Chapter 12 Heat Examples in Rectangles

1 Heat Equation Dirichlet Boundary Conditions

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \qquad (1.1)\\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

1. **Separate Variables** Look for simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

Substituting into (1.1) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$\frac{\psi'(t)}{k\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{\psi'(t)}{k\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.$$

We seek to find all possible constants λ and the corresponding nonzero functions φ and ψ.

We obtain

$$\varphi'' - \lambda\varphi = 0, \qquad \psi' - k\lambda\psi = 0.$$

The solution of the second equation is

$$\psi(t) = Ce^{k\lambda t} \qquad (1.2)$$

where C is an arbitrary constant. Furthermore, the boundary conditions give

$$\varphi(0)\psi(t) = 0, \quad \varphi(\ell)\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi(\ell) = 0. \qquad (1.3)$$

2. **Find Eigenvalues and Eignevectors** The next main step is to find the eigenvalues and eigenfunctions from (1.3). There are, in general, three cases:

   (a) If $\lambda = 0$ then $\varphi(x) = ax + b$ so applying the boundary conditions we get

   $$0 = \varphi(0) = b, \quad 0 = \varphi(\ell) = a\ell \quad \Rightarrow a = b = 0.$$

   Zero is not an eigenvalue.

(b) If $\lambda = \mu^2 > 0$ then

$$\varphi(x) = a\cosh(\mu x) + b\sinh(\mu x).$$

Applying the boundary conditions we have

$$0 = \varphi(0) = a \Rightarrow a = 0 \quad 0 = \varphi(\ell) = b\sinh(\mu\ell) \quad \Rightarrow b = 0.$$

Therefore, there are no positive eigenvalues.
Consider the following alternative argument: If $\varphi''(x) = \lambda\varphi(x)$ then multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

$$\lambda \int_0^\ell \varphi(x)^2\,dx = \int_0^\ell \varphi(x)\varphi''(x)\,dx = -\int_0^\ell \varphi'(x)^2\,dx + \varphi(x)\varphi'(x)\Big|_0^\ell.$$

Since $\varphi(0) = \varphi(\ell) = 0$ we conclude

$$\lambda = -\frac{\displaystyle\int_0^\ell \varphi'(x)^2\,dx}{\displaystyle\int_0^\ell \varphi(x)^2\,dx}$$

and we see that λ must be less than or equal to zero.

(c) So, finally, consider $\lambda = -\mu^2$ so that

$$\varphi(x) = a\cos(\mu x) + b\sin(\mu x).$$

Applying the boundary conditions we have

$$0 = \varphi(0) = a \Rightarrow a = 0 \quad 0 = \varphi(\ell) = b\sin(\mu\ell).$$

From this we conclude $\sin(\mu\ell) = 0$ which implies

$$\mu = \frac{n\pi}{\ell}.$$

So we have eigenfunctions $b_n\sin(\mu_n x)$ and we choose the constant b_n so that

$$\int_0^\ell \varphi_n(x)^2\,dx = 1 \quad \Rightarrow \quad b_n = \sqrt{\frac{2}{\ell}}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad \varphi_n(x) = \sqrt{\frac{2}{\ell}}\sin(\mu_n x), \quad n = 1, 2, \cdots. \tag{1.4}$$

From (1.2) we also have the associated functions $\psi_n(t) = e^{k\lambda_n t}$.

3. **Write Formal Sum** From the above considerations we can conclude that for any integer N and constants $\{b_n\}_{n=0}^N$

$$u_N(x,t) = \sum_{n=1}^N c_n\psi_n(t)\varphi_n(x).$$

satisfies the differential equation in (1.1) and the boundary conditions.

4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants b_n so that the initial condition $u(x,0) = f(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for u as an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{k\lambda_n t} \varphi_n(x)$$

and we try to find $\{c_n\}$ satisfying

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$$

But this nothing more than a Sine expansion of the function φ on the interval $(0,\ell)$.

$$c_n = \int_0^{\ell} \varphi(x)\varphi_n(x)\,dx. \tag{1.5}$$

Example 1.1. As an explicit example for the initial condition consider $\ell = 1$, $k = 1/10$ and $f(x) = x(1-x)$. Let us recall that $\mu_n = \left(\frac{n\pi}{\ell}\right)$ which in this case reduces to $n\pi$.

$$\begin{aligned}
c_n &= \sqrt{2}\int_0^1 x(1-x)\sin(n\pi x)\;dx \\
&= \sqrt{2}\int_0^1 x(1-x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)'\;dx \\
&= \frac{\sqrt{2}}{n\pi}\left[-x(1-x)\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 + \int_0^1 (1-2x)\frac{\cos(n\pi x)}{\mu_n}\,dx\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{n\pi}\int_0^1 (1-2x)\left(\frac{\sin(n\pi x)}{n\pi}\right)'\;dx \\
&= \frac{\sqrt{2}}{n\pi}\left[(1-2x)\frac{\sin(n\pi x)}{n\pi}\bigg|_0^1 - \int_0^1 (-2)\frac{\sin(n\pi x)}{n\pi}\,dx\right] \\
&= \frac{2\sqrt{2}}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx = \frac{2\sqrt{2}}{(n\pi)^2}\left[-\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1\right] = \frac{2\sqrt{2}\,[1-(-1)^n]}{(n\pi)^3}
\end{aligned}$$

We arrive at the solution

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^3} e^{-n^2\pi^2 t/10}\sin(n\pi x)\,. \tag{1.6}$$

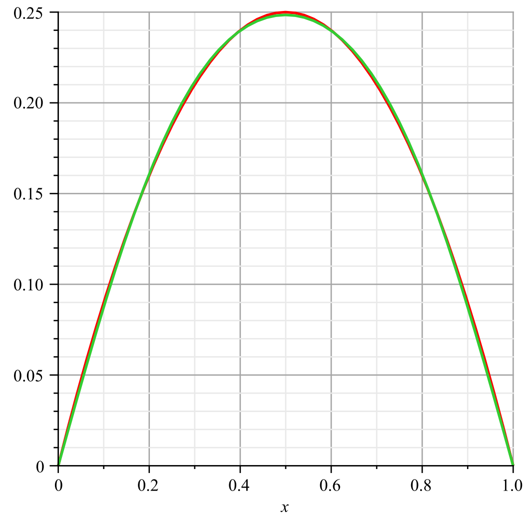
where

$$x(1-x) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\sin(n\pi x)\,.$$

As an example with $N = 3$ we have

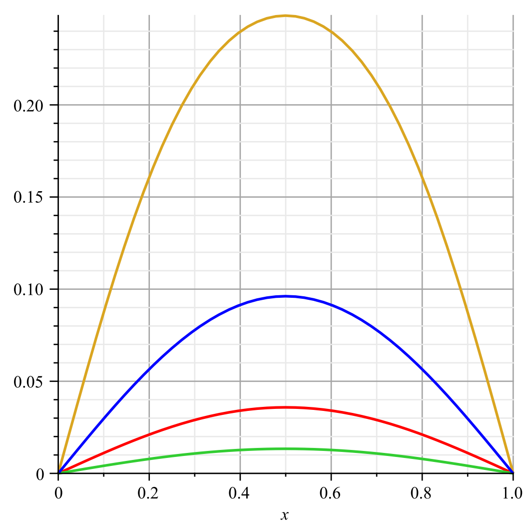
$$x(1-x) \approx \frac{8}{\pi^3}\left(\sin(\pi x) + \frac{\sin(3\pi x)}{27}\right).$$

In the following figure we plot the left and right hand side of the above.



Finally we plot the approximate solution at times $t = 0$, $t =$, $t = 2$, $t = 3$

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{3} \frac{[1-(-1)^n]}{n^3} e^{-n^2\pi^2 t/10} \sin(n\pi x).$$



2 Heat Equation Neumann Boundary Conditions

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ &u_x(0,t) = 0, \quad u_x(\ell,t) = 0 \\ &u(x,0) = f(x) \end{aligned} \tag{2.7}$$

1. **Separate Variables** Look for simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

Substituting into (2.7) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$\frac{\psi'(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{\psi'(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.$$

We seek to find all possible constants λ and the corresponding nonzero functions φ and ψ.

We obtain

$$\varphi'' - \lambda\varphi = 0, \qquad \psi' - \lambda\psi = 0.$$

The solution of the second equation is

$$\psi(t) = Ce^{\lambda t} \tag{2.8}$$

where C is an arbitrary constant. Furthermore, the boundary conditions give

$$\varphi'(0)\psi(t) = 0, \quad \varphi'(\ell)\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi'(0) = 0, \quad \varphi'(\ell) = 0. \tag{2.9}$$

2. **Find Eigenvalues and Eignevectors** The next main step is to find the eigenvalues and eigenfunctions from (2.9). There are, in general, three cases:

   (a) If $\lambda = 0$ then $\varphi(x) = ax + b$ so applying the boundary conditions we get

   $$0 = \varphi'(0) = a, \quad 0 = \varphi'(\ell) = a \quad \Rightarrow a = 0.$$

   Notice that b is still an arbitrary constant. We conclude that $\lambda_0 = 0$ is an eigenvalue with normalized eigenfunction $\varphi_0(x) = 1/\sqrt{\ell}$.

   (b) If $\lambda = \mu^2 > 0$ then

   $$\varphi(x) = a\cosh(\mu x) + b\sinh(\mu x)$$

   and

   $$\varphi'(x) = a\mu\sinh(\mu x) + b\mu\cosh(\mu x).$$

   Applying the boundary conditions we have

   $$0 = \varphi'(0) = b\mu \Rightarrow b = 0 \quad 0 = \varphi'(\ell) = a\mu\sinh(\mu\ell) \quad \Rightarrow a = 0.$$

   Therefore, there are no positive eigenvalues.

   Consider the following alternative argument: If $\varphi''(x) = \lambda\varphi(x)$ then multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

   $$\lambda\int_0^\ell \varphi(x)^2\,dx = \int_0^\ell \varphi(x)\varphi''(x)\,dx = -\int_0^\ell \varphi'(x)^2\,dx + \varphi(x)\varphi'(x)\Big|_0^\ell.$$

Since $\varphi'(0) = \varphi'(\ell) = 0$ we conclude

$$\lambda = -\frac{\displaystyle\int_0^\ell \varphi'(x)^2\,dx}{\displaystyle\int_0^\ell \varphi(x)^2\,dx}$$

and we see that λ must be less than or equal to zero (zero only if $\varphi' = 0$).

(c) So, finally, consider $\lambda = -\mu^2$ so that

$$\varphi(x) = a\cos(\mu x) + b\sin(\mu x)$$

and

$$\varphi'(x) = -a\mu\sin(\mu x) + b\mu\cos(\mu x).$$

Applying the boundary conditions we have

$$0 = \varphi'(0) = b\mu \Rightarrow b = 0 \quad 0 = \varphi'(\ell) = -a\mu\sin(\mu\ell).$$

From this we conclude $\sin(\mu\ell) = 0$ which implies $\mu = \dfrac{n\pi}{\ell}$. So we have eigenfunctions $a_n\cos(\mu_n x)$ and we choose the constant a_n so that

$$\int_0^\ell \varphi_n(x)^2\,dx = 1 \quad \Rightarrow \quad a_n = \sqrt{\frac{2}{\ell}}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad \varphi_n(x) = \sqrt{\frac{2}{\ell}}\cos(\mu_n x), \quad n = 1, 2, \cdots .. \tag{2.10}$$

From (2.8) we also have the associated functions $\psi_n(t) = e^{\lambda_n t}$.

3. **Write Formal Infinite Sum** From the above considerations we can conclude that for any integer N and constants $\{a_n\}_{n=0}^N$

$$u_n(x,t) = c_0 + \sum_{n=1}^N c_n\psi_n(t)\varphi_n(x)$$

satisfies the differential equation in (2.7) and the boundary conditions.

4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants a_n so that the initial condition $u(x,0) = f(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for u as an infinite sum

$$u(x,t) = c_0 + \sum_{n=1}^\infty c_n e^{\lambda_n t}\varphi_n(x)$$

and we try to find $\{a_n\}$ satisfying

$$\varphi(x) = u(x,0) = c_0 + \sum_{n=1}^\infty c_n\varphi_n(x).$$

But this nothing more than a Cosine expansion of the function φ on the interval $(0,\ell)$.

Our work on Fourier series showed us that

$$c_n = \int_0^\ell f(x)\varphi_n(x)\,dx. \tag{2.11}$$

As an explicit example for the initial condition consider $\ell = 1$ and $f(x) = x(1-x)$. In this case (2.11) becomes

$$c_n = \int_0^1 f(x)\varphi_n(x)\,dx.$$

We have

$$\begin{aligned} c_0 &= \int_0^1 \varphi(x)\,dx = \int_0^1 x(1-x)\,dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \frac{1}{6}. \end{aligned}$$

and for $n > 0$

$$\begin{aligned} c_n &= \sqrt{2}\int_0^1 \varphi(x)\cos(n\pi x)\;dx = 2\int_0^1 x(1-x)\cos(n\pi x)\;dx \\ &= \sqrt{2}\int_0^1 x(1-x)\left(\frac{\sin(n\pi x)}{n\pi}\right)'\;dx \\ &= \sqrt{2}\left[x(1-x)\frac{\sin(n\pi x)}{n\pi}\Bigg|_0^1 - \int_0^1 (1-2x)\frac{\sin(n\pi x)}{n\pi}\;dx\right] \\ &= \frac{\sqrt{2}}{n\pi}\int_0^1 (1-2x)\left(\frac{\cos(n\pi x)}{n\pi}\right)'\;dx \\ &= \frac{\sqrt{2}}{n\pi}\left[(1-2x)\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1 - \int_0^1 (-2)\frac{\cos(n\pi x)}{n\pi}\;dx\right] \end{aligned}$$

$$\begin{aligned} &= \frac{\sqrt{2}}{n\pi}\left[-\frac{\cos(n\pi)}{n\pi} - \frac{1}{n\pi}\right] \\ &= \frac{-\sqrt{2}}{(n\pi)^2}((-1)^n + 1) = \begin{cases} \dfrac{-2\sqrt{2}}{(n\pi)^2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}. \end{aligned}$$

In order to eliminate the odd terms in the expansion we introduce a new index, k by $n = 2k$ where $k = 1, 2, \cdots$. So finally we arrive at the solution

$$u(x,t) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}e^{-4k^2\pi^2 t}\cos(2k\pi x). \tag{2.12}$$

As an example with $N = 4$ we have

$$x(1-x) \approx \frac{1}{6} - \frac{1}{\pi^2}\left(\sum_{n=1}^{4}\frac{\cos(2k\pi x)}{k^2}\right).$$

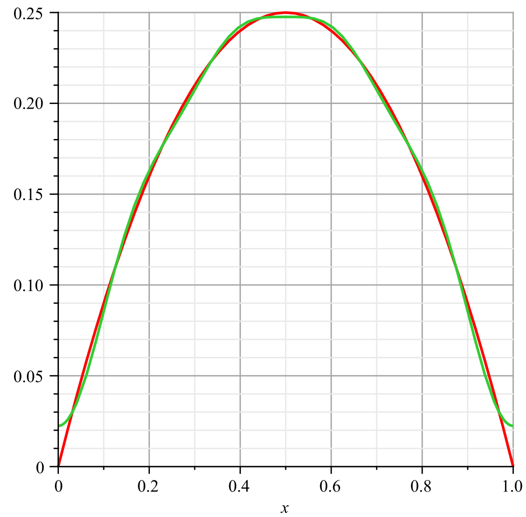
Notice that as $t \to \infty$ the infinite sum converges to zero uniformly in x. Indeed,

$$\left| \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-4k^2\pi^2 t} \cos(2k\pi x) \right| \le e^{-4\pi^2 t} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \, e^{-4\pi^2 t}.$$

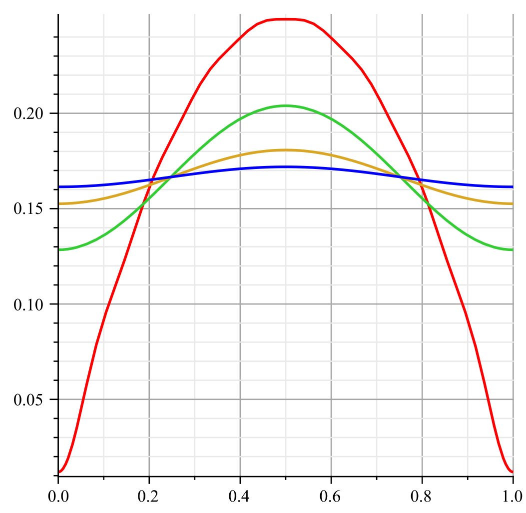
So the solution converges to a nonzero steady state temperature which is exactly the average value of the initial temperature distribution.

$$\lim_{t\to\infty} u(x,t) = \frac{1}{6} = \int_0^1 f(x)\, dx.$$

In the following figure we plot the left and right hand side of the above.



Finally we plot the approximate solution at times $t = 0$, $t = 1/10$, $t = 2/10$, $t = 3/10$.



2.1 Heat Equation Dirichlet-Neumann Boundary Conditions

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\
u(0,t) &= 0, \quad u_x(\ell,t) = 0 \\
u(x,0) &= f(x)
\end{aligned}
$$

Apply **Separation of Variables** to obtain the Sturm-Liouville problem for $\{\lambda_n, \varphi_n(x)\}$. **Find Eigenvalues and Eignevectors** The next main step is to find the eigenvalues and eigenfunctions. There are, in general, three cases:

1. If $\lambda = 0$ then $\varphi(x) = ax + b$ so applying the boundary conditions we get

   $$0 = \varphi(0) = b, \quad 0 = \varphi'(\ell) = a \quad \Rightarrow a = b = 0.$$

   Zero is not an eigenvalue.

2. If $\lambda = \mu^2 > 0$ then

   $$\varphi(x) = a\cosh(\mu x) + b\sinh(\mu x)$$

   and

   $$\varphi'(x) = a\mu\sinh(\mu x) + b\mu\cosh(\mu x).$$

   Applying the boundary conditions we have

   $$0 = \varphi'(0) = a\mu \Rightarrow a = 0 \quad 0 = \varphi'(\ell) = b\mu\cosh(\mu\ell) \quad \Rightarrow b = 0.$$

   Therefore, there are no positive eigenvalues.

   Consider the following alternative argument: If $\varphi''(x) = \lambda\varphi(x)$ then multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

   $$\lambda \int_0^\ell \varphi(x)^2\,dx = \int_0^\ell \varphi(x)\varphi''(x)\,dx = -\int_0^\ell \varphi'(x)^2\,dx + \varphi(x)\varphi'(x)\Big|_0^\ell.$$

   Since $\varphi(0) = \varphi'(\ell) = 0$ we conclude

   $$\lambda = -\frac{\int_0^\ell \varphi'(x)^2\,dx}{\int_0^\ell \varphi(x)^2\,dx}$$

   and we see that λ must be less than or equal to zero.

3. So, finally, consider $\lambda = -\mu^2$ so that

   $$\varphi(x) = a\cos(\mu x) + b\sin(\mu x)$$

   and

   $$\varphi'(x) = -a\mu\sin(\mu x) + b\mu\cos(\mu x).$$

   Applying the boundary conditions we have

   $$0 = \varphi(0) = a\mu \Rightarrow a = 0 \quad 0 = \varphi'(\ell) = b\mu\cos(\mu\ell).$$

From this we conclude $\cos(\mu\ell) = 0$ which implies

$$\mu = \frac{(2n-1)\pi}{2\ell}$$

So we have eigenfunctions $b_n \sin(\mu_n x)$ and we choose the constant b_n so that

$$\int_0^\ell \varphi_n(x)^2\,dx = 1 \quad \Rightarrow \quad b_n = \sqrt{\frac{2}{\ell}}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{(2n-1)\pi}{2\ell}\right)^2, \quad \varphi_n(x) = \sqrt{\frac{2}{\ell}}\sin(\mu_n x), \quad n = 1, 2, \cdots. \tag{2.13}$$

From (1.2) we also have the associated functions $\psi_n(t) = e^{\lambda_n t}$.

Write Formal Infinite Sum From the above considerations we can conclude that for any integer N and constants $\{b_n\}_{n=0}^N$

$$u_n(x,t) = \sum_{n=1}^N c_n \psi_n(t)\varphi_n(x).$$

satisfies the heat equation and the boundary conditions.

Use Fourier Series to Find Coefficients The only problem remaining is to somehow pick the constants c_n so that the initial condition $u(x,0) = f(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for u as an infinite sum

$$u(x,t) = \sum_{n=1}^\infty c_n e^{\lambda_n t}\varphi_n(x)$$

and we try to find $\{b_n\}$ satisfying

$$f(x) = u(x,0) = \sum_{n=1}^\infty c_n \varphi_n(x).$$

But this nothing more than a Sine type expansion of the function φ on the interval $(0,\ell)$ and we have

$$f(x) = \sum_{n=1}^\infty c_n \varphi_n(x).$$

$$c_n = \int_0^\ell f(x)\varphi_n(x)\,dx. \tag{2.14}$$

As an explicit example let $\ell = 1$ so that $\varphi_n(x) = \sqrt{2}\sin(\mu_n x)$ and for the initial condition consider $f(x) = x$. Let us recall that $\mu_n = \left(\frac{(2n-1)\pi}{2\ell}\right)$

$$
\begin{aligned}
c_n &= \frac{\sqrt{2}}{1}\int_0^1 \varphi(x)\varphi_n(x)\,dx = \frac{2}{1}\int_0^1 x\sin\left(\mu_n x\right)\,dx \\
&= \sqrt{2}\int_0^1 x\left(-\frac{\cos\left(\mu_n x\right)}{\mu_n}\right)'\,dx \\
&= \sqrt{2}\left[-x\frac{\cos(\mu_n x)}{\mu_n}\bigg|_0^1 + \int_0^1 \frac{\cos(\mu_n x)}{\mu_n}\,dx\right] \\
&= \sqrt{2}\left[-1\frac{\cos(\mu_n 1)}{\mu_n} + \frac{\sin(\mu_n x)}{\mu_n^2}\bigg|_0^1\right] \\
&= \sqrt{2}\left[-1\frac{\cos((2n-1)\pi/2)}{\mu_n} + \frac{\sin((2n-1)\pi/2)}{\mu_n^2}\right] \\
&= \frac{4\sqrt{2}1(-1)^{n+1}}{(2n-1)^2\pi^2}
\end{aligned}
$$

We arrive at the solution

$$
u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}e^{\lambda_n t}\sin\left(\mu_n x\right). \tag{2.15}
$$