Orthogonal Functions and Fourier Series

1. The *inner product* of two functions f and g on $[a,b]$ is $\langle f,g\rangle = \displaystyle\int_a^b f(x)g(x)dx$.

2. f and g are *orthogonal* on $[a,b]$ is $\langle f,g\rangle = 0$.

3. A collection of functions $\{\varphi_j\}_{j=1}^{\infty}$ is *orthogonal* on $[a,b]$ if $\langle \varphi_j,\varphi_k\rangle = 0$.

4. The *norm* of a function f on $[a,b]$ is $\|f\| = \langle f,f\rangle^{1/2} = 0$.

5. A collection of functions $\{\varphi_j\}_{j=1}^{\infty}$ is *orthonormal* on $[a,b]$ if $\langle \varphi_j,\varphi_k\rangle = \delta_{jk}$, i.e., the functions are orthogonal and the norm of each function is 1.

6. Given an infinite orthogonal set $\{\varphi_j\}_{j=1}^{\infty}$ on $[a,b]$ an *orthogonal series expansion* is $\displaystyle\sum_{j=1}^{\infty} c_j\varphi_j(x)$ where the c_j are constants. Our main consideration is whether a given function f on $[a,b]$ has an infinite orthogonal expansion, i.e.,
$$f(x) = \sum_{j=1}^{\infty} c_j\varphi_j(x)$$
for some constants c_j.

7. The series of the functions
$$\varphi_0 = 1, \quad \varphi_n^{(1)} = \cos\left(\frac{n\pi x}{\ell}\right), \quad \varphi_n^{(2)} = \sin\left(\frac{n\pi x}{\ell}\right), \quad \text{for } n \geq 1$$
are orthogonal on $[-p,p]$. The associated orthogonal series expansion of a function f is
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right)\right]$$
is known as a *Fourier series*. **Note:** we use $a_0/2$ so that the formula for a_0 and a_n (given below) agree.

8. For $f(x) = f(x+2\ell)$ and f is piecewise smooth on $-\ell \leq x \leq \ell$. Then for all $x \in \mathbb{R}$
$$\frac{f(x+)+f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right)\right\},$$
where
$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx,\ a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx,\ b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$
If f is continuous (at x) then
$$\frac{f(x+)+f(x-)}{2} = f(x).$$
Here $f(x+)$ and $f(x-)$ denote the limits from the right and left respectively.

9. Let f and f' be piecewise continuous on $(-p, p)$ (i.e., f and f' are continuous except for possibly a finite number of points at which they may have only jump discontinuities). Then the Fourier series of f for all x satisfies

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right\}.$$

At a point of continuity the series converges to $f(x)$.

10. The following geometric properties are useful for half-range expansions:

    (a) A function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

    (b) Product of two even functions is even

    (c) product of two odd functions is even

    (d) Product of an even and odd function is odd

    (e) Sum of even functions is even

    (f) sum of odd functions is odd

11. **Fourier Cosine:** Given f on $[0, \ell]$

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x)\, dx, \quad a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right)\, dx$$

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

12. **Fourier Sine:** Given f on $[0, \ell]$

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx$$

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$