sx}}{s}\right) dx \\ & \frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}.\end{aligned}$$

So we have

$$U(x,s) = \frac{e^{-sx}}{s}\left(\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}\right) + Ce^{-sx} = \frac{x}{s^2} - \frac{1}{s^3} + Ce^{-sx}.$$

We can evaluate the constant C using the boundary condition

$$0 = U(0,s) = -\frac{1}{s^3} + C \quad \Rightarrow \quad C = \frac{1}{s^3}$$

so we have

$$U(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{e^{-sx}}{s^3}.$$

Taking the inverse Laplace transform we have

$$u(x,t) = xt - \frac{t^2}{2} + H(t-x)\frac{(t-x)^2}{2}$$

where H is the unit step function (or Heaviside function)

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}.$$

Example 2.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = 0, \quad x > 0, \quad t > 0,$$

with boundary and initial condition

$$u(0,t) = 0 \;\; t > 0, \qquad \text{and} \;\; u(x,0) = \sin(x), \quad x > 0.$$

As above we use the notation $U(x,s) = L(u(x,t))(s)$ for the Laplace transform of u. Then applying the Laplace transform to this equation we have

$$\frac{dU}{dx}(x,s) + sU(x,s) - u(x,0) + U(x,s) = 0 \quad \Rightarrow \quad \frac{dU}{dx}(x,s) + (s+1)U(x,s) = \sin(x).$$

This is a constant coefficient first order linear ODE. We solve it by finding the integrating factor

$$\mu = e^{\int (s+1)dx} = e^{(s+1)x}$$

Thus we have

$$\frac{d}{dx}\left[e^{(s+1)x}U(x,s)\right] = e^{(s+1)x}\sin(x).$$

We integrate both sides to get

$$U(x,s) = e^{-(s+1)x}\left(\int e^{(s+1)r}\sin(r)\, dr\right) + Ce^{-(s+1)x}.$$

We can use integration by parts to evaluate the integral:

$$e^{-(s+1)x}\left(\int_0^x e^{(s+1)r}\sin(r)\,dr\right) = \frac{(s+1)\sin(x)-\cos(x)+e^{-(s+1)x)}}{s^2+2s+2}.$$

So we have

$$U(x,s) = \frac{(s+1)\sin(x)-\cos(x)+e^{-(s+1)x)}}{s^2+2s+2} + Ce^{-(s+1)x}.$$

We can evaluate the constant C using the boundary condition

$$0 = U(0,s) = \frac{-1+1}{s^2+2s+2} + C \quad \Rightarrow \quad C = 0.$$

So we have

$$U(x,s) = \frac{(s+1)\sin(x)-\cos(x)+e^{-(s+1)x)}}{s^2+2s+2}.$$

Taking the inverse Laplace transform we have

$$u(x,t) = e^{-t}\cos(t)\sin(x) - e^{-t}\sin(t)\cos(t) + e^{-t}H(t-x)\sin(t-x)$$

This can be written as

$$u(x,t) = e^{-t}\left[\sin(x-t) + H(t-x)\sin(t-x)\right].$$

Example 3.

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0<x<2, \quad t>0,\\ u(0,t) &= 0, \quad u(2,t)=0\\ u(x,0) &= 3\sin(2\pi x).\end{aligned}$$

Take the Laplace transform and apply the initial condition

$$\frac{d^2U}{dx^2}(x,s) = sU(x,s) - u(x,0) = sU(x,s) - 3\sin(2\pi x).$$

We write this equation as a non-homogeneous, second order linear constant coefficient equation for which we can apply the methods from Math 3354.

$$\frac{d^2U}{dx^2}(x,s) - sU(x,s) = -3\sin(2\pi x).$$

The general solution can be written as

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

where $U_h(x,s)$ is the general solution of the homogeneous problem

$$U_h(x,s) = c_1e^{\sqrt{s}x} + c_2e^{-\sqrt{s}x}$$

and $U_p(x,s)$ is any particular solution of the non-homogeneous problem

$$U_p(x,s) = A\cos(2\pi x) + B\sin(2\pi x).$$

We first use the method of undetermined coefficients to find A and B. To this end we have

$$\frac{d}{dx}U_p(x,s) = -2\pi A\sin(2\pi x) + 2\pi B\cos(2\pi x),$$

$$\frac{d^2}{dx^2}U_p(x,s) = -(2\pi)^2 A\cos(2\pi x) + (2\pi)^2 B\sin(2\pi x).$$

Therefore

$$\begin{aligned}\frac{d^2}{dx^2}U_p(x,s) &- sU_p(x,s)\\ &= (-(2\pi)^2 - s)[A\cos(2\pi x) + B\sin(2\pi x)]\\ &= -3\sin(2\pi x).\end{aligned}$$

From this we conclude that

$$-(s+(2\pi)^2)A = 0, \qquad \text{and} \qquad -(s+(2\pi)^2)B = -3,$$

so that

$$A = 0, \quad B = \frac{3}{s+4\pi^2}.$$

Now we have the general solution

$$U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{3}{(s+4\pi^2)}\sin(2\pi x)$$

We note the the Laplace transforms of the boundary conditions give

$$u(0,t) = 0 \quad \Rightarrow \quad U(0,s) = 0, \quad \text{and} \quad u(2,t) = 0 \quad \Rightarrow \quad U(2,s) = 0$$

So we have

$$0 = U(0,s) = c_1 + c_2, \quad 0 = U(2,s) = c_1 e^{\sqrt{s}2} + c_2 e^{-\sqrt{s}2}$$

which gives $c_1 = c_2 = 0$ and we have

$$U(x,s) = \frac{3}{(s+4\pi^2)}\sin(2\pi x).$$

To find our solution we apply the inverse Laplace transform

$$u(x,t) = L^{-1}\left(\frac{3}{(s+4\pi^2)}\sin(2\pi x)\right) = 3e^{-4\pi^2 t}\sin(2\pi x).$$

Just as we would have obtained using eigenfunction expansion methods.

Example 4. Next we consider a similar problem for the 1D wave equation.

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) + \sin(\pi x), \quad 0 < x < 1, \quad t > 0,\\ &u(x,0) = 0, \quad u_t(x,0) = 0\\ &u(0,t) = 0 \quad u(1,t) = 0.\end{aligned}$$

Taking the Laplace transform and applying the initial conditions we obtain

$$\frac{d^2U}{dx^2}(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0) - \frac{\sin(\pi x)}{s} = s^2U(x,s) - \frac{\sin(\pi x)}{s}.$$

We need to solve the constant coefficient non-homogeneous ODE

$$\frac{d^2U}{dx^2}(x,s) - s^2U(x,s) = -\frac{\sin(\pi x)}{s}$$

Once again we know that

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

where $U_h(x,s)$ is the general solution of the homogeneous problem

$$U_h(x,s) = c_1e^{sx} + c_2e^{-sx}$$

and $U_p(x,s)$ is any particular solution of the non-homogeneous problem

$$U_p(x,s) = A\cos(\pi x) + B\sin(\pi x).$$

We apply the method of undetermined coefficients to find A and B. To this end we have

$$\frac{d}{dx}U_p(x,s) = -\pi A\sin(\pi x) + \pi B\cos(\pi x),$$

$$\frac{d^2}{dx^2}U_p(x,s) = -\pi^2 A\cos(\pi x) + \pi^2 B\sin(\pi x).$$

Therefore

$$\begin{aligned}\frac{d^2}{dx^2}U_p(x,s) &- s^2U_p(x,s)\\ &= (-\pi^2 - s^2)[A\cos(\pi x) + B\sin(\pi x)]\\ &= -\frac{\sin(\pi x)}{s}.\end{aligned}$$

From this we conclude that

$$-(s^2+\pi^2)A = 0, \qquad \text{and} \qquad -(s^2+\pi^2)B = -\frac{1}{s},$$

so that

$$A = 0, \quad B = \frac{1}{s(s^2+\pi^2)}.$$

So we have

$$U_p(x,s) = \frac{\sin(\pi x)}{s(s^2+\pi^2)}$$

and

$$U(x,s) = c_1e^{sx} + c_2e^{-sx} + \frac{\sin(\pi x)}{s(s^2+\pi^2)}.$$

Next we apply the BCs to find c_1 and c_2.

$$0 = U(0,s) = c_1 + c_2, \quad \text{and} \quad 0 = U(1,s) = c_1e^{s} + c_2e^{-s}$$

which implies $c_1 = 0$ and $c_2 = 0$. So we arrive at

$$U(x,s) = \frac{\sin(\pi x)}{s(s^2+\pi^2)}.$$

Finally we apply the inverse Laplace transform to obtain

$$\begin{aligned} u(x,t) = L^{-1}(U(x,s)) &= L^{-1}\left(\frac{1}{s(s^2+\pi^2)}\right)\ \sin(\pi x) \\ &= \frac{1}{\pi^2}\ L^{-1}\left(\frac{1}{s} - \frac{s}{(s^2+\pi^2)}\right)\ \sin(\pi x) \\ &= \frac{1}{\pi^2}(1-\cos(\pi t))\ \sin(\pi x). \end{aligned}$$

Here we have done partial fractions

$$\frac{1}{s(s^2+\pi^2)} = \frac{a}{s} + \frac{bs+c}{(s^2+\pi^2)} = \frac{1}{\pi^2}\left(\frac{1}{s} - \frac{s}{(s^2+\pi^2)}\right).$$

Example 5. This example shows the real use of Laplace transforms in solving a problem we could not have solved with our earlier work.

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t), \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x) \\ u(x,t) &\quad \text{bounded.} \end{aligned}$$

Under the assumption that $u(x,t)$ is bounded we know that the Laplace transform exists and, indeed, we have

$$|u(x,t)| \le M \quad \Rightarrow \quad |U(x,s)| \le \int_0^\infty e^{-st}|u(x,t)|\,dt \le M\int_0^\infty e^{-st}\,dt = \frac{M}{s}.$$

Applying the Laplace transform we obtain

$$\frac{d^2U}{dx^2}(x,s) = sU(x,s) - u(x,0) = sU(x,s) - f(x).$$

We write this equation as a non-homogeneous, second order linear constant coefficient equation.

$$\frac{d^2U}{dx^2}(x,s) - sU(x,s) = -f(x).$$

The general solution can be written as

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

where $U_h(x,s)$ is the general solution of the homogeneous problem

$$U_h(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$$

and $U_p(x,s)$ is any particular solution of the non-homogeneous problem. We find it using the method of variation of parameters from Math 3354. For this method we use $U_1 = e^{\sqrt{s}x}$, $U_2 = e^{-\sqrt{s}x}$.

$$W(U_1,U_2) = \begin{vmatrix} U_1(x,s) & U_2(x,s) \\ U_1'(x,s) & U_2'(x,s) \end{vmatrix} = -2\sqrt{s}$$

$$
\begin{aligned}
U_p(x,s) &= \int_0^x \frac{[-U_1(x,s)U_2(\xi,s)+U_2(x,s)U_1(\xi,s])(-f(\xi))}{W(\xi,s)}\,d\xi \\
&= -\frac{1}{2\sqrt{s}}\int_0^x \left[e^{\sqrt{s}x}e^{-\sqrt{s}\xi}+e^{-\sqrt{s}x}e^{\sqrt{s}\xi}\right] f(\xi)\,d\xi \\
&= -\frac{e^{\sqrt{s}x}}{2\sqrt{s}}\int_0^x e^{-\sqrt{s}\xi}f(\xi)\,d\xi + \frac{e^{-\sqrt{s}x}}{2\sqrt{s}}\int_0^x e^{\sqrt{s}\xi}f(\xi)\,d\xi
\end{aligned}
$$

So the general solution can be written as

$$
U(x,s) = \left(c_1 - \frac{1}{2\sqrt{s}}\int_0^x e^{-\sqrt{s}\xi}f(\xi)\,d\xi\right)e^{\sqrt{s}x} + \left(c_2 + \frac{1}{2\sqrt{s}}\int_0^x e^{\sqrt{s}\xi}f(\xi)\,d\xi\right)e^{-\sqrt{s}x}.
$$

Recall our assumption that $u(x,t)$ be bounded for all $-\infty < x < \infty$ implies that $U(x,s)$ is also bounded for all $-\infty < x < \infty$ for any fixed $s > 0$.

Now in order that the first term in the general solution stays bounded as $x \to \infty$ we need

$$
\lim_{x\to\infty}\left(c_1 - \frac{1}{2\sqrt{s}}\int_0^x e^{-\sqrt{s}\xi}f(\xi)\,d\xi\right) = 0
$$

which implies

$$
c_1 = \frac{1}{2\sqrt{s}}\int_0^\infty e^{-\sqrt{s}\xi}f(\xi)\,d\xi.
$$

In exactly the same way we must have

$$
\lim_{x\to-\infty}\left(c_2 + \frac{1}{2\sqrt{s}}\int_0^x e^{\sqrt{s}\xi}f(\xi)\,d\xi\right) = 0
$$

which implies

$$
c_2 = \frac{1}{2\sqrt{s}}\int_{-\infty}^0 e^{-\sqrt{s}\xi}f(\xi)\,d\xi.
$$

Thus

$$
\begin{aligned}
U(x,s) &= \left(\frac{1}{2\sqrt{s}}\int_0^\infty e^{-\sqrt{s}\xi}f(\xi)\,d\xi - \frac{1}{2\sqrt{s}}\int_0^x e^{-\sqrt{s}\xi}f(\xi)\,d\xi\right)e^{\sqrt{s}x} \\
&\quad + \left(\frac{1}{2\sqrt{s}}\int_{-\infty}^0 e^{-\sqrt{s}\xi}f(\xi)\,d\xi + \frac{1}{2\sqrt{s}}\int_0^x e^{\sqrt{s}\xi}f(\xi)\,d\xi\right)e^{-\sqrt{s}x} \\
&= \left(\frac{e^{\sqrt{s}x}}{2\sqrt{s}}\int_x^\infty e^{-\sqrt{s}\xi}f(\xi)\,d\xi\right) + \left(\frac{e^{-\sqrt{s}x}}{2\sqrt{s}}\int_{-\infty}^x e^{\sqrt{s}\xi}f(\xi)\,d\xi\right) \\
&= \frac{1}{2\sqrt{s}}\int_{-\infty}^\infty e^{-\sqrt{s}|x-\xi|}f(\xi)\,d\xi
\end{aligned}
$$

We want to find the inverse Laplace transform

$$
L^{-1}\left(\frac{e^{-\sqrt{s}|x-\xi|}}{2\sqrt{s}}\right).
$$

From our table we have

$$L^{-1}\left(\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right) = \frac{e^{-a^2/(4t)}}{\sqrt{\pi t}}$$

and if we set $a = |x-\xi|$ then we have

$$L^{-1}\left(\frac{e^{-\sqrt{s}|x-\xi|}}{2\sqrt{s}}\right) = \frac{e^{-|x-\xi|^2/(4t)}}{\sqrt{4\pi t}} \equiv K(|x-\xi|, t).$$

So we have

$$\begin{aligned} u(x,t) = L^{-1}(U(x,s)) = L^{-1}\left(\frac{1}{2\sqrt{s}} \int_{-\infty}^{\infty} e^{-\sqrt{s}|x-\xi|} f(\xi)\, d\xi\right) \\ = \int_{-\infty}^{\infty} L^{-1}\left(\frac{e^{-\sqrt{s}|x-\xi|}}{2\sqrt{s}}\right) f(\xi)\, d\xi \\ = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-|x-\xi|^2/(4t)} f(\xi)\, d\xi \\ = \int_{-\infty}^{\infty} K(|x-\xi|, t)\; f(\xi)\, d\xi \end{aligned}$$

The function

$$K(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}$$

is called the "Fundamental Heat Kernel".

Table of Laplace Transforms

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{a}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$
$f'(t)$	$s\mathcal{L}(f) - f(0)$
$f''(t)$	$s^2\mathcal{L}(f) - sf(0) - f'(0)$
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}(s)$
$e^{at} f(t)$	$\mathcal{L}(f)(s-a)$
$u(t-a) = \begin{cases} 0 & t \leq a \\ 1 & t > a \end{cases}$	$\frac{e^{-as}}{s}$
$u(t-a)f(t-a)$	$e^{-as}\mathcal{L}(f)(s)$
$\delta(t-a)$	e^{-as}
$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau$	$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$

The "error function" denoted by erf(x) is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx.$$

Notice that we can use the properties of integrals to deduce that

$$\operatorname{erf}(-x) = -\operatorname{erf}(x).$$

The complementary error function erfc(x) defined by

$$\operatorname{erfc}(x) = \frac{2}{\pi} \int_x^\infty e^{-s^2}\, ds.$$

Notice that

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \left(\int_0^x e^{-x^2}\, dx + \int_x^\infty e^{-s^2}\, ds \right) = 1.$$

Additional Laplace Transforms

$\dfrac{e^{-a^2/(4t)}}{\sqrt{\pi t}}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\dfrac{a e^{-a^2/(4t)}}{2\sqrt{\pi t^3}}$	$e^{-a\sqrt{s}}$
$\operatorname{erf}(t)$	$\dfrac{e^{s^2/4}\operatorname{erfc}(s/2)}{s}$
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
$2\sqrt{\dfrac{t}{\pi}}\, e^{-a^2/(4t)} - a\left\{\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)\right\}$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
$e^{b^2 t + ab}\left\{\operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)\right\}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}$
$-e^{b^2 t + ab}\left\{\operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)\right\} + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{b e^{-a\sqrt{s}}}{s(\sqrt{s}+b)}$