Autonomous Systems

A first order autonomous system is one of the form

$$\frac{d\boldsymbol{X}}{dt}(t) = \boldsymbol{G}(\boldsymbol{X}(t)), \quad \text{where } \boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{G}(\boldsymbol{X}(t)) = \begin{bmatrix} g_1(x_1, \cdots, x_n) \\ \vdots \\ g_n(x_1, \cdots, x_n) \end{bmatrix}. \tag{1}$$

Here $\boldsymbol{G}(\cdot)$ may be a linear or nonlinear matrix valued function of $x_1, \cdots, x_n$. The system is called autonomous provided $\boldsymbol{G}(\cdot)$ depends explicitly only on $x_1, \cdots, x_n$ and not on t.

When $n = 2$ we call the system a Plane autonomous System. In this case we write

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

The vector $\boldsymbol{V}(x, y) = (P(, y), Q(x, y))$ defines a *Vector Field* which is the velocity field of a stream (path of a particle) $\boldsymbol{X}(t) = (x(t), y(t))$, i.e., $\boldsymbol{X}'(t) = \boldsymbol{v}(x(t), y(t))$.

Types of solutions:

1. A **Constant solution** is a solution in the form $x(t) = x_0$, $y(t) = y_0$ or $\boldsymbol{X}(t) = \boldsymbol{X}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.
   A constant solution is called a **critical, stationary or equilibrium** solution.
2. A solution $\boldsymbol{X}(t) = (x(t), y(t))$ that is not constant and never crosses itself is called an **Arc Solution**.
3. A solution $\boldsymbol{X}(t) = (x(t), y(t))$ is called a **Periodic Solution** (or **cycle**) if there is a positive number p, called the *period*, so that $\boldsymbol{X}(t + p) = (x(t + p), y(t + p)) = (x(t), y(t)) = \boldsymbol{X}(t)$ for all t.

Deciding the type:

1. To find Critical points we find all solution pairs (x, y) of the system $P(x, y) = 0,\ Q(x, y) = 0$.
2. To find periodic solutions is very difficult in general.
   (a) A 2D linear system has period solutions if the eigenvalues are pure imaginary, $\lambda = \pm i\beta$:
   $$\boldsymbol{X}(t) = a_1\big[\cos(\beta t)\boldsymbol{w} - \sin(\beta t)\boldsymbol{z}\big] + a_2\big[\cos(\beta t)\boldsymbol{z} + \sin(\beta t)\boldsymbol{w}\big]$$
   This is no longer a sufficient condition if $n > 2$.
   (b) For general nonlinear systems it is very difficult to discover or predict periodic solutions – this is still an active research area in the theory of Dynamical Systems.
   (c) **Transformation to Polar Coordinates** Sometimes it is possible to find periodic solutions by transforming to polar coordinates. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$ then
   $$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right), \quad \frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right).$$

Stability for a 2D Linear System

For a 2D linear system

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Rightarrow \quad \frac{d\boldsymbol{X}}{dt} = A\boldsymbol{X} \text{ with } \boldsymbol{X}(0) = \boldsymbol{X}_0 \tag{2}$$

we set

$$\tau = \text{trace}(\boldsymbol{A}), \quad \Delta = \det(\boldsymbol{A}), \quad D = \tau^2 - 4\Delta$$

then $p(\lambda) = \det(\lambda I - A) = \lambda^2 - \tau\lambda + \Delta = 0$ gives

$$\lambda_1 = \frac{(\tau + \sqrt{D})}{2}, \quad \lambda_2 = \frac{(\tau - \sqrt{D})}{2}.$$

In this case the x-y plane is called the *phase plane* and a collection of trajectories $\boldsymbol{X}(t)$ a *phase portrait*. We are interested in describing the asymptotic behavior of trajectories using properties of the eigenvalues.

Case 1: $\boldsymbol{D > 0}$ implies there are two real distinct eigenvalues $\lambda_1 \neq \lambda_2$ with associated eigenvectors $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. The general solution is

$$\boldsymbol{X}(t) = a_1 e^{\lambda_1 t}\boldsymbol{V}_1 + a_2 e^{\lambda_2 t}\boldsymbol{V}_2. \tag{3}$$

We have the cases:

(a) **Both eigenvalues negative** $D > 0$, $\tau < 0$ and $\Delta > 0$ then $\lambda_2 < \lambda_1 < 0$ and $\boldsymbol{X}(t) \to \boldsymbol{0}$ as $t \to \infty$. We say that $\boldsymbol{0}$ is a **Asymptotically Stable Node**.

(b) **Both eigenvalues positive** $D > 0$, $\tau > 0$ and $\Delta > 0$ then $0 < \lambda_2 < \lambda_1$ and all solutions become unbounded, i.e.,

$$\|\boldsymbol{X}(t)\| = \sqrt{x^2(t) + y^2(t)} \xrightarrow{t \to \infty} \infty.$$

We say that $\boldsymbol{0}$ is a **Unstable Node**.

(c) **Eigenvalues with Opposite signs** $D > 0$, and $\Delta < 0$ then $\lambda_2 < 0 < \lambda_1$ and $\boldsymbol{X}(t)$ may approach zero but most solutions become unbounded as $t \to \infty$. Thus $\boldsymbol{0}$ is an unstable critical point but we say that it is a **Saddle Node**.

Case 2: $\Delta = 0$ implies there is one (double) real eigenvalue λ_0 and there are two possiblities:

(a) If there are two linearly independent eigenvectors v_1 and v_2. The general solution is

$$\boldsymbol{X}(t) = (a_1\boldsymbol{V}_1 + a_2\boldsymbol{V}_2)\, e^{\lambda_0 t}. \tag{4}$$

(b) If there is only one linearly independent eigenvector v_0. Then the general solution is

$$\boldsymbol{X}(t) = (a_1\boldsymbol{V}_0 + a_2(t\boldsymbol{V}_0 + \boldsymbol{P}))\, e^{\lambda_0 t}, \quad \text{where} \quad (\boldsymbol{A} - \lambda_0)\boldsymbol{P} = \boldsymbol{V}_0. \tag{5}$$

For either of these cases we have the following stability results:

i. If $\lambda_0 < 0$ then the origin is **stable** called a **Degenerate Stable Node**.
   A. If there are two eigenvectors $\boldsymbol{X}(t) \to 0$ along $c_1\boldsymbol{V}_1 + c_2\boldsymbol{V}_2$.
   B. If there is one eigenvector $\boldsymbol{X}(t) \to 0$ along $\boldsymbol{V}_1$.

ii. If $\lambda_0 > 0$ then the origin is **unstable** called a **Degenerate Unstable Node**.
   A. If there are two eigenvectors $\|\boldsymbol{X}(t)\| \to \infty$ along $c_1\boldsymbol{V}_1 + c_2\boldsymbol{V}_2$.
   B. If there is one eigenvector $\|\boldsymbol{X}(t)\| \to \infty$ along $\boldsymbol{V}_1$.

Case 3: $\Delta < 0$ implies there are complex eigenvalues $\lambda_0 = \alpha \pm i\beta$. In this case we solve $(\boldsymbol{A} - \lambda_0)\boldsymbol{V} = 0$ where v will contain complex numbers, i.e., $\boldsymbol{V} = \boldsymbol{W} + i\boldsymbol{Z}$ where $\boldsymbol{W}$ and $\boldsymbol{Z}$ are the real and imaginary parts of $\boldsymbol{V}$. Then the general solution can be written as

$$\boldsymbol{X}(t) = a_1 e^{\alpha t}\big[\cos(\beta t)\boldsymbol{W} - \sin(\beta t)\boldsymbol{Z}\big] + a_2 e^{\alpha t}\big[\cos(\beta t)\boldsymbol{Z} + \sin(\beta t)\boldsymbol{W}\big]. \tag{6}$$

(a) **Pure Imaginary eigenvalues** Here $\lambda = \alpha + i\beta$ with $\alpha = 0$.

$$\boldsymbol{X}(t) = a_1\big[\cos(\beta t)\boldsymbol{W} - \sin(\beta t)\boldsymbol{Z}\big] + a_2\big[\cos(\beta t)\boldsymbol{Z} + \sin(\beta t)\boldsymbol{W}\big]$$

from which we see $X(t)$ is periodic with period $p = 2\pi/\beta$. Writing the components as

$$x(t) = c_{11}\cos(\beta t) + c_{12}\sin(\beta t), \quad y(t) = c_{21}\cos(\beta t) + c_{22}\sin(\beta t)$$

where we assume

$$d = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}c_{22} - c_{12}c_{21} \neq 0.$$

Solving these equations for $\cos(\beta t)$ and $\sin(\beta t)$ and using $\cos^2(\beta t) + \sin^2(\beta t) = 1$ we arrive at a quadratic in x and y with coefficients functions of c_{11}, c_{12}, c_{21} c_{22}. After some simplifying it can be written in the form

$$ax^2 + 2hxy + by^2 + xy + dy + f = 0.$$

From analytic geometry we know that this represents an ellipse if $ab - h^2 > 0$. It can be shown that

$$ab - h^2 = \frac{1}{d^2} > 0.$$

So all trajectories are ellipses and the origin is called a **Center**. All trajectories either traverse in the clockwise or counter clockwise direction.

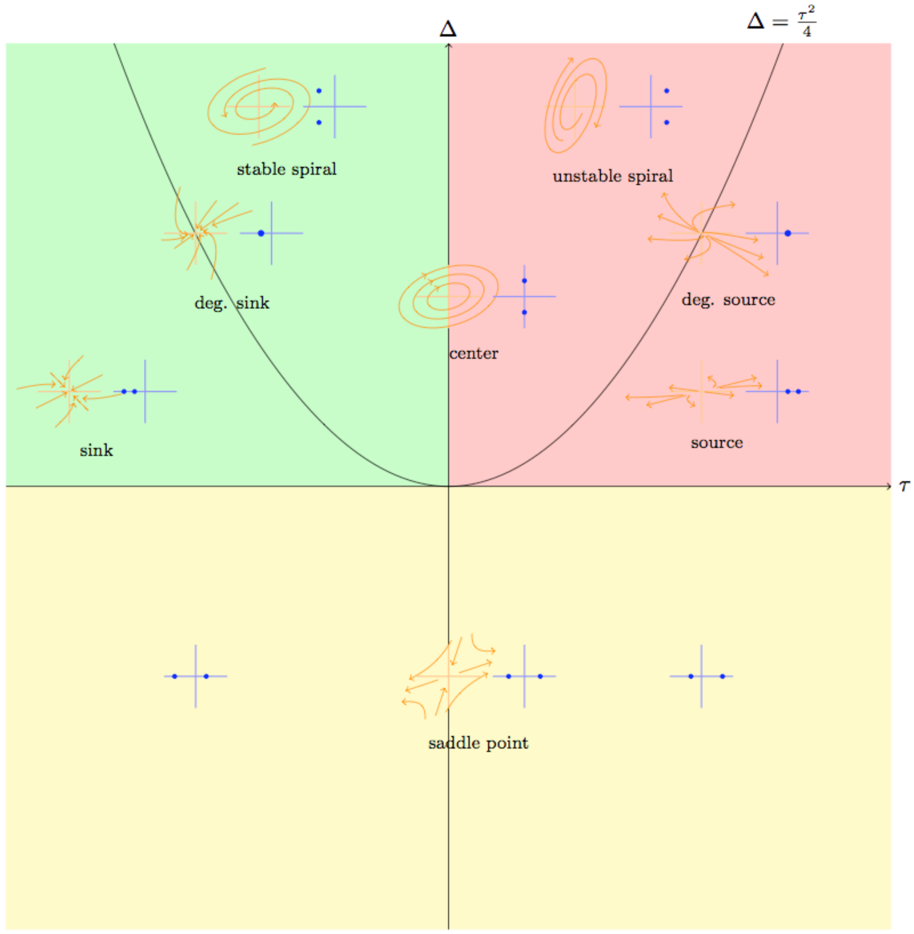
(b) **Eigenvalues have nonzero Real Part** In this case $\alpha \neq 0$ and the solution is

$$x(t) = e^{\alpha t}\left(c_{11}\cos(\beta t) + c_{12}\sin(\beta t)\right), \quad y(t) = e^{\alpha t}\left(c_{21}\cos(\beta t) + c_{22}\sin(\beta t)\right).$$

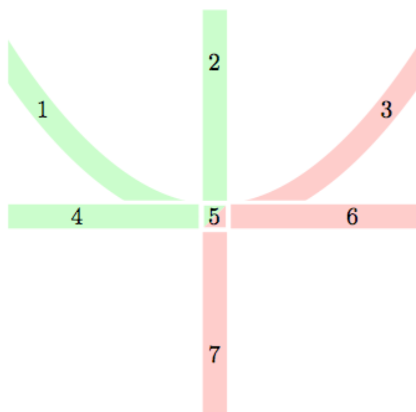
There are two cases:
   i. If $\alpha < 0$ we $\|X(t)\| \to 0$, the solution spirals inward and we say the origin is a **Stable Spiral Point**.
   ii. If $\alpha > 0$ we $\|X(t)\| \to \infty$, the solution spirals outward and we say the origin is a **Unstable Spiral Point**.

General Theorem: Let $\boldsymbol{A}$ be $n \times n$ with $\det(\boldsymbol{A}) \neq 0$ and solution $\boldsymbol{X}(t)$ of $\boldsymbol{X}' = A\boldsymbol{X}$, $\boldsymbol{X}(0) = \boldsymbol{X}_0 \neq 0$.

1. $\lim\limits_{t\to\infty} \boldsymbol{X}(t) = 0 \Leftrightarrow$ the eigenvalues of $\boldsymbol{A}$ all have negative real part, e.g., $\Delta > 0$ and $\tau < 0$.
2. $\boldsymbol{X}$ is periodic $\Leftrightarrow$ the eigenvalues are pure imaginary, e.g., $\Delta < 0$ and $\tau = 0$.
3. In all other cases there exits at least one IC $\boldsymbol{X}_0$ so that $\|\boldsymbol{X}(t)\| \to \infty$ as $t \to \infty$.



Green background means the origin is stable ; areas with a red or yellow background are unstable .



Region 1: $\Delta = \tau^2/4$, $\tau < 0 \Rightarrow$ one negative eigenvalue. Stable; **Region** 2: $\tau = 0$ eigenvalues pure imaginary. Center; **Region** 3: $\Delta = \tau^2/4$, $\tau > 0 \Rightarrow$ one positive eigenvalue. Unstable; **Region** 4: $\tau < 0$ and $\Delta = 0$ so one $\lambda_2 < 0$ and $\lambda_1 = 0$. $\boldsymbol{X} = c_1\boldsymbol{V}_1 + c_2 e^{\lambda_2 t}\boldsymbol{V}_2 \to c_1\boldsymbol{V}_1$. Nonzero stable critical point; **Region** 5: $\Delta = 0$ and $\tau = 0$ so $\lambda = 0$ is a double root. Two cases: $\boldsymbol{X} = c_1\boldsymbol{V}_1 + c_2\boldsymbol{V}_2$ or $\boldsymbol{X}(t) = C_1\boldsymbol{V}_0 + a_2(t\boldsymbol{V}_0 + \boldsymbol{P})$ where $(\boldsymbol{A} - \lambda_0)\boldsymbol{P} = \boldsymbol{V}_0$ which is unstable; **Region** 6: $\Delta = 0$ and $\tau > 0$ so essentially the same as Region 4 except that $\lambda_2 = 0$ and $\lambda_1 > 0$ so the system is unstable; **Region** 7: $\Delta < 0$ and $\tau = 0$ so we have $\lambda_2 < 0 < \lambda_1$; Saddle.

Linearization and Local Stability for Nonlinear System.

Once again first order autonomous system is one of the form

$$\frac{d\boldsymbol{X}}{dt}(t) = \boldsymbol{G}(\boldsymbol{X}(t)), \quad \text{where } \boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{G}(\boldsymbol{X}(t)) = \begin{bmatrix} g_1(x_1, \cdots, x_n) \\ \vdots \\ g_n(x_1, \cdots, x_n) \end{bmatrix}. \tag{7}$$

Definition 1. Let $\boldsymbol{X}_1$ denote a critical point of the autonomous system with solution $\boldsymbol{X}(t)$ for initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0$ where $\boldsymbol{X}_0 \neq \boldsymbol{X}_1$.

1. We say that $\boldsymbol{X}_1$ is a **Stable critical point** when for every $\rho > 0$ there is a corresponding $r > 0$ such that if $\boldsymbol{X}_0$ satisfies $\|\boldsymbol{X}_0 - \boldsymbol{X}_1\| < r$ then $\boldsymbol{X}(t)$ satisfies $\|\boldsymbol{X}_1 - \boldsymbol{X}(t)\| < \rho$ for all $t > 0$.
   If in addition, $\lim_{t\to\infty} \boldsymbol{X}(t) = \boldsymbol{X}_1$ when $\|\boldsymbol{X}_0 - \boldsymbol{X}_1\| < r$ then we say $\boldsymbol{X}_1$ is an **Asymptotically Stable critical point**.
2. We say that $\boldsymbol{X}_1$ is a **Unstable critical point** if it is not stable, i.e., if there is a disk of radius $\rho > 0$ with the property that, for any, $r > 0$ there is at least one $\boldsymbol{X}_0$ satisfying $\|\boldsymbol{X}_0 - \boldsymbol{X}_1\| < r$ for which $\boldsymbol{X}(t)$ satisfies $\|\boldsymbol{X}_1 - \boldsymbol{X}(t)\| > \rho$ for some $t > 0$.

It is very rare that one can obtain explicit solutions of nonlinear systems of ODEs. Thus determining stability or instability must rely on other means. We consider one such method, the method of **Linearization**.

Theorem 1. *Let x_1 be a critical point of $x' = g(x)$ where g is differentiable at x_1.*

1. *If $g'(x_1) < 0$ then x_1 is an Asymptotically Stable critical point.*
2. *If $g'(x_1) > 0$ then x_1 is an Unstable critical point.*

Now consider a plane autonomous system

$$\frac{d}{dt}\boldsymbol{X} = \boldsymbol{G}(\boldsymbol{X}) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} \quad \text{or} \quad \begin{array}{l} x' = P(x,y) \\ y' = Q(x,y) \end{array}.$$

The **Jacobian** matrix for this system is $\boldsymbol{G}' = \begin{bmatrix} \dfrac{\partial P(x,y)}{\partial x} & \dfrac{\partial P(x,y)}{\partial y} \\ \dfrac{\partial Q(x,y)}{\partial x} & \dfrac{\partial Q(x,y)}{\partial y} \end{bmatrix}$ or $\boldsymbol{G}' = \begin{bmatrix} P_x & P_y \\ Q_x & Q_y \end{bmatrix}$.

Theorem 2. *Let x_1 be a critical point of $\boldsymbol{X}' = \boldsymbol{G}(\boldsymbol{X})$ where $\boldsymbol{G}$ is differentiable at $\boldsymbol{X}_1$.*

1. *If the eigenvalues $\boldsymbol{G}'(\boldsymbol{X}_1)$ have negative real parts then $\boldsymbol{X}_1$ is an Asymptotically Stable critical point.*
2. *If $\boldsymbol{G}'(\boldsymbol{X}_1)$ has an eigenvalue with positive real part then $\boldsymbol{X}_1$ is an Unstable critical point.*

In the diagram linear results on stability imply the nonlinear case in all regions except for borderline cases. This means on the parabola $\tau^2 - 4\Delta = 0$, the positive and negative τ axis and the positive Δ axis. Note the negative Δ axis gives a saddle for linear and nonlinear.