Heat & Wave Equation in a Rectangle

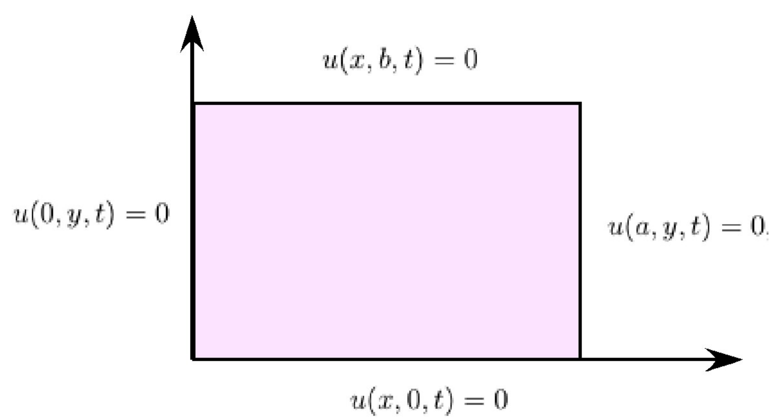
Section 12.8

1 Heat Equation in a Rectangle

In this section we are concerned with application of the method of separation of variables applied to the heat equation in two spatial dimensions. In particular we will consider problems in a rectangle.

Thus we consider

$$u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right), \quad t>0, \quad (x,y) \in [0,a]\times[0,b], \tag{1.1}$$
$$u(0,y,t)=0, \quad u(a,y,t)=0, \quad u(x,0,t)=0, \quad u(x,b,t)=0$$
$$u(x,y,0)=f(x,y)$$



$$u(x,y) = X(x)Y(y)T(t).$$

Substituting into (2.1) and dividing both sides by $kX(x)Y(y)T(t)$ gives

$$\frac{T'(t)}{kT(t)} = \frac{Y''(y)}{Y(y)} + \frac{X''(x)}{X(x)}$$

Since the left side is independent of x, y and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{T'(t)}{kT(t)} = \frac{Y''(y)}{Y(y)} + \frac{X''(x)}{X(x)} = \lambda.$$

We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions T, X and Y. We obtain

$$T'(t) - k\lambda T(t) = 0,$$

and

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}.$$

But since the left hand side depends only on x and the right hand side only on y, we conclude that there is a constant α

$$X'' - \alpha X = 0.$$

On the other hand we could also write

$$\frac{Y''(y)}{Y(y)} = \lambda - \frac{X''(x)}{X(x)}$$

so there exists a constant β so that

$$Y'' - \beta Y = 0.$$

Thus we have

$$\boxed{X'' - \alpha X = 0, \quad Y'' - \beta Y = 0, \quad T'(t) - k\lambda T(t) = 0 \quad \text{and } \lambda = \alpha + \beta.}$$

Furthermore, the boundary conditions give

$$X(0)Y(y) = 0, \quad X(a)Y(y) = 0, \quad \text{for all } y.$$

Since $Y(y)$ is not identically zero we obtain the desired eigenvalue problem

$$X''(x) - \alpha X(x) = 0, \quad X(0) = 0, \quad X(a) = 0. \tag{1.2}$$

We have solved this problem many times and we have $\alpha = -\mu^2$ so that

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Applying the boundary conditions we have

$$0 = X(0) = c_1 \Rightarrow c_1 = 0 \quad 0 = X(a) = c_2 \sin(\mu a).$$

From this we conclude $\sin(\mu a) = 0$ which implies

$$\mu = \frac{n\pi}{a}$$

and therefore

$$\boxed{\alpha_n = -\mu_n^2 = -\left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sqrt{\frac{2}{a}} \sin(\mu_n x), \quad n = 1, 2, \cdots.} \tag{1.3}$$

Now from the boundary condition

$$X(x)Y(0) = 0, \quad X(x)Y(b) = 0 \quad \text{for all } x.$$

This gives the problem

$$Y''(y) - \beta Y(y) = 0, \quad Y(0) = 0, \quad Y(b) = 0. \tag{1.4}$$

This is the same as the problem (2.2) so we obtain eigenvalues and eigenfunctions

$$\boxed{\beta_m = -\nu_m^2 = -\left(\frac{m\pi}{b}\right)^2, \quad Y_m(y) = \sqrt{\frac{2}{b}}\sin(\nu_m y), \quad n = 1, 2, \cdots.} \tag{1.5}$$

So we obtain eigenvalues of the main problem given by

$$\boxed{\lambda_{n,m} = -\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right)} \tag{1.6}$$

and corresponding eigenfunctions

$$\boxed{\varphi_{n,m}(x, y) = \frac{2}{\sqrt{ab}}\sin(\mu_n x)\sin(\nu_m y).}$$

We also find the solution to $T'(t) - k\lambda_{n,m}T(t) = 0$ is given by

$$T(t) = e^{k\lambda_{n,m}t}.$$

So we look for u as an infinite sum

$$\boxed{u(x, y, t) = \frac{2}{\sqrt{ab}}\sum_{n,m=1}^{\infty} c_{n,m}e^{k\lambda_{n,m}t}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).} \tag{1.7}$$

The only problem remaining is to somehow pick the constants $c_{n,m}$ so that the initial condition $u(x, y, 0) = f(x, y)$ is satisfied, i.e.,

$$f(x, y) = u(x, y, 0) = \sum_{n,m=1}^{\infty} c_{n,m}\varphi_{n,m}(x, y). \tag{1.8}$$

with

$$\boxed{c_{n,m} = \left(\frac{2}{\sqrt{ab}}\right)\int_0^b\int_0^a f(x, y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\,dx\,dy}$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

Example 1.1 (Dirichlet BCs)**.** To simplify the problem a bit we set $a = 1$ and $b = 1$. Namely we consider

$$\begin{aligned} &u_t(x, y, t) = k\left(u_{xx}(x, y, t) + u_{yy}(x, y, t)\right), \quad t > 0, \quad (x, y) \in [0, 1] \times [0, 1] \\ &u(0, y, t) = 0, \quad u(1, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, 1, t) = 0 \\ &u(x, y, 0) = x(1 - x)y(1 - y) \end{aligned} \tag{1.9}$$

In this case we obtain eigenvalues

$$\lambda_{n,m} = -\pi^2(n^2+m^2), \quad \alpha_n = -\pi^2 n^2, \quad \beta_m = -\pi^2 m^2, \quad n,m = 1,2,\cdots .$$

The corresponding eigenfunctions are given by

$$X_n(x) = \sqrt{2}\sin(n\pi x), \qquad Y_m(y) = \sqrt{2}\sin(m\pi y).$$

Our solution is given by

$$u(x,y,t) = 2\sum_{n=1}^{\infty} c_{n,m} e^{k\lambda_{n,m}t}\sin(n\pi x)\sin(m\pi y).$$

The coefficients $c_{n,m}$ are obtained from

$$x(1-x)y(1-y) = 2\sum_{n,m=1}^{\infty} c_{n,m}\sin(n\pi x)\sin(m\pi y).$$

We have

$$c_{n,m} = 2\int_0^1\int_0^1 x(1-x)y(1-y)\sin(n\pi x)\sin(m\pi y)\,dx\,dy = \frac{8((-1)^n-1)((-1)^m-1)}{n^3m^3\pi^6}.$$

$$u(x,y,t) = 2\sum_{n,m=1}^{\infty} c_{n,m} e^{k\lambda_{n,m}t}\sin(n\pi x)\sin(m\pi y). \tag{1.10}$$

That is

$$u(x,y,t) = \frac{16}{\pi^6}\sum_{n,m=1}^{\infty}\frac{((-1)^n-1)((-1)^m-1)e^{k\lambda_{n,m}t}}{n^3m^3}\sin(n\pi x)\sin(m\pi y). \tag{1.11}$$

Example 1.2 (Mixed Dirichlet and Neumann BCs)**.** To simplify the problem a bit again set $a = 1$ and $b = 1$. Namely we consider

$$\begin{aligned} &u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right), \quad t>0, \quad (x,y)\in[0,1]\times[0,1] \\ &u(0,y,t) = 0, \quad u(1,y,t) = 0, \quad u_y(x,0,t) = 0, \quad u_y(x,1,t) = 0 \\ &u(x,y,0) = x(1-x)y \end{aligned} \tag{1.12}$$

In this case we obtain eigenvalues

$$\lambda_{n,0} = -\pi^2 n^2, \quad Y_0(y) = 1,$$

and

$$\lambda_{n,m} = -\pi^2(n^2+m^2), \quad \alpha_n = -\pi^2 n^2, \quad \beta_m = -\pi^2 m^2, \quad n,m = 1,2,\cdots,$$

with corresponding eigenfunctions are given by

$$X_n(x) = \sqrt{2}\sin(n\pi x), \qquad Y_m(y) = \sqrt{2}\cos(m\pi y).$$

Our solution is given by

$$u(x,y,t) = \sum_{n=1}^{\infty} c_{n,0} e^{k\lambda_{n,0}t}\sqrt{2}\sin(n\pi x) + \sum_{n,m=1}^{\infty} c_{n,m} e^{k\lambda_{n,m}t}\sqrt{2}\sin(n\pi x)\sqrt{2}\cos(m\pi y).$$

Setting $t = 0$ we obtain

$$x(1-x)y = \sum_{n=1}^{\infty} c_{n,0}\sqrt{2}\sin(n\pi x) + \sum_{n,m=1}^{\infty} c_{n,m}\sqrt{2}\sin(n\pi x)\sqrt{2}\cos(m\pi y).$$

This double Fourier series is evaluated again using orthogonality relations. We have

$$c_{n,m} = 2\int_0^1\int_0^1 x(1-x)y\sin(n\pi x)\cos(m\pi y)\,dx\,dy = \frac{-8((-1)^n-1)((-1)^m-1)}{n^3m^2\pi^5}.$$

Finally we obtain the coefficients $c_{n,0}$ from

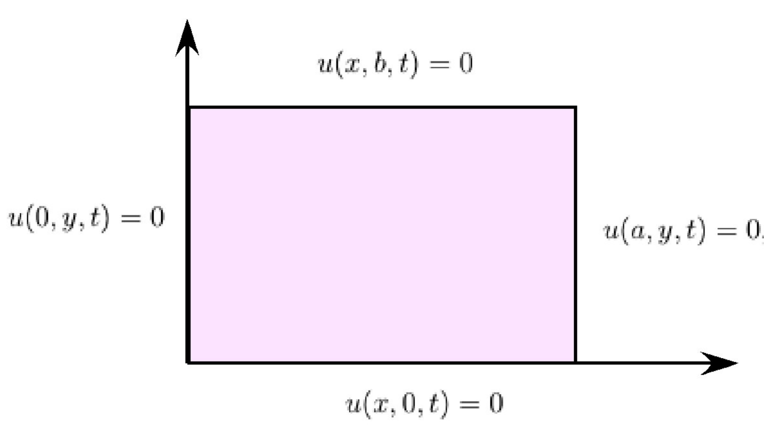
$$c_{n,0} = \sqrt{2}\int_0^1\int_0^1 x(1-x)y\sin(n\pi x)\,dx\,dy = \frac{-\sqrt{2}((-1)^n-1)}{n^3\pi^3}.$$

$$\begin{aligned} u(x,y,t) &= 2\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3\pi^3}e^{k\lambda_{n,0}t}\sin(n\pi x) \\ &\quad -\frac{16}{\pi^5}\sum_{n,m=1}^{\infty}\frac{((-1)^n-1)((-1)^m-1)e^{k\lambda_{n,m}t}}{n^3m^2}\sin(n\pi x)\cos(m\pi y). \end{aligned}$$

2 Wave Equation in Higher Dimensions

In this section we are concerned with application of the method of separation of variables applied to the wave equation in a two dimensional rectangle. Thus we consider

$$\begin{aligned} &u_{tt}(x,y,t) = c^2\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right), \quad t>0, \quad (x,y)\in[0,a]\times[0,b], \qquad (2.1)\\ &u(0,y,t)=0, \quad u(a,y,t)=0, \quad u(x,0,t)=0, \quad u(x,b,t)=0\\ &u(x,y,0)=f(x,y), \quad u_t(x,y,0)=g(x,y) \end{aligned}$$



$$u(x, y) = X(x)Y(y)T(t).$$

Substituting into (2.1) and dividing both sides by $X(x)Y(y)$ gives

$$\frac{T''(t)}{c^2T(t)} = \frac{Y''(y)}{Y(y)} + \frac{X''(x)}{X(x)}$$

Since the left side is independent of x, y and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{T''(t)}{c^2T(t)} = \frac{Y''(y)}{Y(y)} + \frac{X''(x)}{X(x)} = \lambda.$$

We seek to find all possible constants λ and the corresponding nonzero functions T, X and Y. We obtain

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)} \qquad T''(t) - c^2\lambda T(t) = 0.$$

Thus we conclude that there is a constant α

$$X'' - \alpha X = 0.$$

On the other hand we could also write

$$\frac{Y''(y)}{Y(y)} = \lambda - \frac{X''(x)}{X(x)}$$

so there exists a constant β so that

$$Y'' - \beta Y = 0.$$

Furthermore, the boundary conditions give

$$X(0)Y(y) = 0, \quad X(a)Y(y) = 0 \quad \text{for all } y.$$

Since $Y(y)$ is not identically zero we obtain the desired eigenvalue problem

$$X''(x) - \alpha X(x) = 0, \quad X(0) = 0, \quad X(a) = 0. \tag{2.2}$$

We have solved this problem many times and we have $\alpha = -\mu^2$ so that

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Applying the boundary conditions we have

$$0 = X(0) = c_1 \Rightarrow c_1 = 0 \quad 0 = X(a) = c_2 \sin(\mu a).$$

From this we conclude $\sin(\mu a) = 0$ which implies

$$\mu = \frac{n\pi}{a}$$

and therefore

$$\alpha_n = -\mu_n^2 = -\left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sqrt{\frac{2}{a}}\sin(\mu_n x), \quad n = 1, 2, \cdots .. \tag{2.3}$$

Now from the boundary condition

$$X(x)Y(0) = 0, \quad X(x)Y(b) = 0 \quad \text{for all } x.$$

This gives the problem

$$Y''(y) - \beta Y(y) = 0, \quad Y(0) = 0, \quad Y(b) = 0. \tag{2.4}$$

This is the same as the problem (2.2) so we obtain eigenvalues and eigenfunctions

$$\beta_m = -\nu_m^2 = -\left(\frac{m\pi}{b}\right)^2, \quad Y_m(y) = \sqrt{\frac{2}{b}}\sin(\nu_m y), \quad n = 1, 2, \cdots .. \tag{2.5}$$

So we obtain eigenvalues of the main problem given by

$$\lambda_{n,m} = -\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right) \tag{2.6}$$

and corresponding eigenfunctions

$$\varphi_{n,m}(x,y) = \frac{2}{\sqrt{ab}}\sin(\mu_n x)\sin(\nu_m y).$$

We also find the solution to $T''(t) - c^2\lambda_{n,m}T(t) = 0$ is given by

$$T_{n,m}(t) = [a_{n,m}\cos(c\omega_{n,m}t) + b_{n,m}\sin(c\omega_{n,m}t)]$$

where we have defined

$$\omega_{n,m} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}.$$

So we look for u as an infinite sum

$$u(x,y,t) = \frac{2}{\sqrt{ab}}\sum_{n,m=1}^{\infty}[a_{n,m}\cos(c\omega_{n,m}t) + b_{n,m}\sin(c\omega_{n,m}t)]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right). \tag{2.7}$$

We have left to find the constants $a_{n,m}$ and $b_{n,m}$ so that the initial condition $u(x,y,0) = f(x,y)$ and $u_t(x,y,0) = g(x,y)$ are satisfied, i.e.,

$$f(x,y) = u(x,y,0) = \frac{2}{\sqrt{ab}}\sum_{n,m=1}^{\infty} a_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right). \tag{2.8}$$

Thus we conclude that

$$a_{n,m} = \left(\frac{2}{\sqrt{ab}}\right) \int_0^b \int_0^a f(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \, dx\, dy$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

In a similar way we have

$$g(x,y) = u_t(x,y,0) = \frac{2}{\sqrt{ab}} \sum_{n,m=1}^{\infty} c\omega_{n,m} b_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \tag{2.9}$$

with

$$b_{n,m} = \left(\frac{2}{\sqrt{ab}\,\omega_{n,m}}\right) \int_0^b \int_0^a g(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \, dx\, dy$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

Example 2.1. In this example we set $c = 1$, $a = \pi$ and $b = \pi$. Namely we consider

$$\begin{aligned} u_{tt}(x,y,t) &= \left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right), \quad t > 0, \quad (x,y) \in [0,\pi] \times [0,\pi] \\ u(0,y,t) &= 0, \quad u(\pi,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,\pi,t) = 0 \\ u(x,y,0) &= x(\pi - x)y(\pi - y), \quad u_t(x,y,0) = 0. \end{aligned} \tag{2.10}$$

In this case we obtain eigenvalues

$$\lambda_{n,m} = -(n^2 + m^2), \quad \alpha_n = -n^2, \quad \beta_m = -m^2, \quad n, m = 1, 2, \cdots.$$

The corresponding eigenfunctions are given by

$$X_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \qquad Y_m(y) = \sqrt{\frac{2}{\pi}} \sin(my).$$

Our solution is given by

$$u(x,y,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[a_{n,m} \cos(\omega_{n,m} t) + b_{n,m} \sin(\omega_{n,m} t)\right] \sin(nx) \sin(my)$$

where we have defined

$$\omega_{n,m} = \sqrt{n^2 + m^2}.$$

The coefficients $a_{n,m}$ are obtained from

$$x(\pi - x)y(\pi - y) = \frac{2}{\pi} \sum_{n,m=1}^{\infty} a_{n,m} \sin(nx) \sin(my).$$

We have

$$a_{n,m} = \frac{2}{\pi} \int_0^\pi \int_0^\pi x(\pi - x)y(\pi - y) \sin(nx) \sin(my) \, dx\, dy = \frac{8((-1)^n - 1)((-1)^m - 1)}{n^3 m^3 \pi}.$$

Since $u_t(x,y,0) = g(x,y) = 0$ we have

$$b_{n,m} = 0.$$

$$u(x,y,t) = \frac{16}{\pi^2} \sum_{n,m=1}^{\infty} \frac{((-1)^n - 1)((-1)^m - 1)}{n^3 m^3} e^{k\lambda_{n,m} t} \sin(nx) \sin(my). \tag{2.11}$$

Example 2.2. In this example we set $c = 1$, $a = \pi$ and $b = \pi$. Namely we consider

$$\begin{aligned} u_{tt} &= (u_{xx}(x,y,t) + u_{yy}(x,y,t)), \quad t > 0, \quad (x,y) \in [0,\pi] \times [0,\pi] \qquad (2.12)\\ u_x(0,y,t) &= 0, \quad u_x(\pi,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,\pi,t) = 0 \\ u(x,y,0) &= x(\pi - x)y, \qquad u_t(x,y,0) = 0. \end{aligned}$$

We get eigenvalue problem in x given by

$$X'' - \alpha X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0.$$

Therefore we have eigenvalues and eigenvectors

$$\alpha_0 = 0, \quad X_0(x) = \frac{1}{\sqrt{\pi}}, \qquad \alpha_n = -n^2, \quad X_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx), \quad n = 1,2,3,\cdots.$$

The eigenvalue problem in y is given by

$$Y'' - \beta Y = 0, \quad Y(0) = 0, \quad Y(\pi) = 0.$$

The corresponding eigenvalues are

$$\beta_m = -m^2, \quad Y_m(y) = \sqrt{\frac{2}{\pi}} \sin(my), \quad m = 1,2,3,\cdots.$$

In this case we obtain eigenvalues

$$\lambda_{n,m} = -(n^2 + m^2), \quad \alpha_n = -n^2, \quad \beta_m = -m^2, \quad n,m = 1,2,\cdots.$$

The corresponding eigenfunctions are given by

$$\varphi_{n,m}(x,y) = \frac{2}{\pi} \cos(nx) \sin(my).$$

For this example we also have eigenvalues

$$\lambda_{0,m} = -m^2, \quad X_0(x) = \frac{1}{\sqrt{\pi}}.$$

Our solution is given by

$$\begin{aligned} u(x,y,t) = &\frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \left[a_{0,m} \cos(\omega_{0,m} t) + b_{0,m} \sin(\omega_{0,m} t)\right] \sin(my) \\ &+ \frac{2}{\pi} \sum_{n,m=1}^{\infty} \left[a_{n,m} \cos(\omega_{n,m} t) + b_{0,m} \sin(\omega_{0,m} t)\right] \cos(nx) \sin(my). \end{aligned}$$

Setting $t = 0$ we obtain

$$x(\pi - x)y = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty} a_{n,0}\cos(nx) + \frac{2}{\pi}\sum_{n,m=1}^{\infty} a_{n,m}\cos(nx)\sin(my).$$

We have

$$a_{n,m} = \frac{2}{\pi}\int_0^{\pi}\int_0^{\pi} x(1-x)y\cos(nx)\sin(my)\,dx\,dy = \frac{2\pi(-1)^m((-1)^n+1)}{n^2 m}.$$

Finally we obtain the coefficients $a_{n,0}$ from

$$a_{0,m} = \frac{\sqrt{2}}{\pi}\int_0^{\pi}\int_0^{\pi} x(\pi - x)y\sin(my)\,dx\,dy = \frac{\sqrt{2}(-1)^{m+1}\pi^3}{6m}.$$

Finally we arrive at the solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\frac{\pi^2(-1)^{m+1}}{3m}\cos(\omega_{0,m}t)\sin(my)$$
$$+ \sum_{n,m=1}^{\infty}\frac{4((-1)^n+1)((-1)^m)}{n^2 m}\cos(\omega_{n,m}t)\cos(nx)\sin(my)$$

with $\omega_{n,m} = \sqrt{n^2 + m^2}$.