# Chapter 12 PDEs in Rectangles

## 1 2-D Second Order Equations: Separation of Variables

1. A second order linear partial differential equation in two variables $x$ and $y$ is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G. \tag{1}$$

2. If $G = 0$ we say the problem is *homogeneous* otherwise it is *nonhomogeneous*.

3. A *Solution* is a function $u(x, y)$ that has the required differentiability and satisfies the equation. Unlike the case of ODEs the idea of a general solution is not very clear. We will look for particular solutions.

4. A very useful method for looking for solutions is the *Method of Separation of Variables* in which we look for a solution in the form $u(x, y) = \varphi(x)\psi(y)$.

5. (Principle of Superposition) Since the problem is linear, if we find several solutions, say $\{u_j(x, y)\}_{j=1}^n$ then for all constants $\{c_j\}_{j=1}^n$

$$u(x, y) = \sum_{j=1}^n c_j u_j(x, y)$$

is also a solution.

6. Equations in the form (1) can be classified as one three types of equations by
   (a) **Hyperbolic** if $B^2 - 4AC > 0$
   (b) **Parabolic** if $B^2 - 4AC = 0$
   (c) **Elliptic** if $B^2 - 4AC < 0$

7. Main Classical Examples: (replacing $y$ by $t$ in the hyperbolic and parabolic cases)
   (a) **Hyperbolic** Wave equation $\dfrac{\partial^2 u}{\partial t^2}(x, t) = a^2\dfrac{\partial^2 u}{\partial x^2}(x, t)$
   (b) **Parabolic** Heat equation $\dfrac{\partial u}{\partial t}(x, t) = k\dfrac{\partial^2 u}{\partial x^2}(x, t)$
   (c) **Elliptic** Laplace's equation $\dfrac{\partial^2 u}{\partial x^2}(x, y) + \dfrac{\partial^2 u}{\partial y^2}(x, y) = 0$

8. Examples of separation of variables for main examples:
   (a) For $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$ seek $u(x, t) = \varphi(x)\psi(t) \Rightarrow \dfrac{\psi''(t)}{a^2\psi(t)} = \dfrac{\varphi''(x)}{\varphi(x)} = \lambda$ where $\lambda$ is a constant.
   (b) For $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ seek $u(x, t) = \varphi(x)\psi(t) \Rightarrow \dfrac{\psi'(t)}{k\psi(t)} = \dfrac{\varphi''(x)}{\varphi(x)} = \lambda$ where $\lambda$ is a constant.
   (c) For $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ seek $u(x, y) = \varphi(x)\psi(y) \Rightarrow \dfrac{\varphi''(x)}{\varphi(x)} + \dfrac{\psi''(y)}{\psi(y)} = \lambda$ where $\lambda$ is a constant.

9. Boundary Conditions are given on the physical boundary of the spatial domain, i.e., at the ends of a heated rod or vibrating string. The most common BCs are
   (a) **Dirichlet** Homogeneous BC at $x = a$ is $u(a,t) = 0$, Nonhomogeneous BC is $u(a,t) = \gamma(t)$
   (b) **Neumann** Homogeneous BC at $x = a$ is $u_x(a,t) = 0$, Nonhomogeneous BC is $u_x(a,t) = \gamma(t)$
   (c) **Robin** Homogeneous BC at $x = a$ is $u_x(a,t) \pm ku(a,t) = 0$, Nonomogeneous BC is $u_x(a,t) \pm ku(a,t) = \gamma(t)$ where $k > 0$ is a constant and $(\pm)$ is determined by which end of the rod or string.

# 2 Regular Sturm-Liouville problem

One of the most important ideas in functional analysis is contained in the following discussion which is very much related to the idea of eigenvalues and an orthonormal basis of eigenvectors. Consider the boundary value problem.

$$\begin{aligned} &\varphi''(x) - q(x)\varphi(x) = \lambda\varphi(x), \\ &\alpha_1\varphi'(a) + \alpha_2\varphi(a) = 0, \\ &\beta_1\varphi'(b) + \beta_2\varphi(b) = 0 \end{aligned} \tag{2}$$

This is an eigenvalue problem which is referred to as a Regular Sturm-Liouville problem.

**Theorem 2.1.** *The problem (2) has infinitely many eigenpairs $\{\lambda_n, \varphi_n(x)\}$ which satisfy the following properties:*

1. *The eigenvalues are all simple, i.e. they are eigenvalues of multiplicity one which means that $\lambda_j \neq \lambda_k$ for $j \neq k$.*
2. *The eigenvalues are all real and all but a finite number are negative. If $|\alpha_1| + |\alpha_2| \neq 0, |\beta_1| + |\beta_2| \neq 0$ then all (but possibly one) of the eigenvalues are less than or equal to zero.*
3. *If we order the eigenvalues in decreasing order by*
$$\lambda_n < \lambda_{n-1} < \cdots < \lambda_2 < \lambda_1, \quad \textit{and} \quad \lambda_n \to -\infty \quad \textit{as} \quad n \to \infty.$$
4. *The eigenfunctions are all real and $\varphi_n(x)$ has exactly $(n-1)$ zeros in the interval $(a,b)$.*
5. *The eigenfunctions form a* **Complete Orthonormal Set** *in the following sense*
$$\int_a^b \varphi_n(x)\varphi_m(x)\,dx = \delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$
*and if $f(x)$ is piecewise smooth ($PC^{(1)}(a,b)$ in my notation in class), then*
$$\frac{(f(x+) + f(x-))}{2} = \sum_{n=1}^{\infty} c_n\varphi_n(x), \quad a < x < b, \ \textit{where } c_n = \int_a^b f(x)\varphi_n(x)\,dx.$$
*For any $f$ such that* $\|f\| = \left(\int_a^b f^2(x)\,dx\right)^{1/2} < \infty$ *we have* $\left\| f(x) - \sum_{n=1}^{N} c_n\varphi_n(x) \right\| \xrightarrow{N\to\infty} 0$, *i.e.,*
$$f(x) = \sum_{n=1}^{\infty} c_n\varphi_n(x) \qquad \textit{in the sense of} \quad L^2(a,b).$$

# 3 1-D Heat Equation: Eigenvalues and Eigenvectors

Our first PDE is the heat equation on a finite rod $a \le x \le b$.

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad a < x < b, \quad t > 0 \\ u(x,0) &= f(x) \end{aligned}$$

There are three main types of boundary conditions imposed at the ends of the rod. The two main conditions are

$$u(a,t) = 0, \quad u(b,t) = 0 \quad \text{Dirichlet Conditions}$$

$$u_x(a,t) = 0, \quad u_x(b,t) = 0 \quad \text{Neumann Conditions}$$

$$\alpha_1 u_x(a,t) + \alpha_2 u(a,t) = 0, \quad \beta_1 u_x(b,t) + \beta_1 u(b,t) = 0 \quad \text{Robin Conditions}$$

We can also have any combination of these conditions, i.e., we could have a Dirichlet condition at $x = a$ and Neumann condition at $x = b$. Notice that Dirichlet and Neumann BCs are special cases of the Robin BCs.

There is also a more general problem involving two extra terms that correspond to heat conduction and convection.

$$\begin{aligned} u_t(x,t) &= k\big(u_{xx}(x,t) - 2au(x,t)_x + bu(x,t)\big), \quad 0 < x < \ell, \quad t > 0 \\ u(x,0) &= f(x). \end{aligned}$$

Let us consider the heat equation on $a \le x \le b$.

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad a < x < b, \quad t > 0 \\ \alpha_1 u_x(a,t) + \alpha_2 u(a,t) &= 0, \\ \beta_1 u_x(b,t) + \beta_2 u(b,t) &= 0 \\ u(x,0) &= \varphi(x) \end{aligned}$$

Applying separation of variables we seek simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

This gives

$$\frac{\psi'(t)}{k\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

and since the left side is independent of $x$ and the right side is independent of $t$, it follows that the expression must be a constant:

$$\frac{\psi'(t)}{k\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda$$

We seek to find all possible constants $\lambda$ and the corresponding <u>nonzero</u> functions $\varphi$ and $\psi$. The equation $\psi' - k\lambda\psi = 0$ has general solution

$$\psi(t) = Ce^{k\lambda t} \tag{3}$$

where $C$ is an arbitrary constant.

We also obtain

$$\varphi'' - \lambda\varphi = 0$$

Furthermore, the boundary conditions give

$$(\alpha_1\varphi'(a) - \alpha_2\varphi(a))\psi(t) = 0, \quad (\beta_1\varphi'(b) + \beta_2\varphi(b))\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\begin{gathered} \varphi''(x) - \lambda\varphi(x) = 0, \\ \alpha_1\varphi'(a) + \alpha_2\varphi(a) = 0, \\ \beta_1\varphi'(b) + \beta_2\varphi'(b) = 0. \end{gathered}$$

By the Sturm-Liouville Theorem there are infinitely many eigenpairs $\{\lambda_n, \varphi_n(x)\}$ with the $\{\varphi_n\}_{n=1}^{\infty}$ are orthonormal, i.e.,

$$\langle\varphi_n, \varphi_m\rangle = \int_a^b \varphi_n(x)\varphi_m(x)\,dx = \delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}.$$

We seek a solution to heat problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{\lambda_n t}\varphi_n(x). \tag{4}$$

We need two things:

1. When $t = 0$ we find values for $\{c_n\}$ so that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n\varphi_n(x)$$

   given by

$$c_n = \int_a^b f(x)\varphi_n(x)\,dx.$$

2. Next we need for the infinite sum in (7) to represent a solution to the equation. This is formally true since

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} c_n \frac{de^{\lambda_n t}}{dt}\varphi_n(x) \\ &= \sum_{n=1}^{\infty} c_n\lambda_n e^{\lambda_n t}\varphi_n(x) \\ &= \sum_{n=1}^{\infty} c_n e^{\lambda_n t}\frac{d^2\varphi_n}{dx^2}(x) = \frac{\partial^2 u}{\partial x^2}(x,t) \end{aligned}$$

   A rigorous proof in the case $f$ is smooth or if we use convergence in $L^2(a,b)$ and study weak solutions (a topic beyond the scope of this class).

# 4 1-D Wave Equation: Eigenvalues and Eigenvectors

The one dimensional wave equation modeling the displacement of a vibrating string of length $\ell = (b-a)$ covering the interval $a < x < b$ is

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad a < x < b, \quad t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \tag{5}$$

Once again we have the same three main types of boundary conditions imposed at the ends of the string: Dirichlet Conditions, Neumann Conditions, and Robin Conditions. These are all special cases of the general BCs

$$\alpha_1 \varphi'(a) + \alpha_2 \varphi(a) = 0, \beta_1 \varphi'(b) + \beta_2 \varphi(b) = 0.$$

Let us consider the heat equation on $a \le x \le b$.

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad a < x < b, \quad t > 0 \\ \alpha_1 u_x(a,t) + \alpha_2 u(a,t) &= 0, \\ \beta_1 u_x(b,t) + \beta_2 u(b,t) &= 0 \\ u(x,0) = f(x), \quad u_t(x,0) &= g(x) \end{aligned}$$

Applying separation of variables we seek simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

This gives

$$\frac{\psi''(t)}{c^2\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

and since the left side is independent of $x$ and the right side is independent of $t$, it follows that the expression must be a constant:

$$\frac{\psi''(t)}{c^2\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda$$

We seek to find all possible constants $\lambda$ and the corresponding <u>nonzero</u> functions $\varphi$ and $\psi$.

In the $x$ variable we have

$$\varphi'' - \lambda\varphi = 0$$

Furthermore, the boundary conditions give

$$(\alpha_1 \varphi(a) + \alpha_2 \varphi(a))\psi(t) = 0, \quad (\beta_1 \varphi(b) + \beta_2 \varphi(b))\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\begin{aligned} \varphi''(x) - \lambda\varphi(x) &= 0, \\ \alpha_1 \varphi'(a) + \alpha_2 \varphi(a) &= 0, \\ \beta_1 \varphi'(b) + \beta_2 \varphi(b) &= 0. \end{aligned}$$

By the Sturm-Liouville Theorem there are infinitely many eigenpairs $\{\lambda_n, \varphi_n(x)\}$ with the eigenfunctions forming a complete orthonormal set.

In most practical problems we have $\lambda_n = -\mu_n^2$ so the problem for $\psi$ becomes $\psi'' + c^2\mu_n^2\psi = 0$ has general solution

$$\psi_n(t) = a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t) \tag{6}$$

where $a_n$ and $b_n$ are arbitrary constants.

At least for continuous initial conditions $\varphi$ we obtain a solution to wave equation in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)\right) \varphi_n(x). \tag{7}$$

where

1. $f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$ with $a_n = \int_a^b f(x)\varphi_n(x)\,dx$.

2. $g(x) = u_t(x,0) = \sum_{n=1}^{\infty} (c\mu_n) b_n \varphi_n(x)$ with $b_n = (c\mu_n)^{-1} \int_a^b g(x)\varphi_n(x)\,dx$.

3. Next we need for the infinite sum in (7) to represent a solution to the equation. This is formally true since

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \sum_{n=1}^{\infty} c_n \frac{d^2}{dt^2} \left(a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)\right) \varphi_n(x) \\
&= \sum_{n=1}^{\infty} c_n (-c\mu_n)^2 \left(a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)\right) \varphi_n(x) \\
&= \sum_{n=1}^{\infty} c_n (c^2\lambda_n) \left(a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)\right) \varphi_n(x) \\
&= c^2 \sum_{n=1}^{\infty} c_n \left(a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)\right) \frac{d^2 \varphi_n}{dx^2}(x) \\
&= c^2 \frac{\partial^2 u}{\partial x^2}(x,t)
\end{aligned}$$