Pointwise Convergence of Fourier Series

First we give the famous **Reimann-Lebesgue Lemma**

Theorem 1. *Let f be an absolutely Reimann integrable function on $[a, b]$, i.e.,*

$$\int_a^b |f(x)|\, dx < \infty.$$

Then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin(\lambda x)\, dx = 0. \tag{1}$$

Proof. Consider the case in which $f(x) \equiv 1$

$$\int_a^b f(x) \sin(\lambda x)\, dx = \int_a^b \sin(\lambda x)\, dx = \left[-\frac{\cos(\lambda x)}{\lambda} \right]_{x=a}^{x=b} = \frac{(\cos(a\lambda) - \cos(b\lambda))}{\lambda}$$

and this gives

$$\left| \int_a^b f(x) \sin(\lambda x)\, dx \right| \leq \frac{2}{\lambda} \xrightarrow{\lambda \to \infty} 0.$$

Assume now that $f(x)$ is a *piecewise constant* function, i.e.,

$$f(x) = \sum_{j=1}^{K} c_j \mathbf{1}_{I_j}(x), \quad I_j = [x_{j-1}, x_j]\,, \quad a = x_0 < x_1 < \cdots < x_K = b,$$

where

$$\mathbf{1}_{I_j}(x) = \begin{cases} 1, & x \in I_j \\ 0, & x \notin I_j \end{cases}.$$

Then we have

$$\begin{aligned} \int_a^b f(x) \sin(\lambda x)\, dx &= \sum_{j=1}^{K} c_j \int_{I_j} \sin(\lambda x)\, dx \\ &= \sum_{j=1}^{K} c_j \left[-\frac{\cos(\lambda x)}{\lambda} \right]_{x=x_{j-1}}^{x=x_j} \\ &= \sum_{j=1}^{K} c_j \frac{(\cos(x_{j-1}\lambda) - \cos(x_j \lambda))}{\lambda} \end{aligned}$$

which implies

$$\left| \int_a^b f(x) \sin(\lambda x)\, dx \right| \leq \frac{2}{\lambda} \sum_{j=1}^{K} |c_j| \xrightarrow{\lambda \to \infty} 0.$$

Finally take $\epsilon > 0$ arbitrary and let

$$g(x) = \sum_{j=1}^{K} c_j \mathbf{1}_{I_j}(x)$$

be a Reimann sum approximation to $f(x)$ with

$$\int_a^b |f(x) - g(x)|\, dx < \frac{\epsilon}{2}.$$

By the definition of the Reimann integral we can find such a $g(x)$ for every $\epsilon > 0$. Further due to our above calculations we know that for a fixed ϵ we can find λ_0 so that $n > \lambda_0$ implies

$$\left| \int_a^b g(x) \sin(\lambda x)\, dx \right| \leq \frac{\epsilon}{2} \quad \text{for} \quad \lambda > \lambda_0.$$

Then we have

$$\begin{aligned}
\left| \int_a^b f(x) \sin(\lambda x)\, dx \right| &= \left| \int_a^b [f(x) - g(x)] \sin(\lambda x)\, dx + \int_a^b g(x) \sin(\lambda x)\, dx \right| \\
&\leq \int_a^b |f(x) - g(x)|\, |\sin(\lambda x)|\, dx + \left| \int_a^b g(x) \sin(\lambda x)\, dx \right| \\
&\leq \int_a^b |f(x) - g(x)|\ dx + \left| \int_a^b g(x) \sin(\lambda x)\, dx \right| \\
&\leq \frac{\epsilon}{2} + \left| \int_a^b g(x) \sin(\lambda x)\, dx \right| \\
&\leq \epsilon
\end{aligned}$$

where on the last step we have chosen $\lambda > \lambda_0$ as above. Thus for general $f(x)$ we have

$$\left| \int_a^b f(x) \sin(\lambda x)\, dx \right| \xrightarrow{\lambda \to \infty} 0.$$

□

Now we turn to the question of point-wise convergence of the Fourier series. First we recall

Definition 1. A function $f(x)$ is piecewise smooth function on $[-\pi, \pi]$ if f and f' are continuous except possibly for a finite number of points in $[-\pi, \pi]$. Furthermore, at a point x_0 where f has a discontinuity we assume that

$$f(x_0^-) = \lim_{x \uparrow x_0} f(x), \quad f(x_0^+) = \lim_{x \downarrow x_0} f(x)$$

and at a point x_0 where f' has a discontinuity we assume that

$$f_L'(x_0) = \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0^-)}{-h}, \quad f_R(x_0) = \lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0^+)}{h}$$

both exist.

Assumption 1. Let us assume that $f(x)$ is a piecewise smooth function defined on $[-\pi, \pi]$.

In order to prove the desired result we need to introduce the Dirichlet kernel.

Lemma 1.

$$D_N(x) \equiv \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\right) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{\sin[(N+1/2)x]}{2\pi\sin(x/2)} \tag{2}$$

Proof. Recall Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ which implies that $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$ and therefore

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

so we have

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

Therefore

$$\begin{aligned}
\frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\right) &= \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\left[\frac{e^{inx} + e^{-inx}}{2}\right]\right) \\
&= \frac{1}{\pi}\left(\frac{1}{2} + \frac{1}{2}\sum_{n=1}^{N} e^{inx} + \frac{1}{2}\sum_{n=1}^{N} e^{-inx}\right) \\
&= \frac{1}{\pi}\left(\frac{1}{2} + \frac{1}{2}\sum_{n=1}^{N} e^{inx} + \frac{1}{2}\sum_{n=-N}^{-1} e^{inx}\right) \\
&= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx}.
\end{aligned}$$

Now the sum of complex exponentials is a geometric series which can be summed in closed form. Namely, recall

$$\sum_{n=0}^{L} r^n = \frac{1 - r^{N+1}}{1 - r}.$$

So we can write

$$2\pi D_N(x) = \sum_{n=-N}^{N} e^{inx} = e^{-iNx}\sum_{n=0}^{2N} e^{inx}$$

$$= e^{-iNx}\left(\frac{1 - e^{i(2N+1)x}}{1 - e^{ix}}\right)$$

$$= e^{-iNx}\left(\frac{e^{i(N+1/2)x}\left(e^{-i(N+1/2)x} - e^{i(N+1/2)x}\right)}{e^{ix/2}\left(e^{-ix/2} - e^{ix/2}\right)}\right)$$

$$= \left(\frac{e^{-iNx + i(N+1/2)x}}{e^{ix/2}}\right)\left(\frac{-2i\sin[(N+1/2)x]}{-2i\sin[x/2]}\right)$$

$$= \frac{\sin[(N+1/2)x]}{\sin[x/2]}.$$

□

The function

$$D_N(x) = \frac{\sin[(N+1/2)x]}{2\pi\sin(x/2)}$$

is called the Dirichlet Kernel.

Now let us consider a Fourier series. First we define the Nth partial sum S_N by

$$\begin{aligned}
S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N}[a_n\cos(nx) + b_n\sin(nx)] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\left(\frac{e^{nix}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{nix}-e^{-inx}}{2i}\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N}\left[\left(\frac{a_n - ib_n}{2}\right)e^{nix} + \left(\frac{a_n+ib_n}{2}\right)e^{-nix}\right] \\
&\equiv \sum_{n=-N}^{N} c_n e^{nix}
\end{aligned}$$

where

$$c_0 = \frac{a_0}{2}, \quad c_n = \begin{cases} \left(\dfrac{a_n - ib_n}{2}\right), & n > 0 \\ \left(\dfrac{a_n + ib_n}{2}\right), & n < 0 \end{cases}.$$

Next we will use the Dirichlet Kernel to represent S_N. First note that for $n > 0$

$$c_n = \left(\frac{a_n - ib_n}{2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos(nx) - i\sin(nx)]\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

and again for $n > 0$

$$c_{-n} = \left(\frac{a_n + ib_n}{2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos(nx) + i\sin(nx)]\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx,$$

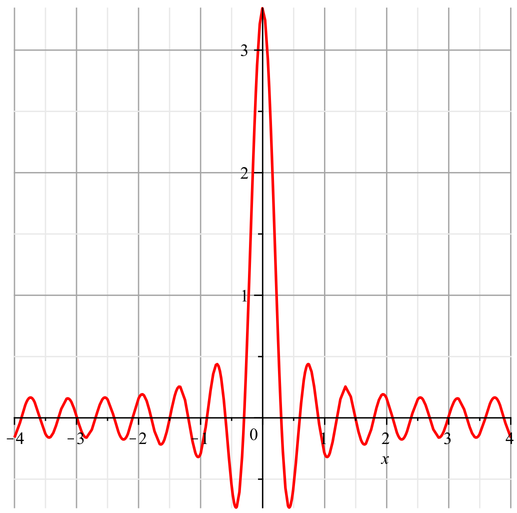
so that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \qquad \text{for all } n = -N, \cdots, N$$

Thus we can write

$$\begin{aligned}
S_N(x) &= \sum_{n=-N}^{N} c_n e^{nix} = \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy\right)e^{nix} \\
&= \int_{-\pi}^{\pi} f(y)\left(\frac{1}{2\pi}\sum_{n=-N}^{N} e^{ni(x-y)}\right)dy \\
&= \int_{-\pi}^{\pi} f(y)D_N(x-y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin[(N+1/2)(x-y)]}{\sin[(x-y)/2]}\,dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\frac{\sin[(N+1/2)y]}{\sin[y/2]}\,dy
\end{aligned}$$

where on the last step we have made the change of variables $y \to x - y$.



Plot of $D_{10}(x)$

So we have written the formula as a convolution with the Dirichlet kernel. Now if we knew that the Dirichlet kernel was a delta sequence and x was a point of continuity of $f(x)$ then we would know that

$$\lim_{N\to\infty} S_N(x) = f(x).$$

But you can notice from the graph that $D_N(x)$ is not positive and we only want to assume the $f(x)$ is piecewise smooth so it may not be continuous at x. Indeed from the graph we can see that $D_N(x)$ is a very oscillatory function. Nevertheless we can still show that, under our assumptions on $f(x)$, there is something like the delta sequence property.

To establish the desired result we first note that

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi \frac{\sin[(N+1/2)x]}{\sin(x/2)}\,dx &= 2\int_0^\pi D_N(x)\,dx \\
&= \frac{1}{\pi}\int_0^\pi \left(1 + 2\sum_{n=1}^N \cos(nx)\right)\,dx \\
&= 1 + \sum_{n=1}^N \int_0^\pi \cos(nx)\,dx \\
&= 1 + \frac{2}{\pi}\sum_{n=1}^N \frac{\sin(nx)}{n}\bigg|_{x=0}^{x=\pi} = 1.
\end{aligned}$$

Thus we have for all N

$$\frac{1}{\pi}\int_0^\pi \frac{\sin[(N+1/2)x]}{\sin(x/2)}\,dx = 1 \tag{3}$$

Let use recall

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\frac{\sin[(N+1/2)y]}{\sin[y/2]}\,dy.$$

Lemma 2. *Under our assumptions on f we have, for any $-\pi \le x_0 \le \pi$,*

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^\pi f(x_0-y)\frac{\sin[(N+1/2)y]}{\sin[y/2]}\,dy = f(x_0^-) \tag{4}$$

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{0} f(x_0 - y) \frac{\sin[(N+1/2)y]}{\sin[y/2]}\, dy = f(x_0^+) \tag{5}$$

Proof.

$$\begin{aligned}
\frac{1}{\pi} \int_0^{\pi} f(x_0 - y) & \frac{\sin[(N+1/2)y]}{\sin[y/2]}\, dy - f(x_0^-) \\
&= \frac{1}{\pi} \int_0^{\pi} \left[f(x_0 - y) - f(x_0^-) \right] \frac{\sin[(N+1/2)y]}{\sin[y/2]}\, dy \\
&= \frac{1}{\pi} \int_0^{\pi} \left(\frac{\left[f(x_0 - y) - f(x_0^-) \right]}{-y} \right) \left(\frac{-y}{\sin[y/2]} \right) \sin[(N+1/2)y]\, dy
\end{aligned}$$

The integrand consists of three terms:

$$\ell_1(y) = \frac{\left[f(x_0 - y) - f(x_0^-) \right]}{-y}, \quad \ell_2(y) = \frac{-y}{\sin[y/2]}, \quad \ell_3(y) = \sin[(N+1/2)y].$$

1. The function $\ell_1(y)$ is piecewise smooth with

$$\lim_{y\to 0^+} = f_L'(x_0).$$

2. The function $\ell_2(y)$ is continuous and bounded.

So the product of the ℓ_1 and ℓ_2 is a function $g(y)$ which is Reimann integrable on $[0, \pi]$ and we can apply Theorem 1 (the Reimann-Lebesgue lemma) to obtain

$$\frac{1}{\pi} \int_0^{\pi} f(x_0 - y) \frac{\sin[(N+1/2)y]}{\sin[y/2]}\, dy - f(x_0^-) = \frac{1}{\pi} \int_0^{\pi} g(y) \sin[(N+1/2)y]\, dy \xrightarrow{N\to\infty} 0$$

and therefore (4) holds.

An almost identical argument shows that (5) holds.

□

Combining the above results (4) and (5) we have

$$\frac{f(x_0^+) + f(x_0^-)}{2} = \lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y) \frac{\sin[(N+1/2)y]}{\sin[y/2]}\, dy. \tag{6}$$

This means

$$\frac{f(x_0^+) + f(x_0^-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$