Constant Coefficient 2D First Order Systems: Eigenvalues & Eigenvectors

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

If we introduce matrices we can write the system in a simple form

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Rightarrow \quad \frac{d\boldsymbol{X}}{dt} = A\boldsymbol{X}. \tag{2}$$

Look for a simple solution $\boldsymbol{X} = e^{\lambda t}\boldsymbol{v}$ with $\boldsymbol{v} \neq 0$ then we have

$$\lambda \boldsymbol{X} = \frac{d\boldsymbol{X}}{dt} = \boldsymbol{A}\boldsymbol{X}, \quad \lambda \boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}$$

which implies

$$(\lambda I - \boldsymbol{A})\boldsymbol{v} = 0. \tag{3}$$

The homogeneous system has a nonzero solution if and only if

$$\det(\lambda I - \boldsymbol{A}) = \lambda^2 - \mathrm{trace}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = 0.$$

The quadratic polynomial is called the *characteristic polynomial*, the roots are called the *eigenvalues* and the associated (nonzero) vectors $\boldsymbol{v}$ are called the *eigenvectors*. Set $\Delta = \det(\boldsymbol{A})$, $\tau = \mathrm{trace}(\boldsymbol{A})$ and

$$D = \tau^2 - 4\Delta.$$

Tthere are three cases depending on the cases $D > 0$, $D = 0$ and $D < 0$.

1. $D > 0$ implies there are two real distinct eigenvalues $\lambda_2 < \lambda_1$ with associated eigenvectors v_1 and v_2. The general solution of (1) is

$$\boldsymbol{X}(t) = a_1 e^{\lambda_1 t}\boldsymbol{V}_1 + a_2 e^{\lambda_2 t}\boldsymbol{V}_2. \tag{4}$$

2. $D = 0$ implies there is real distinct eigenvalue (a double root) λ_0 and there are two possiblities:
   (a) There may be two linearly independent eigenvectors $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. If so the general solution of (1) is

$$\boldsymbol{X}(t) = a_1 e^{\lambda_0 t}\boldsymbol{V}_1 + a_2 e^{\lambda_0 t}\boldsymbol{V}_2. \tag{5}$$

   (b) There may only be one linearly independent eigenvector $\boldsymbol{V}_0$. This is the most complicated case. In this case the general solution of (1) is then

$$\boldsymbol{X}(t) = a_1 e^{\lambda_0 t}\boldsymbol{V}_0 + a_2 e^{\lambda_0 t}(t\boldsymbol{V}_0 + \boldsymbol{P}) \tag{6}$$

   where $\boldsymbol{w}$ is any solution of $(\boldsymbol{A} - \lambda_0)\boldsymbol{P} = \boldsymbol{V}_0$.

3. $D < 0$ implies there are complex eigenvalues $\lambda_0 = \alpha \pm i\beta$. In this case we solve $(\boldsymbol{A} - \lambda_0)\boldsymbol{v} = 0$ where v will contain complex numbers, i.e., $\boldsymbol{v} = \boldsymbol{W} + i\boldsymbol{Z}$ where $\boldsymbol{W}$ and $\boldsymbol{Z}$ are the real and imaginary parts of $\boldsymbol{v}$. Then the general solution can be written as

$$\boldsymbol{X}(t) = a_1 e^{\alpha t}\Big[\cos(\beta t)\boldsymbol{W} - \sin(\beta t)\boldsymbol{Z}\Big] + a_2 e^{\alpha t}\Big[\cos(\beta t)\boldsymbol{Z} + \sin(\beta t)\boldsymbol{W}\Big]. \tag{7}$$

Constant Coefficient First Order Systems: the General Case

The case of a general $n \times n$ matrix is somewhat more complicated. We will not attempt a complete discussion. Consider

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{A}\boldsymbol{X} \quad \text{where} \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}. \tag{8}$$

Once again seeking simple solutions $\boldsymbol{X} = e^{\lambda t}\boldsymbol{V}$ with $\boldsymbol{V} \neq 0$ leads to

$$(\lambda I - \boldsymbol{A})\boldsymbol{V} = 0$$

The homogeneous system has a nonzero solution if and only if

$$p(\lambda) = \det(\lambda I - \boldsymbol{A}) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0.$$

Since we assume that $n > 2$ there are many more possibilities than in the case $n = 2$. We only consider a few cases.

1. **Distinct Real Eigenvalues** If $\boldsymbol{A}$ has n distinct real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ with associated eigenvectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \cdots, \boldsymbol{V}_n$ then the general solution is given by

$$\boldsymbol{X}(t) = c_1 e^{\lambda_1 t}\boldsymbol{V}_1 + c_2 e^{\lambda_2 t}\boldsymbol{V}_2 + \cdots + c_n e^{\lambda_n t}\boldsymbol{V}_n.$$

2. **A Repeated Eigenvalue** Suppose that one of the eigenvalues, say λ_j is an mth order root of $p(\lambda)$ then there are many possibilities.
   (a) If there are m linearly independent eigenvectors $\boldsymbol{V}_{j1}, \boldsymbol{V}_{j2}, \cdots, \boldsymbol{V}_{jm}$ for λ_j then the general solution **contains** an expression of the form

$$\boldsymbol{X}_j(t) = c_{j1} e^{\lambda_j t}\boldsymbol{V}_{j1} + c_{j2} e^{\lambda_j t}\boldsymbol{V}_{j2} + \cdots + c_{jm} e^{\lambda_j t}\boldsymbol{V}_{jm}.$$

   (b) If for example, if λ_j has multiplicity two and there is only one eigenvector $\boldsymbol{V}_j$ then $X_j = a_1 e^{\lambda_j t}\boldsymbol{V}_j + a_2 e^{\lambda_j t}(t\boldsymbol{V}_j + \boldsymbol{P})$ where $(\boldsymbol{A} - \lambda_j)\boldsymbol{P} = \boldsymbol{V}_j$. But if λ_j has multiplicity three and there is only one eigenvector $\boldsymbol{V}_j$ then

$$X_j = a_1 e^{\lambda_j t}\boldsymbol{V}_j + a_2 e^{\lambda_j t}(t\boldsymbol{V}_j + \boldsymbol{P}) + a_3 e^{\lambda_j t}\left(\frac{t^2}{2}\boldsymbol{V}_j + t\boldsymbol{P} + Q\right)$$

   where $(\boldsymbol{A} - \lambda_j)\boldsymbol{P} = \boldsymbol{V}_j$ and $(\boldsymbol{A} - \lambda_j)\boldsymbol{Q} = \boldsymbol{P}$.

3. **A Pair of Complex Eigenvalues** Suppose $\lambda_j = \alpha + i\beta$ is an eigenvalue of multiplicity one (note that also $\alpha - i\beta$ is an eigenvalue if $\boldsymbol{A}$ has real coefficients) with eigenvector $\boldsymbol{V}_j = \boldsymbol{W} + i\boldsymbol{Z}$. The we write

$$\boldsymbol{X}_{j1}(t) = [\boldsymbol{W}\cos(\beta t) - \boldsymbol{Z}\sin(\beta t)]\, e^{\alpha t}, \quad \boldsymbol{X}_{j1}(t) = [\boldsymbol{Z}\cos(\beta t) + \boldsymbol{W}\sin(\beta t)]\, e^{\alpha t}$$

   where $\boldsymbol{W} = \mathrm{Re}\ (\boldsymbol{V}_j)$ and $\boldsymbol{Z} = \mathrm{Im}\ (\boldsymbol{V}_j)$. Then the general solution contains terms

$$\boldsymbol{X}_j(t) = c_{j1}\boldsymbol{X}_{j1}(t) + c_{j2}\boldsymbol{X}_{j2}(t).$$

   Notice this is not the most general case since λ_j could have multiplicity $m > 1$.