7 The Dirichlet Problem in Two Dimensions

7.1 Introduction

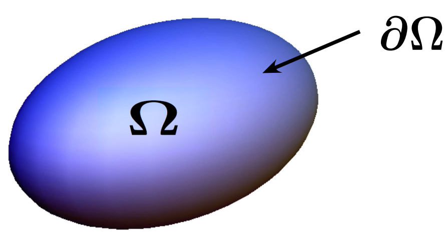
In this section we briefly discuss the solution of elliptic boundary value problems in two dimensional bounded domains. Due to the short time available we will limit considerably the topics covered and emphasize only the most basic elements and ideas.

First we note that Chapter 8 (Solution of the heat equation in two dimensional bounded domains) and 9 (Solution of the wave equation in two dimensional bounded domains) should ideally precede this chapter but due to time constraints and wanting to cover, at some level, examples from the three main types of equations (hyperbolic, parabolic, elliptic) we move on to the typical elliptic problem. As we have already said earlier the main hyperbolic model is the wave equation, the main parabolic equation is the heat equation and the main elliptic problem is Laplace's equation.

To better understand a motivation of Laplace's equation let us first present the higher dimensional analogs of the heat and wave equation in Ω a bounded domain in $\mathbb{R}^n$. In this case the spatial variable $x \in \Omega$ is given by $x = (x_1, x_2, \cdots, x_n)$ and the main differential operator in the spatial variable (replacing d^2/dx^2) is the Laplace operator given in $\mathbb{R}^n$ by

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}. \tag{7.1}$$

Further we denote the boundary of Ω by $\partial\Omega$.



In this case we can write the heat and wave equation as

$$\begin{aligned} u_t(x,t) &= k\Delta u(x,t), \\ u(x,t) &= F(x), \quad x \in \partial\Omega \\ u(x,0) &= \varphi(x) \quad x \in \Omega. \end{aligned}$$

$$\begin{aligned} u_{tt}(x,t) &= c^2\Delta u(x,t), \\ u(x,t) &= F(x), \quad x \in \partial\Omega \\ u(x,0) &= f(x), \ u_t(x,0) = g(x) \quad x \in \Omega. \end{aligned}$$

It can be shown that the solution to the above problem typically converges to a time dependent steady state and it tends to do this rather quickly. So for many applications is makes sense to study to corresponding steady state (time independent) problem. The solution being time independent means its time derivative is zero. This leads to the so-called Laplace equation and the Dirichlet problem.

$$\begin{aligned} \Delta u(x) &= 0, \quad x \in \Omega \\ u(x) &= F(x), \quad x \in \partial\Omega \end{aligned}$$

This is a very important problem in mathematics and engineering and is one of the most studied problems in theory and practice.

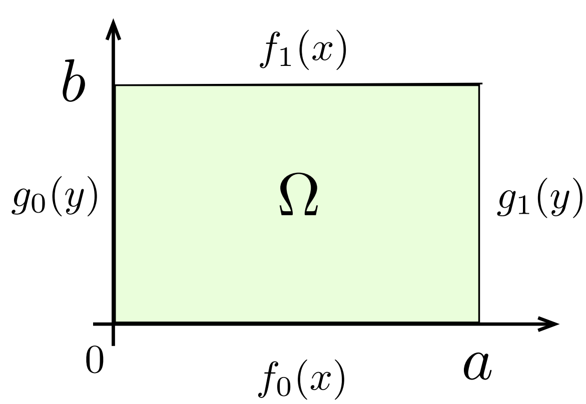
Often in $\mathbb{R}^2$ and $\mathbb{R}^3$ we use the variables x, y and z instead of x_1, x_2 and x_3 and we write $\boldsymbol{x} = (x, y)$ or $\boldsymbol{x} = (x, y, z)$. So with this notation the Dirichlet problem for Laplace's equation becomes

$$\begin{aligned} \Delta u(\boldsymbol{x}) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \text{ in } \ \Omega \\ u(\boldsymbol{x}) &= F(\boldsymbol{x}) \ \ \boldsymbol{x} \in \partial\Omega \end{aligned}$$

$$\begin{aligned} \Delta u(\boldsymbol{x}) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \ \text{ in } \ \Omega \\ u(\boldsymbol{x}) &= F(\boldsymbol{x}) \ \ \boldsymbol{x} \in \partial\Omega \end{aligned}$$

7.2 Dirichlet Problem in a Rectangle

In these notes we will apply the method of separation of variables to obtain solutions to our elliptic problems as an infinite sum involving Fourier coefficients, eigenvalues and eigenvectors. About the simplest cases consist of a 2-Dimensional problem in a rectangle or a disc or annulus. As we have already mentioned these problems arise, for example, in the steady state analysis or heat and wave problems where we are interested in problems with no time dependence, so either u_t or u_{tt} is zero.



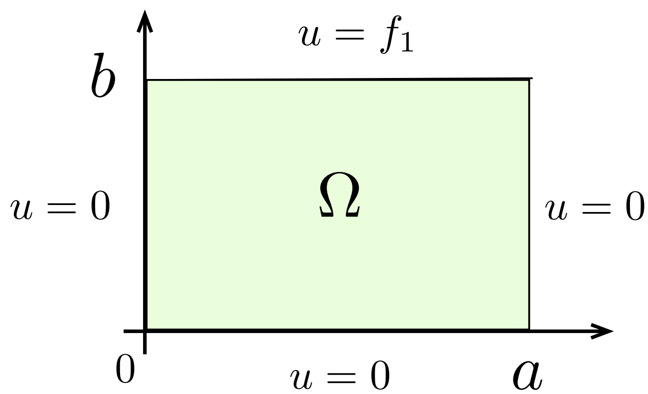
As usual we will start with Dirichlet boundary conditions and a rectangular region Ω. The most general setup in this case is to prescribe a function on each of the four sides of the rectangle as depicted in the figure. Thus we obtain the problem

$$\begin{aligned} &u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \\ &u(0,y) = g_0(y), \quad u(a,y) = g_1(y) \\ &u(x,0) = f_0(x), \quad u(x,b) = f_1(x) \end{aligned} \tag{7.2}$$

Analysis of this problem would become rather messy but the principle of superposition allows us to divide and conquer. We can write the solution to this problem as a sum of solutions to four simpler problems

The Principle of Superposition

We note that for a linear problem it is always possible to replace a single hard problem by several simpler problems. We illustrate this idea with a simple example. Consider the problem



The solution u can be obtained as a sum of the solutions to four simpler problems

$$\begin{aligned} &u^{(1)}_{xx}(x,y) + u^{(1)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \\ &u^{(1)}(0,y) = g_0(y), \quad u^{(1)}(a,y) = 0 \\ &u^{(1)}(x,0) = 0, \quad u^{(1)}(x,b) = 0 \end{aligned} \tag{7.3}$$

$$\begin{aligned} &u^{(2)}_{xx}(x,y) + u^{(2)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \\ &u^{(2)}(0,y) = 0, \quad u^{(2)}(a,y) = 0 \\ &u^{(2)}(x,0) =, \quad u^{(2)}(x,b) = f_1(x) \end{aligned} \tag{7.4}$$

$$\begin{aligned} &u^{(3)}_{xx}(x,y) + u^{(3)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \\ &u^{(3)}(0,y) = 0, \quad u^{(3)}(a,y) = g_1(y) \\ &u^{(3)}(x,0) = 0, \quad u^{(3)}(x,b) = 0 \end{aligned} \tag{7.5}$$

$$\begin{aligned} &u^{(4)}_{xx}(x,y) + u^{(4)}_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \\ &u^{(4)}(0,y) = 0, \quad u^{(4)}(a,y) = 0 \\ &u^{(4)}(x,0) = f_0(x), \quad u^{(4)}(x,b) = 0 \end{aligned} \tag{7.6}$$

7.2.1 Non-Zero Boundary Function of x

To illustrate the method of separation of variables applied to these problems let us consider the BVP in (7.4). In this case the only non-zero boundary term occurs on the top of the box when $y = b$ where we have $u^{(2)}(x, b) = f_1(x)$.

$$\begin{aligned}
\mu_n &= \left(\frac{n\pi}{a}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots, \\
u^{(2)}(x, y) &= \sum_{n=1}^{\infty} b_n \sin(\mu_n x) \sinh(\mu_n y) \\
b_n &= \frac{2}{a \sinh(\mu_n b)} \int_0^a f_1(x) \sin(\mu_n x)\, dx.
\end{aligned}$$

As usual we look for simple solutions in the form

$$u^{(2)}(x, y) = \varphi(x)\psi(y).$$

Substituting into (7.4) and dividing both sides by $\varphi(x)\psi(y)$ gives

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of y, it follows that the expression must be a constant:

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here ψ' means the derivative of ψ with respect to y and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and ψ. We obtain

$$\varphi'' + \lambda\varphi = 0, \qquad \psi'' - \lambda\psi = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(y) = 0, \quad \varphi(a)\psi(y) = 0 \quad \text{for all } y.$$

Since $\psi(y)$ is not identically zero we obtain the eigenvalue problem

$$\varphi''(x) + \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi(a) = 0. \tag{7.7}$$

We have solved a similar problem many times with the main difference being that now the eigenvalues are positive.

$$\mu_n = \left(\frac{n\pi}{a}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots. \tag{7.8}$$

The general solution of

$$\psi''(y) - \mu_n^2 \psi(y) = 0$$

is then

$$\psi(y) = c_1 \cosh(\mu_n y) + c_2 \sinh(\mu_n y) \tag{7.9}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\psi(0) = 0$ implies

$$\psi(y) = \sinh(\mu_n y).$$

So we look for u as an infinite sum

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(\mu_n x) \sinh(\mu_n y) \tag{7.10}$$

The only problem remaining is to somehow pick the constants b_n so that the initial condition $u(x,b) = f_1(x)$ is satisfied.

Setting $y = b$ in (7.10), we seek to obtain $\{b_n\}$ satisfying

$$f_1(x) = u(x,b) = \sum_{n=1}^{\infty} b_n \sin(\mu_n x) \sinh(\mu_n b).$$

This is almost a Sine expansion of the function $f_1(x)$ on the interval $(0,a)$. In particular we obtain

$$\sinh(\mu_n b)\, b_n = \frac{2}{a} \int_0^a f_1(x) \sin(\mu_n x)\, dx. \tag{7.11}$$

We note that to obtain the complete solution to the original problem (7.2) we would need to solve the three other similar problems (7.3), (7.5), (7.6).

Example 7.1. Consider the problem (7.4) with

$$f_1(x) = \begin{cases} x & 0 \le x \le a/2, \\ (\pi - x) & a/2 \le x \le a. \end{cases}$$

For this example (7.10) becomes

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(\mu_n x) \sinh(\mu_n y).$$

In this case we obtain the following results for (7.11) (The explicit integrations are carried out below)

$$\sinh\left(\frac{n\pi b}{a}\right) b_n = \frac{2}{a}\left[\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right) dx + \int_{a/2}^{a} (a-x) \sin\left(\frac{n\pi x}{a}\right) dx\right]$$

$$= \frac{2}{a}\left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4a \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

Which implies

$$b_n = \frac{4a \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh(\mu_n b)}.$$

Then we arrive at the solution

$$u(x,y) = \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh(\mu_n b)} \right] \sin(\mu_n x) \sinh(\mu_n y).$$

To obtain the above formulas we needed to carry two integration by parts. We do them separately in the following.

$$\begin{aligned}
\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right) dx &= \int_0^{a/2} x \left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)' dx \\
&= x\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)\Big|_0^{a/2} - \int_0^{a/2} \left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right) dx \\
&= a/2\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) + \frac{a}{n\pi}\int_0^{a/2} \cos\left(\frac{n\pi x}{a}\right) dx \\
&= -\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\Big|_0^{a/2} \\
&= -\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

and

$$\begin{aligned}
\int_{a/2}^{a} (a-x)\sin\left(\frac{n\pi x}{a}\right) dx &= \int_{a/2}^{a} (a-x)\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)' dx \\
&= (a-x)\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)\Big|_{a/2}^{a} - \int_{a/2}^{a} (-1)\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right) dx \\
&= a/2\left(\frac{a}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) - \frac{a}{n\pi}\int_{a/2}^{a} \cos\left(\frac{n\pi x}{a}\right) dx \\
&= \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\Big|_{a/2}^{a} \\
&= \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

Thus we have

$$\sinh(\mu_n b)\, b_n = \frac{2}{a}\left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right]$$

or finally,

$$b_n = \frac{4a \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh(\mu_n b)}.$$

7.2.2 Non-Zero Boundary Function of y

To illustrate the method of separation of variables applied to these problems let us consider the BVP in (7.5). In this case the only non-zero boundary term occurs on the right hand side of the box when $x = a$ where we have $u^{(3)}(a, y) = g_1(y)$.

$$\begin{aligned} \mu_n &= \left(\frac{n\pi}{b}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sin(\mu_n y), \quad n = 1, 2, \cdots \\ u^{(3)}(x, y) &= \sum_{n=1}^{\infty} b_n \sin(\mu_n y) \sinh(\mu_n x) \\ b_n &= \frac{2}{b \sinh(\mu_n a)} \int_0^b g_1(y) \sin(\mu_n y)\, dy. \end{aligned}$$

To obtain this formula we proceed as usual and look for simple solutions in the form

$$u^{(3)}(x, y) = \varphi(y)\psi(x).$$

The main difference here is that in this case we interchange the roles of x and y since we will want to do a Fourier series in y this time instead of x. Substituting into (7.5) and dividing both sides by $\varphi(y)\psi(x)$ gives

$$\frac{\psi''(x)}{\psi(x)} = \frac{-\varphi''(y)}{\varphi(y)}$$

Since the left side is independent of y and the right side is independent of x, it follows that the expression must be a constant:

$$\frac{\psi''(x)}{\psi(x)} = \frac{-\varphi''(y)}{\varphi(y)} = \lambda.$$

(Here ψ' means the derivative of ψ with respect to x and φ' means means the derivative of φ with respect to y.) We seek to find all possible constants λ and the corresponding nonzero functions φ and ψ. We obtain

$$\varphi''(y) + \lambda\varphi(y) = 0, \qquad \psi''(x) - \lambda\psi(x) = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(x) = 0, \quad \varphi(b)\psi(x) = 0 \quad \text{for all } x.$$

Since $\psi(x)$ is not identically zero we obtain the eigenvalue problem

$$\varphi''(y) + \lambda\varphi(y) = 0, \quad \varphi(0) = 0, \quad \varphi(b) = 0. \tag{7.12}$$

Once again we note the main difference now is that the eigenvalues λ_n are positive.

$$\mu_n = \left(\frac{n\pi}{b}\right), \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sin(\mu_n y), \quad n = 1, 2, \cdots . \tag{7.13}$$

The general solution of

$$\psi''(x) - \mu_n^2\psi(x) = 0$$

is then

$$\psi(y) = c_1 \cosh(\mu_n x) + c_2 \sinh(\mu_n x) \tag{7.14}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\psi(0) = 0$ implies

$$\psi(x) = \sinh(\mu_n x) .$$

So we look for u as an infinite sum

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin(\mu_n y) \sinh(\mu_n x) \tag{7.15}$$

The only problem remaining is to somehow pick the constants b_n so that the initial condition $u(a, y) = g_1(y)$ is satisfied.

Setting $x = a$ in (7.10), we seek to obtain $\{b_n\}$ satisfying

$$g_1(y) = u(a, y) = \sum_{n=1}^{\infty} b_n \sin(\mu_n y) \sinh(\mu_n a) .$$

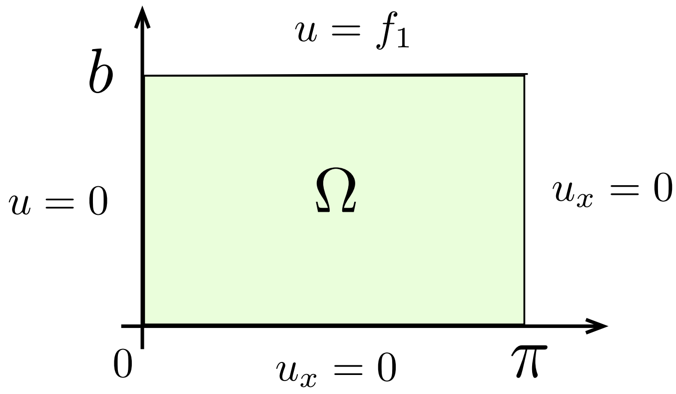
This is almost a Sine expansion of the function $g_1(y)$ on the interval $(0, b)$. In particular we obtain

$$\sinh(\mu_n a)\, b_n = \frac{2}{b} \int_0^b g_1(y) \sin(\mu_n y) \; dy. \tag{7.16}$$

7.3 The Laplace Equation with other Boundary Conditions

Next we consider a slightly different problem involving a mixture of Dirichlet and Neumann boundary conditions. To simplify the problem a bit we set $a = \pi$ Namely we consider

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) = 0, &\quad (x, y) \in [0, \pi] \times [0, b], \\ u(0, y) = 0, &\quad u_x(\pi, y) = 0 \\ u_y(x, 0) = 0, &\quad u(x, b) = f_1(x) \end{aligned} \tag{7.17}$$



Look for simple solutions in the form

$$u(x, y) = \varphi(x)\psi(y).$$

Substituting into (7.17) and dividing both sides by $\varphi(x)\psi(y)$ gives

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of y, it follows that the expression must be a constant:

$$\frac{\psi''(y)}{\psi(y)} = \frac{-\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here ψ' means the derivative of ψ with respect to y and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and ψ. We obtain

$$\varphi'' + \lambda\varphi = 0, \qquad \psi'' - \lambda\psi = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(y) = 0, \quad \varphi'(\pi)\psi(y) = 0 \quad \text{for all } y.$$

Since $\psi(y)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) + \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi'(\pi) = 0. \tag{7.18}$$

$$\mu_n = \frac{(2n-1)}{2}, \quad \lambda_n = \mu_n^2, \quad \varphi_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots. \tag{7.19}$$

The general solution of $\psi'' - \mu_n^2\psi = 0$ is

$$\psi(y) = c_1 \cosh(\mu_n y) + c_2 \sinh(\mu_n y) \tag{7.20}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\psi'(0) = 0$ implies

$$\psi(y) = \cosh(\mu_n y).$$

So we look for u as an infinite sum

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x) \cosh(\mu_n y) \tag{7.21}$$

Finally we need to find the constants a_n so that

$$f_1(x) = u(x,b) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x) \cosh(\mu_n b).$$

This is almost a Cosine expansion of the function $f_1(x)$ on the interval $(0,\pi)$. In particular we obtain

$$\cosh(\mu_n b)\, a_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin(\mu_n x)\, dx. \tag{7.22}$$

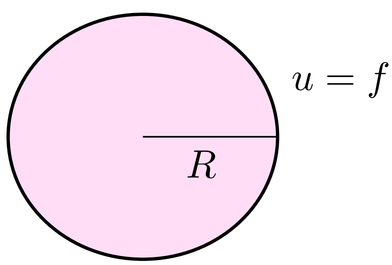
7.4 A Dirichlet Problem in a Disk

Next we consider a problem in a different geometry. In particular, we look at Laplace's equation in a disk. Consider the problem

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad x^2 + y^2 < C^2 \tag{7.23}$$

$$u(x,y) = f(x,y), \quad x^2 + y^2 = C^2,$$

$$u(x,y) \text{ bounded on } x^2 + y^2 \le C^2. \tag{7.24}$$



It turns out that we cannot solve this problem using separation of variables as it is written. But, as you will see, if we change coordinates to polar coordinates then separation of variables works fine.

To this end recall that polar coordinates are given by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

or

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x).$$

So we need to translate

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

into the variables r and θ. First we use $r^2 = x^2 + y^2$ and implicit differentiation to compute

$$2rr_x = 2x \;\Rightarrow\; r_x = \frac{x}{r}, \quad 2rr_y = 2y \;\Rightarrow\; r_y = \frac{y}{r}$$

so we have

$$r_x = \frac{x}{r} = \cos(\theta) \;\text{ and }\; r_y = \frac{y}{r} = \sin(\theta). \tag{7.25}$$

Similarly, differentiating $y = r\sin(\theta)$ with respect to x and using (7.25) we have

$$\begin{aligned} 0 &= r_x \sin(\theta) + r\cos(\theta)\theta_x & (7.26)\\ &= \cos(\theta)\sin(\theta) + r\cos(\theta)\theta_x & (7.27)\end{aligned}$$

which implies

$$\theta_x = -\frac{\sin(\theta)}{r}. \tag{7.28}$$

Differentiating $x = r\cos(\theta)$ with respect to y and using (7.25) we have

$$\begin{aligned} 0 &= r_y \cos(\theta) - r\sin(\theta)\theta_y & (7.29)\\ &= \cos(\theta)\sin(\theta) - r\sin(\theta)\theta_y & (7.30)\end{aligned}$$

and we obtain

$$\theta_y = \frac{\cos(\theta)}{r}. \tag{7.31}$$

Next, using the chain rule and (7.25) we compute

$$u_x = u_r r_x + u_\theta \theta_x = u_r \cos(\theta) - u_\theta \frac{\sin(\theta)}{r} \tag{7.32}$$

and

$$u_y = u_r r_y + u_\theta \theta_y u_r \sin(\theta) + u_\theta \frac{\cos(\theta)}{r}. \tag{7.33}$$

Using the formulas (7.32) and (7.33) we now compute the second derivatives:

$$\begin{aligned} u_{xx} &= (u_{rr} r_x + u_{r\theta}\theta_x)\cos(\theta) - (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\frac{\sin(\theta)}{r} \\ &\quad + u_r(\cos(\theta))_x - u_\theta\left(\frac{\sin(\theta)}{r}\right)_x \\ &= \left(u_{rr}\cos(\theta) + u_{r\theta}\left(-\frac{\sin(\theta)}{r}\right)\right)\cos(\theta) \\ &\quad - \left(u_{\theta r}\cos(\theta) + u_{\theta\theta}\left(-\frac{\sin(\theta)}{r}\right)\right)\frac{\sin(\theta)}{r} \\ &\quad + u_r(-\sin(\theta))\theta_x - u_\theta\left(\frac{(r\cos(\theta)\theta_x - \sin(\theta) r_x)}{r^2}\right) \\ &= u_{rr}\cos^2(\theta) - 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\sin^2(\theta)}{r^2}\right) \\ &\quad + u_r\frac{\sin^2(\theta)}{r} - u_\theta\left(\frac{-\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta)}{r^2}\right). \end{aligned}$$

Finally we arrive at

$$u_{xx} = u_{rr}\cos^2(\theta) - 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\sin^2(\theta)}{r^2}\right) \tag{7.34}$$
$$+ u_r\frac{\sin^2(\theta)}{r} + u_\theta\left(\frac{(2\sin(\theta)\cos(\theta))}{r^2}\right)$$

Exactly the same type of calculation which begins with

$$u_y = u_r r_y + u_\theta \theta_y = u_r\sin(\theta) + u_\theta\left(\frac{\cos(\theta)}{r}\right)$$

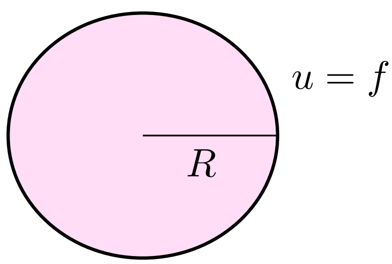
leads to

$$u_{yy} = u_{rr}\sin^2(\theta) + 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\cos^2(\theta)}{r^2}\right) \tag{7.35}$$
$$+ u_r\frac{\cos^2(\theta)}{r} - u_\theta\left(\frac{(2\sin(\theta)\cos(\theta))}{r^2}\right)$$

Now adding (7.34) and (7.35) leads to

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{7.36}$$

With this we can rewrite the problem (7.23) as



$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < C, \ -\pi \le \theta \le \pi \tag{7.37}$$
$$u(C,\theta) = f(\theta), \quad -\pi \le \theta \le \pi$$
$$u(r,\theta) \text{ bounded.} \tag{7.38}$$

Separation of variables proceeds as follows. We seek simple solutions to (7.37) in the form

$$u = \varphi(\theta)\psi(r).$$

Substituting this into the equation in (7.37) we have

$$\left(\varphi(\theta)\psi(r)\right)_{rr} + \frac{1}{r}\left(\varphi(\theta)\psi(r)\right)_r + \frac{1}{r^2}\left(\varphi(\theta)\psi(r)\right)_{\theta\theta} = 0$$

$$\varphi(\theta)\psi''(r) + \frac{1}{r}\varphi(\theta)\psi'(r) + \frac{1}{r^2}\varphi''(\theta)\psi(r) = 0.$$

Next we divide by $\varphi(\theta)\psi(r)$ and multiply by r^2 to obtain

$$\frac{r^2(\psi'' + (1/r)\psi')}{\psi} = -\frac{\varphi''}{\varphi}$$

which as usual in separation of variables must equal a constant λ.

Recall the solutions for so-called Euler-Cauchy equations:

$$r^2\psi'' + ar\psi' + b\psi = 0$$

Consider the change of variables $s = \ln(r)$ or $r = e^s$. By the chain rule

$$\frac{d\psi}{dr} = \frac{d\psi}{ds}\frac{ds}{dr} = \frac{1}{r}\frac{d\psi}{ds}$$

and

$$\frac{d^2\psi}{dr^2} = \frac{d}{ds}\left(\frac{d\psi}{dr}\frac{1}{r}\right) = \frac{1}{r^2}\frac{d\psi^2}{ds^2} - \frac{1}{r^2}\frac{dR}{ds}$$

So the equation becomes

$$r^2\left(\frac{1}{r^2}\frac{d\psi^2}{ds^2} - \frac{1}{r^2}\frac{d\psi}{ds}\right) + ar\frac{1}{r}\frac{d\psi}{ds} + b\psi = 0$$

which simplifies to

$$\frac{d\psi^2}{ds^2} + (a-1)\frac{d\psi}{ds} + b\psi = 0.$$

This is a constant coefficient equation and we recall from ODEs that there are three possibilities for the solutions depending on the roots of the characteristic equation. In the present case we have $a = 1$ and $b = \lambda$.

Thus we obtain the pair of BVPs for ODEs:

$$\varphi''(\theta) + \lambda\varphi(\theta) = 0, \quad \varphi(-\pi) = \varphi(\pi), \quad \varphi'(-\pi) = \varphi'(\pi)$$

For $\lambda = 0$ we have $\varphi(\theta) = 1$. For nonzero λ we have $\lambda = \mu^2$ and the general solution is $\varphi(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$.

The first boundary condition implies $B\sin(\mu\pi) = B\sin(-\mu\pi)$ or $\sin(\mu\pi) = 0$. This implies $\mu = m$ an integer. The second boundary condition implies $-\mu A\sin(\mu\pi) = -\mu A\sin(-\mu\pi)$ which again gives $\mu = m$ so that $\lambda = m^2$ and both A and B are arbitrary. Thus we have $\lambda = m^2$ and

$$\varphi(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta)$$

With $\lambda = m^2$ with $m = 0, 1, 2, \cdots$ as above

$$r^2\psi''(r) + r\psi'(r) - m^2\psi(r) = 0$$

for which we seek solutions in the form $\psi(r) = r^c$. When $m = 0$ we get

$$r^2\psi''(r) + r\psi'(r) = 0$$

which is an Euler-Cauchy problem with general solution $\psi(r) = a + b\ln(r)$.

But in order for the solution to be bounded we need $b = 0$ so ψ is an arbitrary constant, say $\psi = 1$.

For $m \neq 0$ we have again an Euler-Cauchy problem with general solution

$$\psi = ar^m + br^{-m}.$$

Once again, for ψ bounded we need $b = 0$, so we take

$$\psi = r^m.$$

Combining these results we seek our general solution in the form

$$u(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m \left[a_m \cos(m\theta) + b_m \sin(m\theta)\right] \tag{7.39}$$

Now we need

$$f(\theta) = u(R,\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} R^m \left[a_m \cos(m\theta) + b_m \sin(m\theta)\right].$$

This is a general Fourier series and we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_m = \frac{1}{\pi R^m}\int_{-\pi}^{\pi} f(\theta)\cos(m\pi\theta)\, d\theta$$

$$b_m = \frac{1}{\pi R^m}\int_{-\pi}^{\pi} f(\theta)\sin(m\pi\theta)\, d\theta$$

Example 7.2.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \ -\pi \le \theta \le \pi$$
$$u(R,\theta) = \cos^2(\theta),$$
$$u(r,\theta) \text{ bounded.}$$

Using the trig identity

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

and our usual orthogonality conditions the solution is given by

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}r^2\cos(2\theta)$$

Example 7.3 (ellipweb3)**.**

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < 1, \ -\pi \le \theta \le \pi \\ u(R,\theta) &= \sin(3\theta), \\ u(r,\theta) &\text{ bounded.} \end{aligned}$$

$$u(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m \left[a_m\cos(m\theta) + b_m\sin(m\theta)\right]$$

Now we need

$$\sin(3\theta) = u(1,\theta) = a_0 + \sum_{m=1}^{\infty}\left[a_m\cos(m\theta) + b_m\sin(m\theta)\right].$$

But we can argue (using our knowledge of orthogonality) that the solution is given by

$$u(r,\theta) = r^3\sin(3\theta).$$

Notice that this solution can be transformed back into rectangular coordinates but it would be a mess.

8 Assignment

1. Solve the Dirichlet problem $\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, & (x,y)\in[0,1]\times[0,1], \\ u(0,y)=0, \quad u(1,y)=1 \\ u(x,0)=0, \quad u(x,1)=0 \end{cases}$

2. Solve the Dirichlet problem $\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, & (x,y)\in[0,1]\times[0,1], \\ u(0,y)=0, \quad u(1,y)=2y \\ u(x,0)=0, \quad u(x,1)=2x \end{cases}$

3. Solve the Dirichlet problem $\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, & (x,y)\in[0,1]\times[0,1], \\ u(0,y)=-y^2, \quad u(1,y)=1-y^2 \\ u(x,0)=x^2, \quad u(x,1)=x^2-1 \end{cases}$

4. Solve the Dirichlet problem $\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, & (x,y)\in[0,1]\times[0,1], \\ u(0,y)=0, \quad u(1,y)=1 \\ u(x,0)=0, \quad u(x,1)=0 \end{cases}$

5. Solve the Dirichlet problem $\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, & (x,y)\in[0,1]\times[0,\pi], \\ u(0,y)=0, \quad u(1,y)=0 \\ u(x,0)=\sin(\pi x), \quad u(x,\pi)=0 \end{cases}$

6. Solve the Dirichlet problem in the disk $0<r<1$ and $0\le\theta\le 2\pi$

$$\begin{cases} \Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, \\ u(1,\theta)=-\sin(\theta)+2\cos(\theta) \end{cases}$$

7. Solve the Dirichlet problem in the disk $0<r<1$ and $0\le\theta\le 2\pi$

$$\begin{cases} \Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, \\ u(1,\theta)=5 \end{cases}$$

8. Solve the Dirichlet problem in the disk $0<r<1$ and $0\le\theta\le 2\pi$

$$\begin{cases} \Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, \\ u(1,\theta)=2\sin(2\theta) \end{cases}$$

9. Solve the Dirichlet problem in the disk $0<r<1$ and $0\le\theta\le 2\pi$

$$\begin{cases} \Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, \\ u(1,\theta)=\cos(3\theta) \end{cases}$$

10. Solve the Dirichlet problem in the disk $0<r<1$ and $0\le\theta\le 2\pi$

$$\begin{cases} \Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, \\ u(1,\theta)=\begin{cases} 1 & 0\le\theta<\pi \\ 0 & \pi\le\theta<2\pi \end{cases} \end{cases}$$