Math 4354 Fall 2008 Sec. 1 Test 1 Name ____________________

1. Solve the initial value problem $u_t + 4u_x = u^2$ with $u(x,0) = x$ for $x \in \mathbb{R}$.

ANS:

$$\begin{array}{lll} dx/ds = 4 & dt/ds = 1 & du/ds = u^2 \\ x(0) = r & t(0) = 0 & u(0) = r \\ \Rightarrow x = 4s + r & t = s & u^{-1}(s) - u^{-1}(0) = s \end{array}$$

So we have $r = x - 4t$, $u(0) = r$ so solving $u^{-1}(s) - u^{-1}(0) = s$ for u we get

$$u = \frac{r}{1 - sr} = \frac{(x-4t)}{1 - t(x-4t)}$$

2. Solve the initial value problem $u_t + 2txu_x = u$ with $u(x,0) = x^2$.

ANS:

$$\begin{array}{lll} dt/ds = 1 & dx/ds = 2sx & du/ds = u \\ t(0) = 0 & x(0) = r & u(0) = r^2 \\ \Rightarrow t = s & x = re^{s^2} & \ln(u) - \ln(r^2) = s \end{array}$$

So we have $x = re^{s^2} \quad \Rightarrow \quad r = xe^{-s^2}$ so solving $\ln(u) - \ln(r^2) = s$ for u we get

$$u = r^2 e^s = e^t (xe^{-t^2})^2$$

3. Use D'alemberts formula to solve
$$u_{tt}(x,t) = u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = \sin(x), \quad u_t(x,0) = \cos(x)$$

ANS:

$$\begin{aligned} u(x,t) &= \frac{\sin(x-t) + \sin(x+t)}{2} + \frac{1}{2}\int_{(x-t)}^{(x+t)} \cos(s)\,ds \\ &= \frac{\sin(x-t) + \sin(x+t)}{2} + \frac{1}{2}\sin(s)\Big|_{(x-t)}^{(x+t)} \\ &= \frac{\sin(x-t) + \sin(x+t)}{2} + \frac{1}{2}(\sin(x+t) - \sin(x-t)) \\ &= \sin(x+t) \end{aligned}$$

4. The initial displacement of an an infinite string with wave speed $c = 2$ and initial condition $u(x, 0) = (1 + x^2)^{-1}$, for $-\infty < x < \infty$. With what velocity $u_t(x, 0)$ must the string start if the resulting motion consists only of a right traveling wave.

ANS:

We know $u(x,t) = F(x+2t) + G(x-2t)$ but our assumption that u is a right traveling wave means that $u(x,t) = G(x-2t)$. Now we also know that $u(x,0) = (1+x^2)^{-1} = G(x)$ so

$$u(x,t) = (1 + (x-2t)^2)^{-1}$$

which implies

$$u_t(x,t) = -(1 + (x-2t)^2)^{-2}[2(x-2t)(-2)] = \frac{4(x-2t)}{(1+(x-2t)^2)^2}$$

so $u_t(x,0) = \dfrac{4x}{(1+x^2)^2}$.

5. Show that for all $t > 0$ and all $x \in \mathbb{R}$ $\quad S(x,t) = \dfrac{1}{\sqrt{4k\pi t}} e^{-x^2/(4kt)}$ satisfies the heat equation $S_t(x,t) = kS_{xx}(x,t)$.

ANS:

$S_x = \left(\dfrac{-2x}{4kt}\right) S$ so $S_{xx} = \left(\dfrac{-2}{4kt}\right) S + \left(\dfrac{-2x}{4kt}\right) S_x = \left[\left(\dfrac{2x}{4kt}\right)^2 - \dfrac{2}{(4kt)}\right] S$,

$$kS_{xx} = \left[\left(\frac{x^2}{4kt^2}\right) - \frac{1}{(2t)}\right] S.$$

So our goal is to show that k times this expression is the same as S_t. We compute S_t.

$$\begin{aligned} S_t &= \left(\frac{1}{\sqrt{4k\pi t}}\right)_t e^{-x^2/(4kt)} + \frac{1}{\sqrt{4k\pi t}} \left(e^{-x^2/(4kt)}\right)_t \\ &= \left[(-1/2)(4k\pi t)^{-3/2}(4k\pi)\right] e^{-x^2/(4kt)} + \frac{1}{\sqrt{4k\pi t}} \left(e^{-x^2/(4kt)} \frac{x^2}{4kt^2}\right) \end{aligned}$$

Looking at the form of our desired answer we factor out S to get

$$S_t = \left[\frac{-4k\pi}{8k\pi t} + \frac{x^2}{4kt^2}\right] S = \left[\frac{x^2}{4kt^2} - \frac{1}{2t}\right] S = kS_{xx}$$

6. Suppose that $u(x,t)$ is a bounded strict solution to the heat equation on the whole line with $k = 1$ and with initial condition $u(x,0) = \arctan(x)$. Let $Q = \{(x,t) : -\infty < x < \infty,\ t \geq 0\}$. Find M_0 so that $\max_{(x,t)\in Q} |u(x,t)| \leq M_0$. (Give reasons for your answer).

ANS:

By the maximum principle we have

$$\sup_{(x,t)\in Q} |u(x,t)| = \sup_{x\in\mathbb{R}} |u(x,0)| = \sup_{x\in\mathbb{R}} |\arctan(x)| = \frac{\pi}{2}.$$