5 Wave Equation on Finite Interval

5.1 Wave Equation Dirichlet Boundary Conditions

$$
\begin{aligned}
u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\
u(0,t) &= 0, \quad u(\ell,t) = 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x)
\end{aligned}
\tag{5.1}
$$

First we present the solution to this problem and then provide a detailed derivation.

$$
u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(\mu_n ct) + b_n \sin(\mu_n ct) \right) \sin(\mu_n x)
$$

$$
a_n = \frac{2}{\ell} \int_0^\ell f(x) \sin(\mu_n x)\, dx.
$$

$$
b_n = \frac{2}{n\pi c} \int_0^\ell g(x) \sin(\mu_n x)\, dx.
$$

Look for simple solutions in the form

$$
u(x,t) = \varphi(x)\psi(t).
$$

Substituting into (5.1) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$
\frac{\ddot{\psi}(t)}{c^2\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}
$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$
\frac{\ddot{\psi}(t)}{c^2\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.
$$

(Here $\ddot{\psi}$ means the derivative of ψ with respect to t and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and T. We obtain

$$
\varphi'' - \lambda\varphi = 0, \qquad \ddot{\psi} - c^2\lambda\psi = 0.
$$

Furthermore, the boundary conditions give

$$
\varphi(0)\psi(t) = 0, \quad \varphi(\ell)\psi(t) = 0 \quad \text{for all } t.
$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi(\ell) = 0. \tag{5.2}$$

We have solved this problem many times and we have $\lambda = -\mu^2$ so that

$$\varphi(x) = a\cos(\mu x) + b\sin(\mu x).$$

Applying the boundary conditions we have

$$0 = \varphi(0) = a \Rightarrow a = 0 \quad 0 = \varphi(\ell) = b\sin(\mu\ell).$$

From this we conclude $\sin(\mu\ell) = 0$ which implies

$$\mu_n = \frac{n\pi}{\ell}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad \varphi_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots .. \tag{5.3}$$

The solution of $\ddot{\psi} - c^2\lambda_n\psi = 0$ is then

$$\psi(t) = a\cos(\mu_n\, ct) + b\sin(\mu_n\, ct) \tag{5.4}$$

where a and b are arbitrary constants.

Next we look for u as an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n\cos(\mu_n\, ct) + b_n\sin(\mu_n\, ct)\right)\sin(\mu_n x) \tag{5.5}$$

The only problem remaining is to somehow pick the constants a_n and b_n so that the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are satisfied.

Setting $t = 0$ in (5.5), we seek to obtain $\{a_n\}$ satisfying

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(\mu_n\, x).$$

This gives a Sine expansion for the function $f(x)$ on the interval $(0,\ell)$.

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\mu_n\, x)\, dx. \tag{5.6}$$

Next we differentiate (formally) (5.5) with respect to t to obtain

$$u_t(x,t) = \sum_{n=1}^{\infty} (c\mu_n)\left(-a_n\sin(\mu_n\, ct) + b_n\cos(\mu_n\, ct)\right)\sin(\mu_n\, x).$$

Setting $t = 0$ in this expression, we seek to obtain $\{b_n\}$ satisfying

$$g(x) = u_t(x, 0) = \sum_{n=1}^{\infty} b_n \,(c\mu_n) \sin\left(\mu_n\, x\right).$$

This is almost a Sine expansion of the function $g(x)$ on the interval $(0, \ell)$. Namely we obtain

$$b_n = \left(\frac{1}{c\mu_n}\right) \frac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\mu_n\, x\right)\, dx.$$

Our after simplifying

$$b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \sin\left(\mu_n\, x\right)\, dx. \tag{5.7}$$

5.2 Compare Heat and Wave Solutions

At this point we note an important difference between the heat and wave equation solutions. For the heat equation the solutions were of the form

$$\sum_{n=1}^{\infty} c_n e^{\lambda_n t} \varphi_n(x)$$

and, at least for $t > 0$, there is no question about the convergence of this series due to exponential decay of the terms $e^{\lambda_n t}$ as $n \to \infty$. But for the wave equation the series does not include such terms. Indeed, the individual terms look like

$$\left(a_n \cos\left(\mu_n ct\right) + b_n \sin\left(\mu_n ct\right)\right) \sin\left(\mu_n\, x\right)$$

and these do not decay rapidly. Therefore to justify that we do have a solution to (5.1) we must take another approach. First notice that

$$\sin\left(\mu_n x\right) \cos\left(\mu_n ct\right) \tag{5.8}$$
$$= \frac{1}{2}\left[\sin\left(\mu_n(x + ct)\right) + \sin\left(\mu_n(x - ct)\right)\right]$$

and

$$\sin\left(\mu_n x\right) \sin\left(\mu_n ct\right) \tag{5.9}$$
$$= \frac{1}{2}\left[\cos\left(\mu_n(x - ct)\right) - \cos\left(\mu_n(x + ct)\right)\right]$$

$$= \frac{n\pi}{2\ell} \int_{x-ct}^{x+ct} \sin\left(\mu_n \xi\right)\, d\xi. \tag{5.10}$$

Using (5.8), (5.9), (5.6) and (5.7) we can rewrite (5.5) as

$$u(x,t) = \sum_{n=1}^{\infty} f_n \frac{\left[\sin\left(\mu_n(x + ct)\right) + \sin\left(\mu_n(x - ct)\right)\right]}{2} + \frac{1}{2c} \sum_{n=1}^{\infty} \int_{x-ct}^{x+ct} g_n \sin\left(\mu_n \xi\right)\, d\xi. \tag{5.11}$$

This expression can be reduced further as follows: Let $\overline{F}_0(x)$ and $\overline{G}_0(x)$ denote the odd 2ℓ-periodic extensions of f and g, i.e., first let

$$F_0(x) = \begin{cases} f(x) & 0 < x < \ell \\ -f(-x) & -\ell < x < 0 \end{cases}, \qquad G_0(x) = \begin{cases} g(x) & 0 < x < \ell \\ -g(-x) & -\ell < x < 0 \end{cases}.$$

Then we extend F_0 and G_0 to be periodic functions on $\mathbb{R}$, which we denote by $\overline{F}_0(x)$ and $\overline{G}_0(x)$, respectively. Then the Fourier Sine series of $\overline{F}_0(x)$ is

$$\overline{F}_0(x) = \sum_{n=1}^{\infty} f_n \sin(\mu_n x)$$

where

$$f_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \overline{F}_0(x)\sin(\mu_n x)\,dx = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\mu_n x)\,dx = a_n.$$

Similarly, the Fourier Sine series of $\overline{G}_0(x)$ is

$$\overline{G}_0(x) = \sum_{n=1}^{\infty} g_n \sin(\mu_n x)$$

where

$$g_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \overline{G}_0(x)\sin(\mu_n x)\,dx = \frac{2}{\ell}\int_0^{\ell} g(x)\sin(\mu_n x)\,dx = \frac{n\pi c}{\ell} b_n.$$

In other words the series (5.11) can be written as

$$u(x,t) = \frac{1}{2}\left[\overline{F}_0(x+ct) + \overline{F}_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \overline{G}_0(\xi)\,\xi. \tag{5.12}$$

Now we notice that, for example, if $f(x)$ is C^2 and $g(x)$ is C^1 on $[0,\ell]$, then (5.12) gives the solution to (5.1).

Let us use (5.1), assuming f and g are sufficiently smooth, to check the conditions. First we note that (5.12) formally satisfies the wave equation from our work on the D'Alembert form of the solution. Next we note that

$$u(x,0) = \frac{1}{2}\left[\overline{F}_0(x) + \overline{F}_0(x)\right] = \overline{F}_0(x) = f(x), \quad 0 < x < \ell.$$

$$u_t(x,0) = \frac{1}{2}\left[\overline{G}_0(x) + \overline{G}_0(x)\right] = \overline{G}_0(x) = g(x), \quad 0 < x < \ell.$$

Example 5.1.

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= f(x) = \begin{cases} x, & 0 \le x \le \ell/2 \\ \ell - x, & \ell/2 \le x \le \ell \end{cases} \\ u_t(x,0) &= g(x) = 0 \end{aligned} \tag{5.13}$$

For this example (5.5) becomes

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\mu_n\, ct\right) \sin\left(\mu_n\, x\right).$$

In this case we have

$$a_n = \frac{2}{\ell}\left[\int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx + \int_{\ell/2}^{\ell} (\ell - x) \sin\left(\mu_n\, x\right)\, dx\right]$$

$$= \begin{cases} 0 & n = 2, 4, \cdots \\ \dfrac{4}{\ell} \displaystyle\int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx = \dfrac{4\ell}{n^2\pi^2} \sin\left(\dfrac{n\pi}{2}\right) & n = 1, 3, \cdots \end{cases}$$

Here we have used the following fact. Using the change of variables $s = \ell - x$ we can write

$$\begin{aligned}
\int_{\ell/2}^{\ell} (\ell - x) \sin\left(\mu_n\, x\right)\, dx &= -\int_{\ell/2}^{0} s \sin\left(\mu_n(\ell - s)\right)\, ds \\
&= \int_0^{\ell/2} s \sin\left(n\pi - \mu_n\, s\right)\, ds \\
&= (-1)^{n+1} \int_0^{\ell/2} s \sin\left(\mu_n\, s\right)\, ds \\
&= (-1)^{n+1} \int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{2}{\ell}\left[\int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx + \int_{\ell/2}^{\ell} (\ell - x) \sin\left(\mu_n\, x\right)\, dx\right] \\
&= \frac{2}{\ell}\left[\int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx + (-1)^{n+1} \int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx\right] \\
&= \begin{cases} 0 & n = 2, 4, \cdots \\ \dfrac{4}{\ell} \displaystyle\int_0^{\ell/2} x \sin\left(\mu_n\, x\right)\, dx & n = 1, 3, \cdots \end{cases}
\end{aligned}$$

Finally, keeping in mind that n is odd, we employ integration by parts to evaluate the integral.

$$
\begin{aligned}
\int_0^{\ell/2} x \sin(\mu_n x)\, dx &= \int_0^{\ell/2} x \left(-\frac{\ell}{n\pi}\cos(\mu_n x)\right)' dx \\
&= x\left(-\frac{\ell}{n\pi}\cos(\mu_n x)\right)\Big|_0^{\ell/2} - \int_0^{\ell/2}\left(-\frac{\ell}{n\pi}\cos(\mu_n x)\right) dx \\
&= \frac{\ell}{n\pi}\int_0^{\ell/2} \cos(\mu_n x)\, dx \\
&= \left(\frac{\ell}{n\pi}\right)^2 \sin(\mu_n x)\Big|_0^{\ell/2} \\
&= \left(\frac{\ell}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right).
\end{aligned}
$$

So setting $n = 2k-1$ for $k = 1, 2, \cdots$, we arrive at

We arrive at the solution

$$
u(x,t) = \frac{4\ell}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi c t}{\ell}\right) \sin\left(\frac{(2k-1)\pi x}{\ell}\right). \tag{5.14}
$$

5.3 Wave Equation Neumann Boundary Conditions

$$
\begin{aligned}
u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\
u_x(0,t) &= 0, \quad u_x(\ell,t) = 0 \\
u(x,0) &= f(x) \\
u_t(x,0) &= g(x)
\end{aligned} \tag{5.15}
$$

First we present the solution to this problem and then provide a detailed derivation.

$$
\begin{aligned}
u(x,t) &= \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(c\mu_n t) + b_n \sin(c\mu_n t)\right)\cos(\mu_n x) \\
a_n &= \frac{2}{\ell}\int_0^{\ell} f(x)\cos(\mu_n x)\, dx, \quad a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\, dx \\
b_n &= \frac{2}{n\pi c}\int_0^{\ell} g(x)\cos(\mu_n x)\, dx, \quad b_0 = \frac{2}{\ell}\int_0^{\ell} g(x)\, dx.
\end{aligned}
$$

Look for simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

Substituting into (5.15) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$\frac{\ddot{\psi}(t)}{c^2\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{\ddot{\psi}(t)}{c^2\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here $\dot{\psi}$ means the derivative of ψ with respect to t and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and T. We obtain

$$\varphi'' - \lambda\varphi = 0, \qquad \ddot{\psi} - c^2\lambda\psi = 0.$$

Furthermore, the boundary conditions give

$$\varphi(0)\psi(t) = 0, \quad \varphi(\ell)\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi(\ell) = 0. \tag{5.16}$$

First we note that $\lambda_0 = 0$ with $\varphi_0(x) = 1$ is an eigenpair and the corresponding t equation is $\ddot{\psi}(t) = 0$ so we have any constant times $\psi_0(t) = a + bt$. In keeping with the form of our formulas for Fourier series we write

$$\varphi_0(x)\psi_0(t) = \frac{a_0 + b_0 t}{2}.$$

We have solved this problem a few times. What we found was that $\lambda = 0$ is an eigevalue with eigenfunction $\varphi = 1$ and for $\lambda = -\mu^2$ we have

$$\varphi(x) = a\cos(\mu x) + b\sin(\mu x).$$

Applying the boundary conditions we have

$$0 = \varphi'(0) = b \Rightarrow a = 0 \quad 0 = \varphi'(\ell) = a\sin(\mu\ell).$$

From this we conclude $\sin(\mu\ell) = 0$ which implies

$$\mu_n = \frac{n\pi}{\ell}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad \varphi_n(x) = \cos(\mu_n x), \quad n = 1, 2, \cdots .. \tag{5.17}$$

The solution of $\ddot{\psi} - c^2\lambda_n\psi = 0$ is then

$$\psi(t) = a\cos(\mu_n t) + b\sin(\mu_n t) \tag{5.18}$$

where a and b are arbitrary constants.

Next we look for u as an infinite sum

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\mu_n t) + b_n \sin(\mu_n t)\right) \cos(\mu_n x) \tag{5.19}$$

The problem remaining is to somehow find the constants a_n and b_n so that the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are satisfied.

Setting $t = 0$ in (5.19), we seek to obtain $\{a_n\}$ satisfying

$$f(x) = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\mu_n x).$$

This nothing more than a Cosine expansion of the function $f(x)$ on the interval $(0, \ell)$.

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos(\mu_n x)\, dx, \quad a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\, dx. \tag{5.20}$$

Next we differentiate (formally) (5.19) with respect to t to obtain

$$u_t(x,t) = \sum_{n=1}^{\infty} (\mu_n c)\left(-a_n \sin(\mu_n ct) + b_n \cos(\mu_n ct)\right)\cos(\mu_n x).$$

Setting $t = 0$ in this expression, we seek to obtain $\{b_n\}$ satisfying

$$g(x) = u_t(x,0) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n (\mu_n c)\cos(\mu_n x).$$

Except for an extra factor this is a Fourier Cosine expansion of the function $g(x)$ on the interval $(0, \ell)$. Namely we obtain

$$b_n = (\mu_n c)^{-1}\frac{2}{\ell}\int_0^{\ell} g(x)\cos(\mu_n x)\, dx, \quad b_0 = \frac{2}{\ell}\int_0^{\ell} g(x)\, dx.$$

After simplifying we have

$$b_n = \frac{2}{n\pi c}\int_0^{\ell} g(x)\cos(\mu_n x)\, dx, \quad b_0 = \frac{2}{\ell}\int_0^{\ell} g(x)\, dx. \tag{5.21}$$

Example 5.2. Consider the following example

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u_x(0,t) &= 0, \quad u_x(\ell,t) = 0 \\ u(x,0) &= f(x) = (\ell x^2/2) - (x^3/3) \\ u_t(x,0) &= g(x) = 0 \end{aligned} \tag{5.22}$$

In this case we see that the constants $b_n = 0$ for all n since $g(x) = 0$. We need only compute the coefficients a_n. To this end we need to calculate

$$\begin{aligned} a_n &= \frac{2}{\ell} \int_0^\ell f(x) \cos(\mu_n x)\, dx. \\ &= \frac{2}{\ell} \int_0^\ell \left[(\ell x^2/2) - (x^3/3) \right] \cos(\mu_n x)\, dx. \end{aligned}$$

This example arises in an application to torsional oscillations of a circular steel shaft where $u(x,t)$ represents the angle of twist of the shaft at point x at time t. The reason for telling you this is to explain why I take a more difficult looking f than usual. To do this problem I could proceed using integration by parts but I have decided to provide a formula due to Kronecker.

(Kronecker) If p_m is a polynomial of degree m and g is continuous on $[a,b]$. Then

$$\int_a^b p_m(x) g(x)\, dx = \sum_{j=0}^{m} p^{(j)}(x) G_{j+1}(x) \Big|_a^b,$$

where G_1 is an antiderivative of g, i.e., $G_1'(x) = g(x)$ and, in general, G_{j+1} is an antiderivative for G_j, i.e., $G_{j+1}' = G_j$.

Note it is easy to see, by repeated integration, that if $g(x) = \cos(\mu_n x)$, then

$$G_j(x) = (-1)^{(j+1)} \left(\frac{\ell}{n\pi} \right)^j \begin{cases} \cos(\mu_n x), & k \text{ even} \\ \sin(\mu_n x), & k \text{ odd} \end{cases}$$

For our example $p_m(x) = f(x) = (\ell x^2/2) - (x^3/3)$ so we have

$$\begin{aligned} f(x) &= (\ell x^2/2) - (x^3/3) \\ f'(x) &= \ell x - x^2 \\ f''(x) &= \ell - 2x \\ f'''(x) &= -2. \end{aligned}$$

Applying Kronecker's formula we have

$$
\begin{aligned}
a_n =& \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\mu_n x\right)\, dx. \\
=& \frac{2}{\ell}\Bigg[f(x)\left(\frac{1}{\mu_n}\right)\sin\left(\mu_n x\right) - f'(x)\left(\frac{1}{\mu_n}\right)^2\cos\left(\mu_n x\right) \\
& + f''(x)\left(\frac{1}{\mu_n}\right)^3\sin\left(\mu_n x\right) - f'''(x)\left(\frac{\ell}{n\pi}\right)^4\cos\left(\mu_n x\right)\Bigg]\Bigg|_0^\ell \\
=& \frac{2}{\ell}\left[-f'''(x)\left(\frac{1}{\mu_n}\right)^4\cos\left(\mu_n x\right)\right]\Bigg|_0^\ell \\
=& \frac{4}{\ell}\left(\frac{1}{\mu_n}\right)^4\left[(-1)^n - 1\right] \\
=& \frac{4\ell^3}{n^4\pi^4}\begin{cases} 0, & n = 2, 4, \cdots, \\ -2 & n = 1, 3, \cdots. \end{cases}
\end{aligned}
$$

Next we have

$$
a_0 = \frac{2}{\ell}\int_0^\ell f(x)\, dx = \frac{2}{\ell}\left(\frac{\ell x^3}{6} - \frac{x^4}{12}\right)\Bigg|_0^\ell = \frac{\ell^3}{6}.
$$

At this point we can set $n = (2k-1)$ for $k = 1, 2, \cdots$ and write

$$
a_{2k-1} = \frac{-8\ell^3}{(2k-1)^4\pi^4}.
$$

Finally we have

$$
u(x,t) = \frac{\ell^3}{12} - \frac{8\ell^3}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\cos\left(\frac{(2k-1)\pi ct}{\ell}\right)\cos\left(\frac{(2k-1)\pi x}{\ell}\right).
$$

If, for example, $\ell = \pi$ this becomes

$$
u(x,t) = \frac{\pi^3}{12} - \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\cos\left((2k-1)ct\right)\cos\left((2k-1)x\right).
$$

6 Assignment

1. Solve the BVP for the wave equation $u_{tt}(x,t) = u_{xx}(x,t)$ with

   (a) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $f(x) = \sin(x)\cos(x)$, $g(x) = \sin(x)$.

   (b) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $f(x) = 0$, $g(x) = 4\sin^3(x)$.

   (c) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $f(x) = x(\pi - x)$, $g(x) = 0$.

   (d) BC: $u_x(0,t) = 0,\ u_x(\pi,t) = 0$, IC: $f(x) = 0$, $g(x) = 1$.

2. Solve the initial boundary value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0$$
$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0$$

3. Solve the initial boundary value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0$$
$$u(x,0) = 0, \quad u_t(x,0) = \sin(\pi x)$$

4. Solve the initial boundary value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$
$$u_x(0,t) = 0, \quad u_x(1,t) = 0$$
$$u(x,0) = \cos(\pi x), \quad u_t(x,0) = \cos(2\pi x)$$

5. Solve the initial boundary value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0$$
$$u(x,0) = x, \quad u_t(x,0) = 0$$

6. Solve the initial value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$
$$u_x(0,t) = 0, \quad u_x(1,t) = 0$$
$$u(x,0) = x^2, \quad u_t(x,0) = 0$$

7. Solve the initial value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0$$
$$u(0,t) = 0, \quad u_x(\pi,t) = 0$$
$$u(x,0) = x, \quad u_t(x,0) = 0$$

8. Solve the initial value problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0$$
$$u(0,t) = 0, \quad u_x(\pi,t) = 0$$
$$u(x,0) = 0, \quad u_t(x,0) = x$$