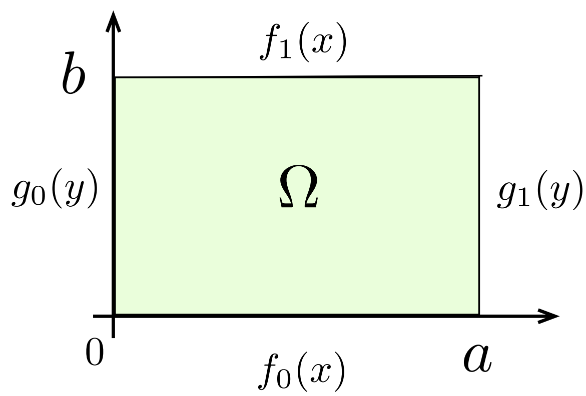
The Dirichlet Problem in a Two Dimensional Rectangle

Section 12.5

1 Dirichlet Problem in a Rectangle

In these notes we will apply the method of separation of variables to obtain solutions to elliptic problems in a rectangle as an infinite sum involving Fourier coefficients, eigenvalues and eigenvectors. These problems represent the simplest cases consisting of the Dirichlet Problem in a 2-Dimensional in a rectangle.



As usual we will start with simplest boundary conditions – Dirichlet boundary conditions – and a rectangular region Ω. The most general setup in this case is to prescribe a function on each of the four sides of the rectangle as depicted in the figure. Thus we obtain the problem

$$\begin{aligned} &u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \\ &u(0,y) = g_0(y), \quad u(a,y) = g_1(y) \\ &u(x,0) = f_0(x), \quad u(x,b) = f_1(x) \end{aligned} \tag{1.1}$$

Analysis of this problem would become rather messy but the principle of superposition allows us to divide and conquer. We can write the solution to this problem as a sum of solutions to four simpler problems

The Principle of Superposition

We note that for a linear problem it is always possible to replace a single hard problem by several simpler problems. More specifically, we can write the solution to a hard problem as the sum of the solutions to several simpler problems. For example the solution u to (1.1) can be obtained as a sum of the solutions to four simpler problems

$$\begin{aligned} &u^{(1)}_{xx}(x,y) + u^{(1)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a]\times[0,b], \\ &u^{(1)}(0,y) = g_0(y), \quad u^{(1)}(a,y) = 0 \\ &u^{(1)}(x,0) = 0, \quad u^{(1)}(x,b) = 0 \end{aligned} \tag{1.2}$$

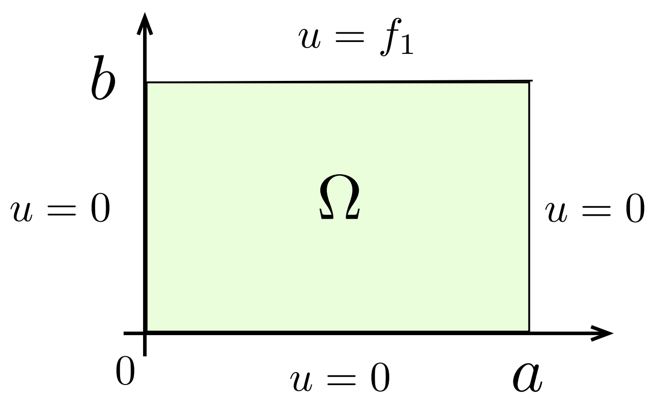
$$\begin{aligned} &u^{(2)}_{xx}(x,y) + u^{(2)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a]\times[0,b], \\ &u^{(2)}(0,y) = 0, \quad u^{(2)}(a,y) = 0 \\ &u^{(2)}(x,0) =, \quad u^{(2)}(x,b) = f_1(x) \end{aligned} \tag{1.3}$$

$$\begin{aligned} &u^{(3)}_{xx}(x,y) + u^{(3)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a]\times[0,b], \\ &u^{(3)}(0,y) = 0, \quad u^{(3)}(a,y) = g_1(y) \\ &u^{(3)}(x,0) = 0, \quad u^{(3)}(x,b) = 0 \end{aligned} \tag{1.4}$$

$$\begin{aligned} &u^{(4)}_{xx}(x,y) + u^{(4)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a]\times[0,b], \\ &u^{(4)}(0,y) = 0, \quad u^{(4)}(a,y) = 0 \\ &u^{(4)}(x,0) = f_0(x), \quad u^{(4)}(x,b) = 0 \end{aligned} \tag{1.5}$$

2 Non-Zero Boundary Function of x

To illustrate the method of separation of variables applied to these problems let us consider the BVP in (1.3). In this case the only non-zero boundary term occurs on the top of the box when $y = b$ where we have $u^{(2)}(x,b) = f_1(x)$.



$$\mu_n = \left(\frac{n\pi}{a}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sqrt{\frac{2}{a}}\sin(\mu_n x), \quad n = 1, 2, \cdots,$$

$$u^{(2)}(x,y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n y)\,\varphi_n(x)$$

$$b_n = \frac{1}{\sinh(\mu_n b)} \int_0^a f_1(x)\varphi_n(x)\,dx.$$

As usual we look for simple solutions in the form

$$u^{(2)}(x,y) = \varphi(x)\psi(y).$$

Substituting into (1.3) and dividing both sides by $\varphi(x)\psi(y)$ gives

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of y, it follows that the expression must be a constant:

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here ψ' means the derivative of ψ with respect to y and φ' means means the derivative of φ with respect to x.)

N.B. *Notice that we are using the negative of the sign we used for λ in the heat and wave equation. This is in keeping with standard practice for the Dirichlet problem. So in this case our eigenvalues will be $\lambda_n = \mu_n^2$ instead of $-\mu_n^2$.*

We seek to find all possible constants λ and the corresponding nonzero functions φ and ψ. We obtain

$$\varphi'' + \lambda\varphi = 0, \qquad \psi'' - \lambda\psi = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(y) = 0, \quad \varphi(a)\psi(y) = 0 \quad \text{for all } y.$$

Since $\psi(y)$ is not identically zero we obtain the eigenvalue problem

$$\varphi''(x) + \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi(a) = 0. \tag{2.1}$$

We have solved a similar problem many times with the main difference being that now the eigenvalues are positive.

$$\mu_n = \left(\frac{n\pi}{a}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sqrt{\frac{2}{a}}\sin(\mu_n x), \quad n = 1, 2, \cdots . \tag{2.2}$$

The general solution of

$$\psi''(y) - \mu_n^2\psi(y) = 0$$

is then

$$\psi(y) = c_1 \cosh(\mu_n y) + c_2 \sinh(\mu_n y) \tag{2.3}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\psi(0) = 0$ implies

$$\psi(y) = \sinh(\mu_n y).$$

So we look for u as an infinite sum

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n y)\, \varphi_n(x) \tag{2.4}$$

The only thing remaining is to somehow pick the constants b_n so that the initial condition $u(x,b) = f_1(x)$ is satisfied.

Setting $y = b$ in (2.4), we seek to obtain $\{b_n\}$ satisfying

$$f_1(x) = u(x,b) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n b)\, \varphi_n(x).$$

This is almost a Sine expansion of the function $f_1(x)$ on the interval $(0,a)$. In particular we obtain

$$\sinh(\mu_n b)\, b_n = \int_0^a f_1(x)\varphi_n(x)\, dx$$

or

$$b_n = \frac{1}{\sinh(\mu_n b)} \int_0^a f_1(x)\varphi_n(x)\, dx. \tag{2.5}$$

N.B. *In order to obtain the complete solution to the original problem* (1.1) *we would need to solve the three other similar problems* (1.2), (1.4), (1.5).

Let us now consider an explicit example for $f_1(x)$:

Example 2.1. Consider the problem (1.3) with

$$f_1(x) = \begin{cases} x & 0 \le x \le a/2, \\ (\pi - x) & a/2 \le x \le a. \end{cases}$$

For this example (2.4) becomes

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n y) \sinh(\mu_n y)\, \varphi_n(x), \quad \text{with} \quad \varphi_n(x) = \sqrt{\frac{2}{a}} \sin(\mu_n x).$$

In this case we obtain the following results for (2.5) (The explicit integrations are carried out below)

$$\sinh\left(\frac{n\pi b}{a}\right) b_n = \sqrt{\frac{2}{a}} \left[\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right) dx + \int_{a/2}^{a} (a-x) \sin\left(\frac{n\pi x}{a}\right) dx \right]$$

$$= \sqrt{\frac{2}{a}} \left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \sqrt{\frac{2}{a}} \left(\frac{2a^2 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \right)$$

Which implies

$$b_n = \sqrt{\frac{2}{a}} \left(\frac{2a^2 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh(\mu_n b)} \right).$$

Then we arrive at the solution

$$u(x,y) = \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\sin\left(\frac{n\pi}{2}\right) \sinh(\mu_n y)}{n^2\pi^2 \sinh(\mu_n b)} \right] \sin(\mu_n x).$$

To obtain the above formulas we needed to carryout two integration by parts. We do them separately in the following.

$$\begin{aligned}
\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right) dx &= \int_0^{a/2} x \left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right)' dx \\
&= x\left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right)\Big|_0^{a/2} - \int_0^{a/2} \left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right) dx \\
&= a/2\left(-\frac{a}{n\pi} \cos\left(\frac{n\pi}{2}\right)\right) + \frac{a}{n\pi}\int_0^{a/2} \cos\left(\frac{n\pi x}{a}\right) dx \\
&= -\frac{a^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\Big|_0^{a/2} \\
&= -\frac{a^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

and

$$\begin{aligned}
\int_{a/2}^{a} (a-x) \sin\left(\frac{n\pi x}{a}\right) dx &= \int_{a/2}^{a} (a-x) \left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right)' dx \\
&= (a-x)\left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right)\Big|_{a/2}^{a} - \int_{a/2}^{a} (-1)\left(-\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right) dx \\
&= a/2\left(\frac{a}{n\pi} \cos\left(\frac{n\pi}{2}\right)\right) - \frac{a}{n\pi}\int_{a/2}^{a} \cos\left(\frac{n\pi x}{a}\right) dx \\
&= \frac{a^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) - \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\Big|_{a/2}^{a} \\
&= \frac{a^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

Thus we have

$$\sinh(\mu_n b)\, b_n = \sqrt{\frac{2}{a}} \left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right]$$

or finally,

$$b_n = \sqrt{\frac{2}{a}} \left(\frac{2a^2 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh(\mu_n b)} \right).$$

3 Non-Zero Boundary Function of y

As another example illustrating the method of separation of variables applied to these problems let us consider the BVP in (1.4). In this case the only non-zero boundary term occurs on the right hand side of the box when $x = a$ where we have $u^{(3)}(a, y) = g_1(y)$.

$$\mu_n = \left(\frac{n\pi}{b}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(y) = \sqrt{\frac{2}{b}} \sin(\mu_n y), \quad n = 1, 2, \cdots$$

$$u^{(3)}(x, y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n x) \sin(\mu_n y)$$

$$b_n = \frac{1}{\sinh(\mu_n a)} \int_0^b g_1(y)\varphi_n(y)\, dy.$$

To obtain this formula we proceed as usual and look for simple solutions in the form

$$u^{(3)}(x, y) = \varphi(y)\psi(x).$$

The main difference here is that in this case we interchange the roles of x and y since we will want to do a Fourier series in y this time instead of x. Substituting into (1.4) and dividing both sides by $\varphi(y)\psi(x)$ gives

$$\frac{\psi''(x)}{\psi(x)} = \frac{-\varphi''(y)}{\varphi(y)}$$

Since the left side is independent of y and the right side is independent of x, it follows that the expression must be a constant:

$$\frac{\psi''(x)}{\psi(x)} = \frac{-\varphi''(y)}{\varphi(y)} = \lambda.$$

(Here ψ' means the derivative of ψ with respect to x and φ' means means the derivative of φ with respect to y.) We seek to find all possible constants λ and the corresponding nonzero functions φ and ψ. We obtain

$$\varphi''(y) + \lambda\varphi(y) = 0, \qquad \psi''(x) - \lambda\psi(x) = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(x) = 0, \quad \varphi(b)\psi(x) = 0 \quad \text{for all } x.$$

Since $\psi(x)$ is not identically zero we obtain the eigenvalue problem

$$\varphi''(y) + \lambda\varphi(y) = 0, \quad \varphi(0) = 0, \quad \varphi(b) = 0. \tag{3.1}$$

Once again we note the main difference now is that the eigenvalues λ_n are positive.

$$\mu_n = \left(\frac{n\pi}{b}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(y) = \sqrt{\frac{2}{b}}\sin(\mu_n y), \quad n = 1, 2, \cdots. \tag{3.2}$$

The general solution of

$$\psi''(x) - \mu_n^2\psi(x) = 0$$

is then

$$\psi(y) = c_1\cosh(\mu_n x) + c_2\sinh(\mu_n x) \tag{3.3}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\psi(0) = 0$ implies

$$\psi(x) = \sinh(\mu_n x).$$

So we look for u as an infinite sum

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n x)\, \varphi_n(y) \tag{3.4}$$

The only remaining part is to find the b_n so that the initial condition $u(a, y) = g_1(y)$ is satisfied.

Setting $x = a$ in (2.4), we seek to obtain $\{b_n\}$ satisfying

$$g_1(y) = u(a, y) = \sum_{n=1}^{\infty} b_n \sinh(\mu_n a)\sin(\mu_n y).$$

This is almost a Sine expansion of the function $g_1(y)$ on the interval $(0, b)$. In particular we obtain

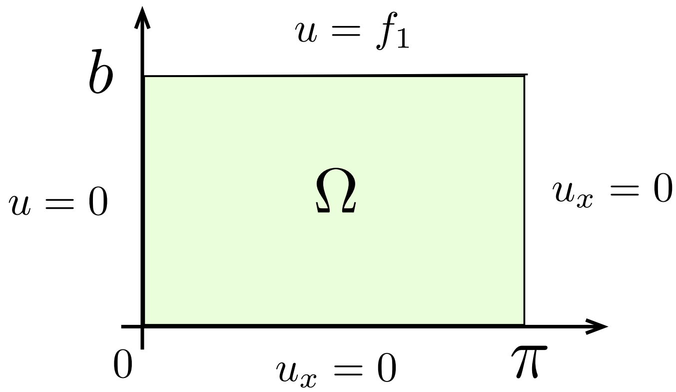
$$\sinh(\mu_n a)\, b_n = \int_0^b g_1(y)\varphi_n(y)\, dy. \tag{3.5}$$

4 The Laplace Equation with other Boundary Conditions

Next we consider a slightly different problem involving a mixture of Dirichlet and Neumann boundary conditions. To simplify the problem a bit we set $a = \pi$ and keep b any number.

Namely we consider

$$\begin{aligned} &u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad (x,y) \in [0,\pi] \times [0,b], \\ &u(0,y) = 0, \quad u_x(\pi,y) = 0 \\ &u_y(x,0) = 0, \quad u(x,b) = f_1(x) \end{aligned} \tag{4.1}$$



Look for simple solutions in the form

$$u(x,y) = \varphi(x)\psi(y).$$

Substituting into (4.1) and dividing both sides by $\varphi(x)\psi(y)$ gives

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of y, it follows that the expression must be a constant:

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here ψ' means the derivative of ψ with respect to y and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and ψ. We obtain

$$\varphi'' + \lambda\varphi = 0, \qquad \psi'' - \lambda\psi = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(y) = 0, \quad \varphi'(\pi)\psi(y) = 0 \quad \text{for all } y.$$

Since $\psi(y)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) + \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi'(\pi) = 0. \tag{4.2}$$

$$\mu_n = \frac{(2n-1)}{2}, \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sqrt{\frac{2}{\pi}}\sin(\mu_n x), \quad n = 1, 2, \cdots . \tag{4.3}$$

The general solution of $\psi'' - \mu_n^2 \psi = 0$ is

$$\psi(y) = c_1 \cosh(\mu_n y) + c_2 \sinh(\mu_n y) \tag{4.4}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\psi'(0) = 0$ implies

$$\psi(y) = \cosh(\mu_n y)\,.$$

So we look for u as an infinite sum

$$u(x, y) = \sum_{n=1}^{\infty} a_n \cosh(\mu_n y)\, \varphi_n(x). \tag{4.5}$$

Finally we need to find the constants a_n so that

$$f_1(x) = u(x, b) = \sum_{n=1}^{\infty} a_n \cosh(\mu_n b)\, \varphi_n(x).$$

As usual we obtain an expansion of the function $f_1(x)$ on the interval $(0, \pi)$in the form

$$\cosh(\mu_n b)\, a_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f_1(x) \sin(\mu_n x)\, dx. \tag{4.6}$$