We consider an example of initial boundary value problem for the wave equation on the seni-line with Dirichlet boundary condition at $x = 0$:

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t), \quad 0 < x < \infty, \quad t > 0 \\ u(0,t) &= 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < \infty, \\ u_t(x,t) &= 0, \quad 0 < x < \infty, . \end{aligned} \tag{1}$$

Let $f(x)$ be given by

$$f(x) = \begin{cases} \dfrac{(1-\cos(\pi x))}{2}, & 2 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

Assuming that $f(x) = 0$ for $x < 0$, we can write the odd extension $\widetilde{f}(x)$ of $f(x)$ is

$$\widetilde{f}(x) = f(x) - f(-x).$$

The solution $\widetilde{u}(x,t)$ of

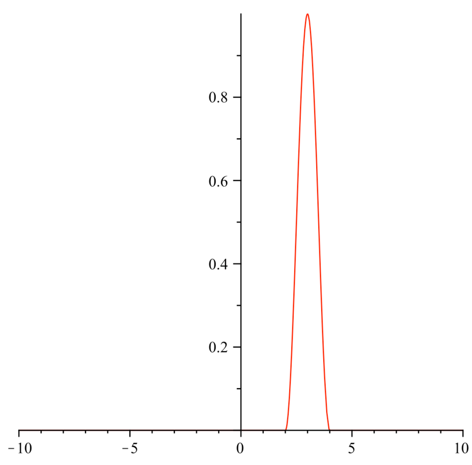
$$\begin{aligned} \widetilde{u}_{tt}(x,t) &= \widetilde{u}_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0 \\ \widetilde{u}(x,0) &= \widetilde{f}(x), \quad -\infty < x < \infty, \\ \widetilde{u}_t(x,t) &= 0, \quad -\infty < x < \infty, . \end{aligned} \tag{2}$$

of

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x+t) + \widetilde{f}(x-t)}{2}.$$

When we restrict this solution to $x > 0$ we obtain (recall $f(x) = 0$ for $x < 0$)

$$u(x,t) = \frac{f(x+t) + f(x-t) - f(t-x)}{2}. \tag{3}$$



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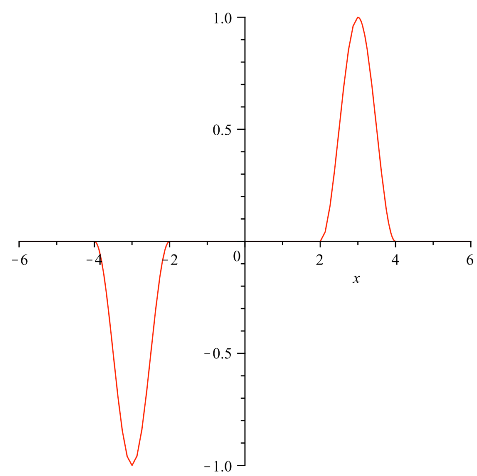
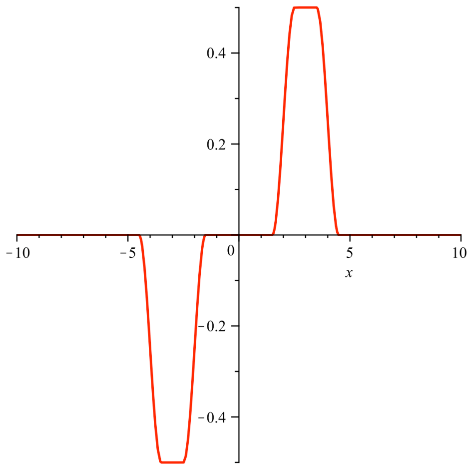
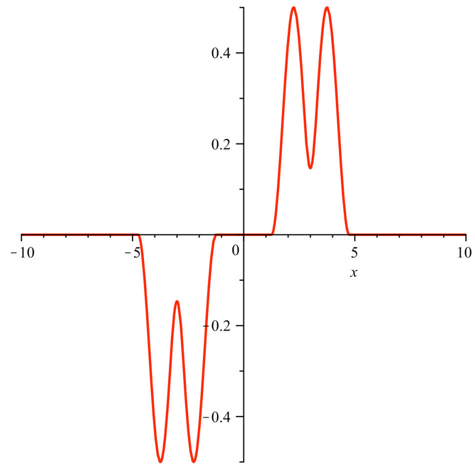


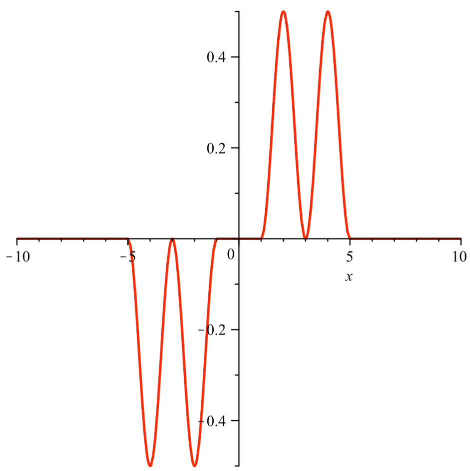
Odd Extension



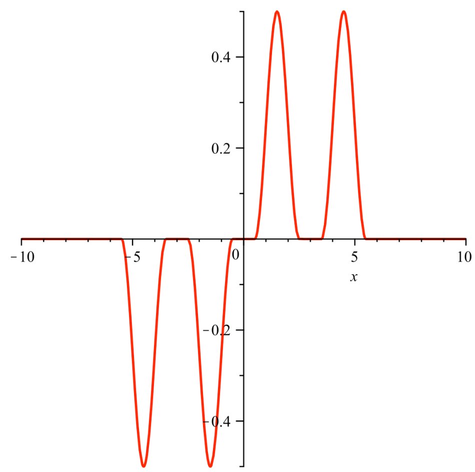
$t = .5$



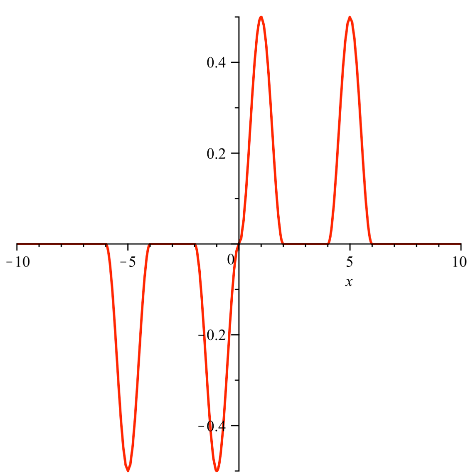
$t = .75$



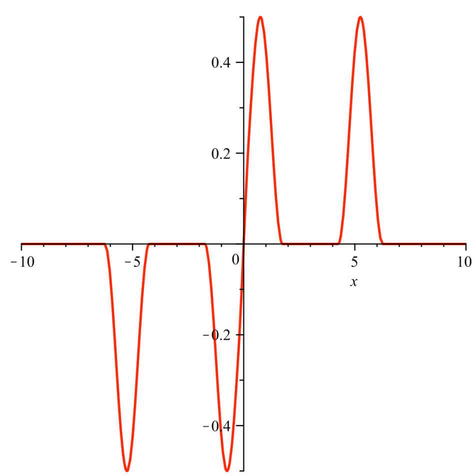
$t = 1$



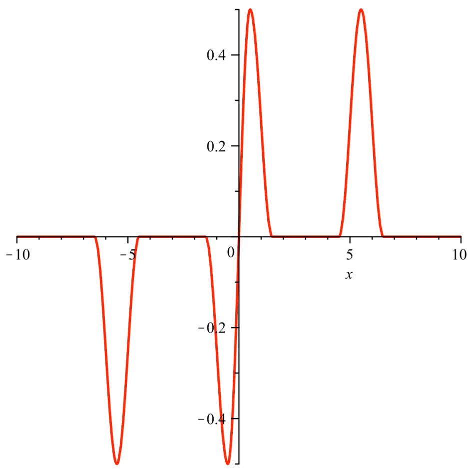
$t = 1.5$



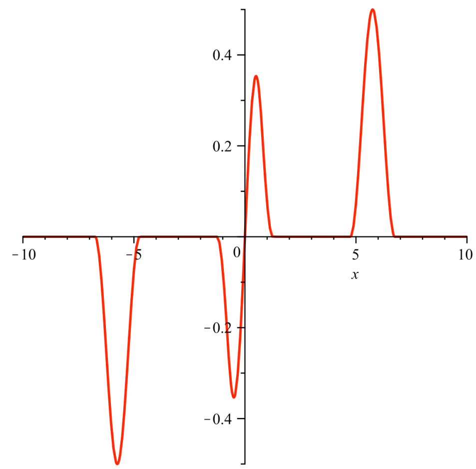
$t = 2$



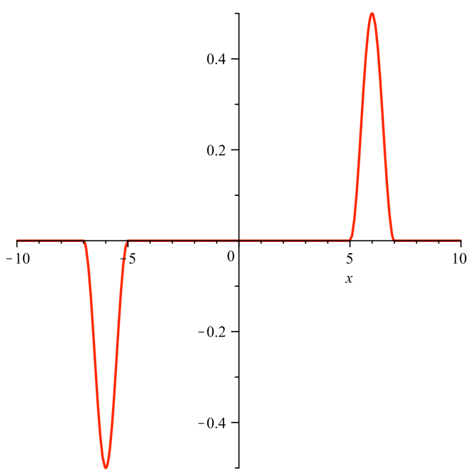
$t = 2.25$



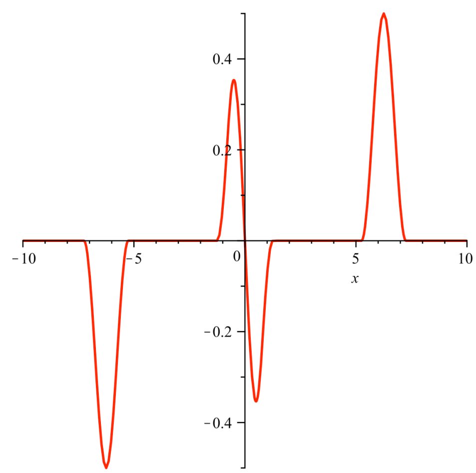
$t = 2.5$



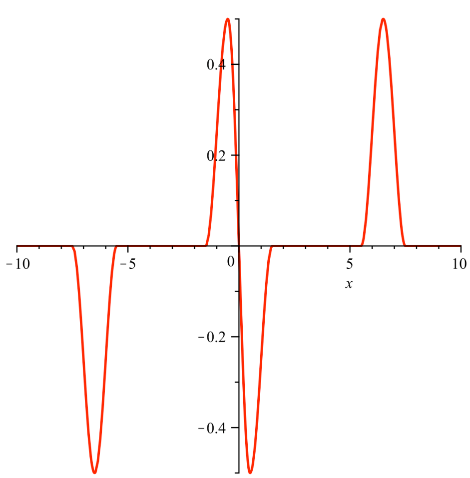
$t = 2.75$



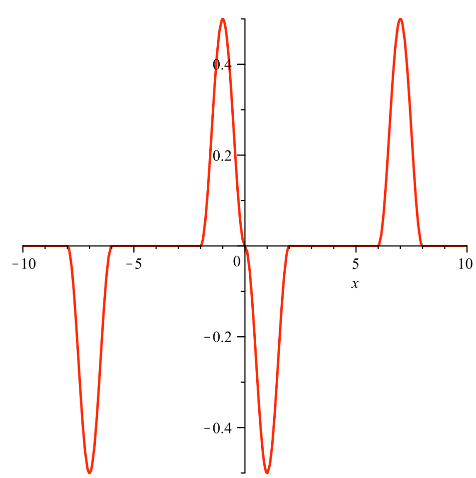
$t = 3$



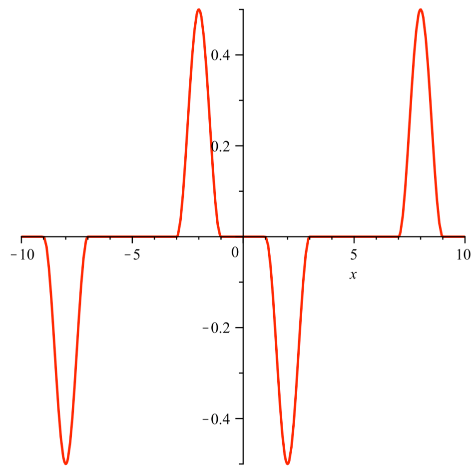
$t = 3.25$



$t = 3.5$



$t = 4$



$t = 5$

In order to compute the solution at a positive value of t we use the formula (3) where

$$f(x-t) = \begin{cases} \dfrac{(1-\cos(\pi(x-t)))}{2}, & 2 < (x-t) < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{(1-\cos(\pi(x-t)))}{2}, & 2+t < x < 4+t \\ 0, & \text{otherwise} \end{cases}$$

Notice that this term corresponds to a wave traveling to the right. The term $f(x+t)$ is a wave traveling to the left while the term $-f(t-x)$ is a wave traveling to the right. As the two terms meet they cancel each other in such a way that the value of $f(x+t) - f(t-x)$ is always zero at $x = 0$ i.e., $f(0+t) - f(t-0) = 0$. Thus the solution always satisfies the boundary condition at $x = 0$.

$$f(x+t) = \begin{cases} \dfrac{(1-\cos(\pi(x+t)))}{2}, & 2 < (x+t) < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$-f(t-x) = \begin{cases} -\dfrac{(1-\cos(\pi(t-x)))}{2}, & 2 < (t-x) < 4 \\ 0, & \text{otherwise} \end{cases}$$

Let us consider the special case of $t = 3$. In this case the above formulas become

$$f(x-3) = \begin{cases} \dfrac{(1-\cos(\pi(x-3)))}{2}, & 5 < x < 7 \\ 0, & \text{otherwise} \end{cases}$$

See the picture above when $t = 3$.

According to our picture at $t = 3$ we should see the wave traveling left and the one traveling right completely cancel. To see this we note that

$$f(x+3) = \begin{cases} \dfrac{(1-\cos(\pi(x+3)))}{2}, & 2 < (x+3) < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$-f(3-x) = \begin{cases} -\dfrac{(1-\cos(\pi(3-x)))}{2}, & 2 < (3-x) < 4 \\ 0, & \text{otherwise} \end{cases}.$$

Recalling that $f(x) = 0$ for $x < 0$ these can be written as

$$f(x+3) = \begin{cases} \dfrac{(1-\cos(\pi(x+3)))}{2}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$-f(3-x) = \begin{cases} -\dfrac{(1-\cos(\pi(3-x)))}{2}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

So on the interval $0 < x < 1$ we have

$$\begin{aligned} f(x+3) - f(3-x) &= \frac{1}{2}\left(\frac{(1-\cos(\pi(x+3)))}{2} - \frac{(1-\cos(\pi(3-x)))}{2}\right) \\ &= \frac{1}{2}\left(\cos(\pi(3-x)) - \cos(\pi(x+3))\right) \\ &= \frac{1}{2}\left(\cos(\pi x - 3\pi) - \cos(\pi x + 3\pi)\right) \\ &= \frac{1}{2}\left(-\cos(\pi x) + \cos(\pi x)\right) = 0 \end{aligned}$$

It is by no means obvious whether the problem has conserved energy. Recall our definition of energy.

$$e(t) = \frac{1}{2}\int_0^\infty [u_t^2(x,t) + u_x^2(x,t)]\, dx$$

and at $t = 0$ this becomes

$$\begin{aligned} e(0) &= \frac{1}{2}\int_0^\infty f'(x)^2\, dx \\ &= \frac{1}{2}\int_0^\infty \left[\frac{\pi \sin(\pi x)}{2}\right]^2 dx \\ &= \frac{1}{2} \times \left(\frac{\pi}{2}\right)^2 \frac{1}{2}\int_2^4 (1 - \sin(2\pi x))\, dx \\ &= \frac{1}{2} \times \left(\frac{\pi}{2}\right)^2 \times \frac{1}{2} \times 2 = \left(\frac{\pi^2}{8}\right). \end{aligned}$$

Now to check that $e(3) = e(0)$ we need to compute the energy at time $t = 3$. This is much more complicated even for this simple example. Recall from (3) that

$$u(x,t) = \frac{f(x+t) + f(x-t) - f(t-x)}{2}.$$

So we need to compute

$$u_x(x,t) = \frac{f'(x+t) + f('x-t) + f'(t-x)}{2},$$

and

$$u_t(x,t) = \frac{f'(x+t) - f('x-t) - f'(t-x)}{2}.$$

First we note that

$$f'(x) = \begin{cases} \frac{\pi}{2}\sin(\pi x), & 2 < x < 4 \\ 0, & \text{otherwise} \end{cases}.$$

At $t = 3$ we have

$$f'(x+3) = \begin{cases} \frac{\pi}{2}\sin(\pi(x+3)), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f'(3-x) = \begin{cases} \frac{\pi}{2}\sin(\pi(3-x)), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

We also have

$$f'(x-t) = \begin{cases} \frac{\pi}{2}\sin(\pi(x-3)), & 5 < x < 7 \\ 0, & \text{otherwise} \end{cases}.$$

So on the interval $0 < x < 1$ we have

$$\begin{aligned}
u_x^2(x,3) &= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \left[\sin(\pi(x+3)) + \sin(\pi(3-x))\right]^2 \\
&= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \left[\sin(\pi x + 3\pi)) + \sin(3\pi - \pi x))\right]^2 \\
&= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \left[-\sin(\pi x)) - \sin(-\pi x))\right]^2 \\
&= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \left[-\sin(\pi x)) + \sin(\pi x))\right]^2 = 0.
\end{aligned}$$

and

$$\begin{aligned}
u_t^2(x,3) &= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 [\sin(\pi(x+3)) - \sin(\pi(3-x))]^2 \\
&= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 [\sin(\pi x + 3\pi)) - \sin(3\pi - \pi x))]^2 \\
&= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 [-\sin(\pi x)) + \sin(-\pi x))]^2 \\
&= \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 [-\sin(\pi x)) - \sin(\pi x))]^2 \\
&= \left(\frac{\pi}{2}\right)^2 \sin^2(\pi x).
\end{aligned}$$

On the interval $5 < x < 7$ we have

$$u_x^2(x,3) = \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \sin^2(\pi(x-3)) = \frac{1}{4} \times \frac{1}{2} \times \left(\frac{\pi}{2}\right)^2 [1 - \cos(2\pi x)]$$

$$u_t^2(x,3) = \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \sin^2(\pi(x-3)) = \frac{1}{4} \times \frac{1}{2} \times \left(\frac{\pi}{2}\right)^2 [1 - \cos(2\pi x)].$$

So we can compute

$$\begin{aligned}
e(3) &= \frac{1}{2} \int_0^\infty [u_x^2(x,3) + u_t^2(x,3)]\, dx \\
&= \frac{1}{2} \left(\int_0^1 [u_x^2(x,3) + u_t^2(x,3)]\, dx + \int_5^7 [u_x^2(x,3) + u_t^2(x,3)]\, dx \right) \\
&= \frac{1}{2} \left(\int_0^1 \left(\frac{\pi}{2}\right)^2 \frac{1}{2}[1 - \cos(2\pi x)]\, dx + \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \int_5^7 [1 - \cos(2\pi x)]\, dx \right) \\
&= \frac{1}{2} \left(\left(\frac{\pi}{2}\right)^2 \frac{1}{2} + 2 \times \frac{1}{4} \times \left(\frac{\pi}{2}\right)^2 \right) \\
&= \frac{1}{2} \times \left(\frac{\pi}{2}\right)^2 = \frac{\pi}{8}.
\end{aligned}$$

This is exactly the same as the value computed for $e(0)$.