Formulas Cullen Zill AEM Chapters 3

I. (Linear, Homogeneous, Constant Coefficients)

$\boxed{y'' + Ay' + By = 0} \quad \Rightarrow \quad \text{try} \quad \boxed{y = e^{rx}}$

Characteristic Equation $\boxed{r^2 + Ar + B = 0}$ has roots $r_1,\ r_2$.

Three Cases:

1. Real distinct roots $r_1 \neq r_2 \Rightarrow$ (general solution) $\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$
2. Real double root $r = r_1 = r_2 \Rightarrow$ (general solution) $\boxed{y = c_1 e^{rx} + c_2 x e^{rx}}$
3. Complet roots $r = \alpha \pm i\beta \Rightarrow$ (general solution) $\boxed{y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)}$

II. (Euler Equation, $x > 0$) $\boxed{x^2 y'' + Axy' + By = 0} \quad \Rightarrow \quad \text{try} \quad \boxed{y = x^r}$

(same as $x = e^t \Rightarrow t = \ln(x)$ and change variables to get $\dfrac{d^2y}{dt^2} + (A-1)\dfrac{dy}{dt} + By = 0$)

Characteristic Equation $\boxed{r^2 + (A-1)r + B = 0}$ has roots $r_1,\ r_2$.

Three Cases:

1. Real distinct roots $r_1 \neq r_2 \Rightarrow$ (general solution) $\boxed{y = c_1 x^{r_1} + c_2 x^{r_2}}$
2. Real double root $r = r_1 = r_2 \Rightarrow$ (general solution) $\boxed{y = c_1 x^r + c_2 \ln(x) x^r}$
3. Complet roots $r = \alpha \pm i\beta \Rightarrow$ (general solution) $\boxed{y = c_1 x^{\alpha}\cos(\beta\ln(x)) + c_2 x^{\alpha}\sin(\beta\ln(x))}$

III. (Nonhomogeneous Linear) $\boxed{y'' + P(x)y' + Q(x)y = R(x)}$ General solution: $\boxed{y = y_h + y_p}$

y_h is the general solution of

$$y'' + P(x)y' + Q(x)y = 0$$

and

y_p is any particular solution of

$$y'' + P(x)y' + Q(x)y = R(x)$$

There are two methods:

A. (Undetermined Coefficients) Guess the form of y_p from $R(x)$. This method requires that P and Q to be constants and R is a sum of terms of the form x^k, $x^k e^{\alpha x}$, $x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$.

B. (Variation of Parameters) Look for a particular solution in the form $y_p = uy_1 + vy_2$. This approach leads to

$$\boxed{y_p(x) = -y_1(x)\int^t \frac{y_2(s)R(s)}{W(s)}\,ds + y_2(x)\int^t \frac{y_1(s)R(s)}{W(s)}\,ds, \qquad W(s) = \det\begin{bmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s)\end{bmatrix}}$$

IV. (Reduction of Order) Suppose that y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$. A second solution can be found in the form $y_2(x) = v(x)y_1(x)$ where $\boxed{v = \int \exp\left(-\int^x P(s)\,ds\right)dx}$

and $\boxed{P = \dfrac{(2y_1' + p(x)y_1)}{y_1}}$.

A homogeneous linear differential equation with constant real coefficients of order n has the form

$$\boxed{y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0. \qquad (*)}$$

We can introduce the notation $D = \dfrac{d}{dx}$ and write the above equation as

$$P(D)y \equiv \left(D^n + a_{n-1}D^{(n-1)} + \cdots + a_0\right) y = 0.$$

By the fundamental theorem of algebra we can factor $P(D)$ as

$$\boxed{(D - r_1)^{m_1} \cdots (D - r_k)^{m_k} (D^2 - 2\alpha_1 D + \alpha_1^2 + \beta_1^2)^{p_1} \cdots (D^2 - 2\alpha_\ell D + \alpha_\ell^2 + \beta_\ell^2)^{p_\ell},}$$

where $\displaystyle\sum_{j=1}^{k} m_j + 2\sum_{j=1}^{\ell} p_j = n.$

There are two types of factors $(D-r)^k$ and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^k$:

The general solution of $(D-r)^k y = 0$ is $y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{rx}$

The general solution of $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^k y = 0$ is

$$y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{\alpha x}\cos(\beta x) + \left(d_1 + d_2 x + \cdots + d_k x^{(k-1)}\right) e^{\alpha x}\sin(\beta x).$$

The general solution contains one such term for each term in the factorization.

We can also argue as before and seek solutions of $(*)$ in the form $y = e^{rx}$ to get a characteristic polynomial

$$r^n + a_{n-1}r^{(n-1)} + \cdots + a_0 = 0.$$

In either case we find the general solution consists of a sum of n expressions each term of which contains an arbitrary constant and a term that looks like x^k, $x^k e^{rx}$, $x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$.