4 1-D Boundary Value Problems Heat Equation

The main purpose of this chapter is to study boundary value problems for the heat equation on a finite rod $a \le x \le b$.

$$u_t(x,t) = k u_{xx}(x,t), \quad a < x < b, \quad t > 0$$
$$u(x,0) = \varphi(x)$$

The main new ingredient is that physical constraints called boundary conditions must be imposed at the ends of the rod. The two main conditions are

$$u(a,t) = 0, \quad u(b,t) = 0 \quad \text{Dirichlet Conditions}$$

$$u_x(a,t) = 0, \quad u_x(b,t) = 0 \quad \text{Neumann Conditions}$$

We can also have any combination of these conditions, i.e., we could have a Dirichlet condition at $x = a$ and Neumann condition at $x = b$.

The is one additional important boundary condition

$$u_x(a,t) - k_0 u(a,t) = 0, \quad u_x(b,t) + k_1 u(b,t) = 0 \quad \text{Robin Conditions}$$

Finally we will a more general problem involving two extra terms that correspond to heat conduction and convection.

$$u_t(x,t) = k\big(u_{xx}(x,t) - 2a u(x,t)_x + b u(x,t)\big), \quad 0 < x < \ell, \quad t > 0$$
$$u(x,0) = \varphi(x).$$

The basic methodology presented here is the idea of eigenvalues and eigenvector expansions as presented in a linear algebra or differential equations class when studying linear systems of ODEs. The basic idea can be described by an example. Let X be an n-vector valued function and A an $n \times n$ matrix. Also let X_0 denote a constant initial condition n-vector. Then to solve the initial value problem

$$\frac{d}{dt} X = AX, \quad X(0) = X_0$$

we find the eigenvalues $\{\lambda_j\}_{j=1}^n$ and eigenvectors $\{\Phi_j\}_{j=1}^n$ of A (i.e. $A\Phi_j = \lambda_j \Phi_j$) where we also assume

$$\langle \Phi_j, \Phi_k \rangle = \Phi^\top \Phi_k = \delta_{jk} \equiv \begin{cases} 1 & j = k \\ 0 & j \ne k \end{cases}.$$

Then, assuming there are n linearly independent eigenvectors, the solution can be written

$$X(t) = \sum_{j=1}^n e^{\lambda_j t} \langle X_0, \Phi_j \rangle \Phi_j.$$

The generalization of this idea to the one dimensional heat equation involves the generalized theory of Fourier series. This method due to Fourier was develop to solve the heat equation and it is one of the most successful ideas in mathematics. We will begin our study with classical Fourier series and then turn to the heat equation and Fourier's idea of separation of variables.

4.1 Fourier Series

A series of the functions

$$\phi_0 = \frac{1}{2}, \quad \phi_n^{(1)} = \cos(nx), \quad \phi_n^{(2)} = \sin(nx), \quad \text{for } n \geq 1$$

written in a series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)$$

is known as a *Fourier series.* (We choose $\phi_0 = \frac{1}{2}$ so all of the functions have the same norm.) A fairly general class of functions can be expanded in Fourier series. Let $f(x)$ be a function defined on $-\pi < x < \pi$. Assume that $f(x)$ can be expanded in a Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big). \tag{4.1}$$

Here the "$\sim$" means "has the Fourier series". We have not said if the series converges yet. For now let's assume that the series converges uniformly so we can replace the $\sim$ with an $=$.

We integrate Equation 4.1 from $-\pi$ to π to determine a_0.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\, dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\, dx \\
&= \pi a_0 + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\, dx + b_n \int_{-\pi}^{\pi} \sin(nx)\, dx \right) \\
&= \pi a_0
\end{aligned}$$

So that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx.$$

Multiplying by $\cos(mx)$ and integrating will enable us to solve for a_m.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = \frac{1}{2}a_0 &\int_{-\pi}^{\pi} \cos(mx)\, dx \\
&+ \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx + b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx \right)
\end{aligned}$$

All but one of the terms on the right side vanishes due to the orthogonality of the functions.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx &= a_m \int_{-\pi}^{\pi} \cos(mx)\cos(mx)\, dx \\
&= a_m \int_{-\pi}^{\pi} \frac{1}{2}\left(1 + \cos(2mx)\right)\, dx \\
&= \pi a_m
\end{aligned}$$

So that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \quad m = 0, 1, 2, \ldots$$

Similarly, we can multiply by $\sin(mx)$ and integrate to solve for b_m. The result is

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \quad m = 1, 2, 3, \ldots.$$

a_n and b_n are called *Fourier coefficients.*

Although we will not show it, Fourier series converge for a fairly general class of functions. Let $f(x^-)$ denote the left limit of $f(x)$ and $f(x^+)$ denote the right limit.

Example 4.1. For the function defined

$$f(x) = \begin{cases} 0 & \text{for } x < 0, \\ x+1 & \text{for } x \geq 0, \end{cases}$$

the left and right limits at $x = 0$ are

$$f(0^-) = 0, \qquad f(0^+) = 1.$$

Theorem 4.1. *Let $f(x)$ be a 2π-periodic function for which $\int_{-\pi}^{\pi} |f(x)|\,dx$ exists. Define the Fourier coefficients*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

If x is an interior point of an interval on which $f(x)$ has limited total fluctuation, then the Fourier series of $f(x)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big),$$

converges to $\frac{1}{2}(f(x^-) + f(x^+))$. If f is continuous at x, then the series converges to $f(x)$.

Example 4.2. Consider the function defined by

$$f(x) = \begin{cases} -x & \text{for } -\pi \leq x < 0 \\ \pi - 2x & \text{for } 0 \leq x < \pi. \end{cases}$$

The Fourier series converges to the function defined by

$$\hat{f}(x) = \begin{cases} 0 & \text{for } x = -\pi \\ -x & \text{for } -\pi < x < 0 \\ \pi/2 & \text{for } x = 0 \\ \pi - 2x & \text{for } 0 < x < \pi. \end{cases}$$

The function $\hat{f}(x)$ is plotted in the following figure.

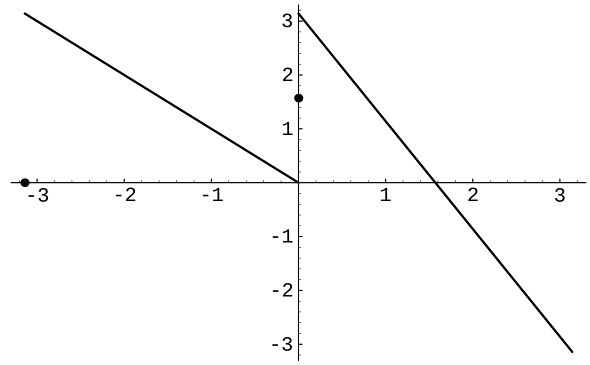


Figure 1: Graph of $\hat{f}(x)$.

A lot can be learned about the Fourier coefficients from the geometry of the function. For example, if $f(x)$ is an even function, $(f(-x) = f(x))$, then there will not be any sine terms in the Fourier series for $f(x)$. The Fourier sine coefficient is

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Since $f(x)$ is an even function and $\sin(nx)$ is odd, $f(x)\sin(nx)$ is odd. b_n is the integral of an odd function from $-\pi$ to π and is thus zero. Also we can simplify the cosine coefficients,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx)\, dx. \end{aligned}$$

Example 4.3. Consider the function defined on $[0, \pi)$ by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi/2 \\ \pi - x & \text{for } \pi/2 \le x < \pi. \end{cases}$$

The Fourier cosine coefficients for this function are

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{0}^{\pi/2} x \cos(nx)\, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cos(nx)\, dx \\ &= \begin{cases} \dfrac{\pi}{4} & \text{for } n = 0, \\ \dfrac{8}{\pi n^2} \cos\left(\dfrac{n\pi}{2}\right) \sin^2\left(\dfrac{n\pi}{4}\right) & \text{for } n \ge 1. \end{cases} \end{aligned}$$

In Figure 3 the even periodic extension of $f(x)$ is plotted in a dashed line and the sum of the first five nonzero terms in the Fourier series are plotted in a solid line.

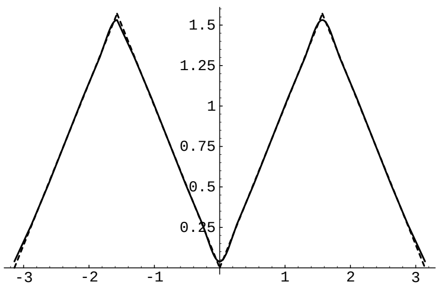


Figure 3: Fourier Cosine Series.

If, on the other hand, $f(x)$ is an odd function, $(f(-x) = -f(x))$, then there will not be any cosine terms in the Fourier series. Since $f(x)\cos(nx)$ is an odd function, the cosine coefficients will be zero. Since $f(x)\sin(nx)$ is an even function,we can rewrite the sine coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

Example 4.4. Consider the function defined on $[0, \pi)$ by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi/2 \\ \pi - x & \text{for } \pi/2 \le x < \pi. \end{cases}$$

The Fourier sine coefficients for this function are

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx \\ &= \frac{16}{\pi n^2}\cos\left(\frac{n\pi}{4}\right)\sin^3\left(\frac{n\pi}{4}\right) \end{aligned}$$

In Figure 4 the odd periodic extension of $f(x)$ is plotted in a dashed line and the sum of the first five nonzero terms in the Fourier sine series are plotted in a solid line.

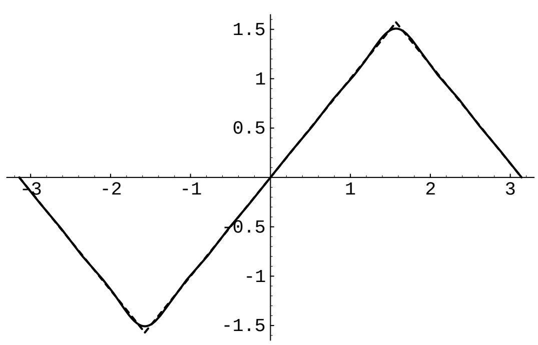


Figure 4: Fourier Sine Series.

4.2 Fourier series for arbitrary interval

The above formulas can easily be extended to periodic functions with an arbitrary period 2ℓ defined on $[-\ell,\ell]$ and, using even and odd extensions, we can even write Fourier series expansions for functions defined on an interval $[0,\ell]$. We present these formulas below.

1. For $f(x) = f(x+2\ell)$ and f is piecewise smooth on $-\ell \le x \le \ell$. Then for all $x \in \mathbb{R}$

   $$\frac{f(x+)+f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right\},$$

   where

   $$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx,$$

   $$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx$$

   $$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$

   If f is continuous (at x) then

   $$\frac{f(x+)+f(x-)}{2} = f(x).$$

2. **Fourier Cosine:** Given f on $[0,\ell]$

   $$a_0 = \frac{1}{\ell}\int_0^{\ell} f(x)\,dx, \quad a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx$$

   $$\frac{f(x+)+f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

3. **Fourier Sine:** Given f on $[0,\ell]$

   $$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx$$

   $$\frac{f(x+)+f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

4.3 Heat Equation Dirichlet Boundary Conditions

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned} \tag{4.2}$$

1. **Separate Variables** Look for simple solutions in the form

   $$u(x,t) = \varphi(x)\psi(t).$$

   Substituting into (4.14) and dividing both sides by $\varphi(x)\psi(t)$ gives

   $$\frac{\dot{\psi}(t)}{k\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

   Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

   $$\frac{\dot{\psi}(t)}{k\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.$$

   (Here $\dot{\psi}$ means the derivative of ψ with respect to t and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and ψ.

   We obtain

   $$\varphi'' - \lambda\varphi = 0, \qquad \dot{\psi} - k\lambda\psi = 0.$$

   The solution of the second equation is

   $$\psi(t) = Ce^{k\lambda t} \tag{4.3}$$

   where C is an arbitrary constant. Furthermore, the boundary conditions give

   $$\varphi(0)\psi(t) = 0, \quad \varphi(\ell)\psi(t) = 0 \quad \text{for all } t.$$

   Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

   $$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi(\ell) = 0. \tag{4.4}$$

2. **Find Eigenvalues and Eignevectors** The next main step is to find the eigenvalues and eigenfunctions from (4.16). There are, in general, three cases:

   (a) If $\lambda = 0$ then $\varphi(x) = ax + b$ so applying the boundary conditions we get

   $$0 = \varphi(0) = b, \quad 0 = \varphi(\ell) = a\ell \quad \Rightarrow a = b = 0.$$

   Zero is not an eigenvalue.

(b) If $\lambda = \mu^2 > 0$ then
$$\varphi(x) = a\cosh(\mu x) + b\sinh(\mu x).$$
Applying the boundary conditions we have
$$0 = \varphi(0) = a \Rightarrow a = 0 \quad 0 = \varphi(\ell) = b\sinh(\mu\ell) \quad \Rightarrow b = 0.$$
Therefore, there are no positive eigenvalues.

Consider the following alternative argument: If $\varphi''(x) = \lambda\varphi(x)$ then multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have
$$\lambda \int_0^\ell \varphi(x)^2\, dx = \int_0^\ell \varphi(x)\varphi''(x)\, dx = -\int_0^\ell \varphi'(x)^2\, dx + \varphi(x)\varphi'(x)\Big|_0^\ell .$$
Since $\varphi(0) = \varphi(\ell) = 0$ we conclude
$$\lambda = -\frac{\displaystyle\int_0^\ell \varphi'(x)^2\, dx}{\displaystyle\int_0^\ell \varphi(x)^2\, dx}$$
and we see that λ must be less than or equal to zero.

(c) So, finally, consider $\lambda = -\mu^2$ so that
$$\varphi(x) = a\cos(\mu x) + b\sin(\mu x).$$
Applying the boundary conditions we have
$$0 = \varphi(0) = a \Rightarrow a = 0 \quad 0 = \varphi(\ell) = b\sin(\mu\ell).$$
From this we conclude $\sin(\mu\ell) = 0$ which implies
$$\mu = \frac{n\pi}{\ell}$$
and therefore
$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad \varphi_n(x) = b_n\sin(\mu_n x), \quad n = 1, 2, \cdots .. \tag{4.5}$$
From (4.15) we also have the associated functions $\psi_n(t) = e^{k\lambda_n t}$.

3. **Write Formal Sum** From the above considerations we can conclude that for any integer N and constants $\{b_n\}_{n=0}^N$
$$u_N(x,t) = \sum_{n=1}^N b_n\psi_n(t)\varphi_n(x) = \sum_{n=1}^N b_n e^{k\lambda_n t}\sin\left(\frac{n\pi x}{\ell}\right).$$
satisfies the differential equation in (4.14) and the boundary conditions.

4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants b_n so that the initial condition $u(x,0) = \varphi(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for u as an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin\left(\frac{n\pi x}{\ell}\right)$$

and we try to find $\{b_n\}$ satisfying

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

But this nothing more than a Sine expansion of the function φ on the interval $(0,\ell)$.

$$b_n = \frac{2}{\ell}\int_0^{\ell} \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx. \tag{4.6}$$

Example 4.5. As an explicit example for the initial condition consider $\ell = 1$, $k = 1/10$ and $\varphi(x) = x(1-x)$. Let us recall that $\mu_n = \left(\frac{n\pi}{\ell}\right)$ which in this case reduces to $n\pi$.

$$\begin{aligned}
b_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\, dx \\
&= 2\int_0^1 x(1-x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)'\, dx \\
&= \frac{2}{n\pi}\left[-x(1-x)\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1 + \int_0^1 (1-2x)\frac{\cos(n\pi x)}{\mu_n}\, dx\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{n\pi}\int_0^1 (1-2x)\left(\frac{\sin(n\pi x)}{n\pi}\right)'\, dx \\
&= \frac{2}{n\pi}\left[(1-2x)\frac{\sin(n\pi x)}{n\pi}\Bigg|_0^1 - \int_0^1 (-2)\frac{\sin(n\pi x)}{n\pi}\, dx\right] \\
&= \frac{4}{(n\pi)^2}\int_0^1 \sin(n\pi x)\, dx = \frac{4}{(n\pi)^2}\left[-\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1\right] = \frac{4\left[1-(-1)^n\right]}{(n\pi)^3}
\end{aligned}$$

We arrive at the solution

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{\left[1-(-1)^n\right]}{n^3} e^{-n^2\pi^2 t/10} \sin(n\pi x). \tag{4.7}$$

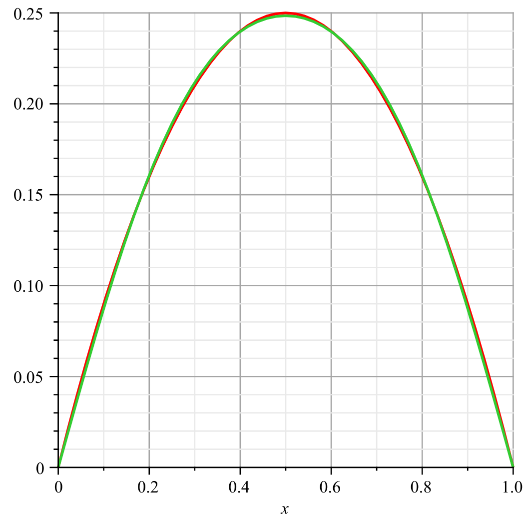
where

$$x(1-x) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{\left[1-(-1)^n\right]}{n^3} \sin(n\pi x).$$

As an example with $N = 3$ we have

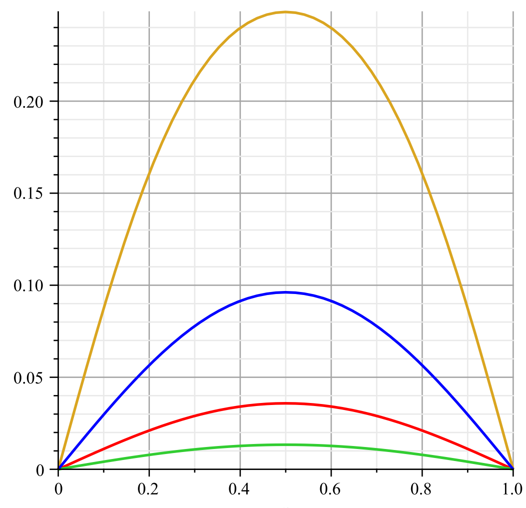
$$x(1-x) \approx \frac{8}{\pi^3}\left(\sin(\pi x) + \frac{\sin(3\pi x)}{27}\right).$$

In the following figure we plot the left and right hand side of the above.



Finally we plot the approximate solution at times $t = 0$, $t =$, $t = 2$, $t = 3$

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{3}\frac{[1-(-1)^n]}{n^3}e^{-n^2\pi^2 t/10}\sin(n\pi x).$$



4.4 Heat Equation Neumann Boundary Conditions

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u_x(0,t) &= 0, \quad u_x(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned} \tag{4.8}$$

1. **Separate Variables** Look for simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

Substituting into (4.8) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$\frac{\dot{\psi}(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{\dot{\psi}(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here $\dot{\psi}$ means the derivative of ψ with respect to t and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and ψ.

We obtain

$$\varphi'' - \lambda\varphi = 0, \qquad \dot{\psi} - \lambda\psi = 0.$$

The solution of the second equation is

$$\psi(t) = Ce^{\lambda t} \tag{4.9}$$

where C is an arbitrary constant. Furthermore, the boundary conditions give

$$\varphi'(0)\psi(t) = 0, \quad \varphi'(\ell)\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi'(0) = 0, \quad \varphi'(\ell) = 0. \tag{4.10}$$

2. **Find Eigenvalues and Eignevectors** The next main step is to find the eigenvalues and eigenfunctions from (4.10). There are, in general, three cases:

   (a) If $\lambda = 0$ then $\varphi(x) = ax + b$ so applying the boundary conditions we get

   $$0 = \varphi'(0) = a, \quad 0 = \varphi'(\ell) = a \quad \Rightarrow a = 0.$$

   Notice that b is still an arbitrary constant. We conclude that $\lambda_0 = 0$ is an eigenvalue with eigenfunction $\varphi_0(x) = 1$.

   (b) If $\lambda = \mu^2 > 0$ then

   $$\varphi(x) = a\cosh(\mu x) + b\sinh(\mu x)$$

   and

   $$\varphi'(x) = a\mu\sinh(\mu x) + b\mu\cosh(\mu x).$$

   Applying the boundary conditions we have

   $$0 = \varphi'(0) = b\mu \Rightarrow b = 0 \quad 0 = \varphi'(\ell) = a\mu\sinh(\mu\ell) \quad \Rightarrow a = 0.$$

Therefore, there are no positive eigenvalues.

Consider the following alternative argument: If $\varphi''(x) = \lambda\varphi(x)$ then multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

$$\lambda \int_0^\ell \varphi(x)^2\,dx = \int_0^\ell \varphi(x)\varphi''(x)\,dx = -\int_0^\ell \varphi'(x)^2\,dx + \varphi(x)\varphi'(x)\Big|_0^\ell.$$

Since $\varphi'(0) = \varphi'(\ell) = 0$ we conclude

$$\lambda = -\frac{\displaystyle\int_0^\ell \varphi'(x)^2\,dx}{\displaystyle\int_0^\ell \varphi(x)^2\,dx}$$

and we see that λ must be less than or equal to zero (zero only if $\varphi' = 0$).

(c) So, finally, consider $\lambda = -\mu^2$ so that

$$\varphi(x) = a\cos(\mu x) + b\sin(\mu x)$$

and

$$\varphi'(x) = -a\mu\sin(\mu x) + b\mu\cos(\mu x).$$

Applying the boundary conditions we have

$$0 = \varphi'(0) = b\mu \Rightarrow b = 0 \quad 0 = \varphi'(\ell) = -a\mu\sin(\mu\ell).$$

From this we conclude $\sin(\mu\ell) = 0$ which implies $\mu = \dfrac{n\pi}{\ell}$ and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad \varphi_n(x) = \cos(\mu_n x), \quad n = 1, 2, \cdots .. \tag{4.11}$$

From (4.9) we also have the associated functions $\psi_n(t) = e^{\lambda_n t}$.

3. **Write Formal Infinite Sum** From the above considerations we can conclude that for any integer N and constants $\{a_n\}_{n=0}^N$

$$u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^N a_n\psi_n(t)\varphi_n(x) = a_0 + \sum_{n=1}^N a_n e^{\lambda_n t}\cos\left(\frac{n\pi x}{\ell}\right).$$

satisfies the differential equation in (4.8) and the boundary conditions.

4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants a_n so that the initial condition $u(x,0) = \varphi(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for u as an infinite sum

$$u(x,t) = a_0 + \sum_{n=1}^\infty a_n e^{\lambda_n t}\cos\left(\frac{n\pi x}{\ell}\right)$$

and we try to find $\{a_n\}$ satisfying

$$\varphi(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right).$$

But this nothing more than a Cosine expansion of the function φ on the interval $(0,\ell)$. Our work on Fourier series showed us that

$$a_0 = \frac{2}{\ell}\int_0^\ell \varphi(x)\,dx, \quad a_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx. \tag{4.12}$$

As an explicit example for the initial condition consider $\ell = 1$ and $\varphi(x) = x(1-x)$. In this case (4.12) becomes

$$a_0 = 2\int_0^1 \varphi(x)\,dx, \quad a_n = 2\int_0^1 \varphi(x)\cos(n\pi x)\,dx.$$

We have

$$\begin{aligned} a_0 &= 2\int_0^1 \varphi(x)\,dx = 2\int_0^1 x(1-x)\,dx \\ &= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \frac{1}{3}. \end{aligned}$$

and

$$\begin{aligned} a_n &= 2\int_0^1 \varphi(x)\cos(n\pi x)\,dx = 2\int_0^1 x(1-x)\cos(n\pi x)\,dx \\ &= 2\int_0^1 x(1-x)\left(\frac{\sin(n\pi x)}{n\pi}\right)'\,dx \\ &= 2\left[x(1-x)\frac{\sin(n\pi x)}{n\pi}\Bigg|_0^1 - \int_0^1 (1-2x)\frac{\sin(n\pi x)}{n\pi}\,dx\right] \\ &= \frac{2}{n\pi}\int_0^1 (1-2x)\left(\frac{\cos(n\pi x)}{n\pi}\right)'\,dx \\ &= \frac{2}{n\pi}\left[(1-2x)\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1 - \int_0^1 (-2)\frac{\cos(n\pi x)}{n\pi}\,dx\right] \\ &\quad = \frac{2}{n\pi}\left[-\frac{\cos(n\pi)}{n\pi} - \frac{1}{n\pi}\right] \\ &\quad = \frac{-2}{(n\pi)^2}((-1)^n + 1) = \begin{cases} \dfrac{-4}{(n\pi)^2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}. \end{aligned}$$

In order to eliminate the odd terms in the expansion we introduce a new index, k by $n = 2k$ where $k = 1, 2, \cdots$. So finally we arrive at the solution

$$u(x,t) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}e^{-4k^2\pi^2 t}\cos(2k\pi x). \tag{4.13}$$

As an example with $N = 4$ we have

$$x(1-x) \approx \frac{1}{6} - \frac{1}{\pi^2}\left(\sum_{n=1}^{4}\frac{\cos(2k\pi x)}{k^2}\right).$$

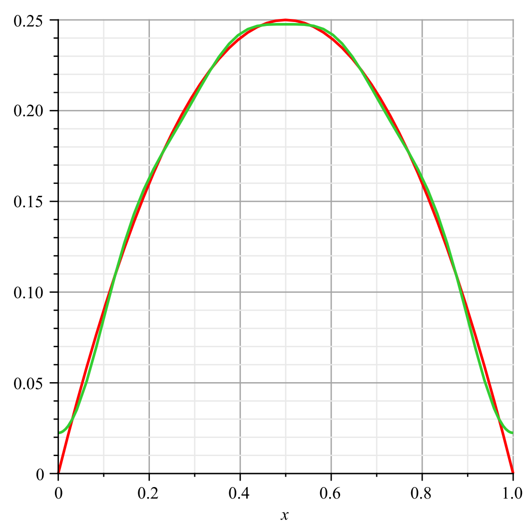
Notice that as $t \to \infty$ the infinite sum converges to zero uniformly in x. Indeed,

$$\left|\sum_{k=1}^{\infty}\frac{1}{k^2}e^{-4k^2\pi^2 t}\cos(2k\pi x)\right| \le e^{-4\pi^2 t}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}\ e^{-4\pi^2 t}.$$

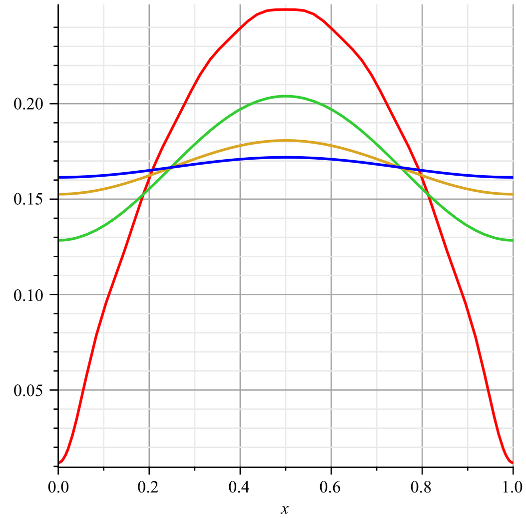
So the solution converges to a nonzero steady state temperature which is exactly the average value of the initial temperature distribution.

$$\lim_{t\to\infty} u(x,t) = \frac{1}{6} = \int_0^1 \varphi(x)\,dx.$$

In the following figure we plot the left and right hand side of the above.



Finally we plot the approximate solution at times $t = 0$, $t = 1/10$, $t = 2/10$, $t = 3/10$.



4.5 Heat Equation Dirichlet-Neumann Boundary Conditions

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u_x(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned} \tag{4.14}$$

1. **Separate Variables** Look for simple solutions in the form

$$u(x,t) = \varphi(x)\psi(t).$$

Substituting into (4.14) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$\frac{\dot{\psi}(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{\dot{\psi}(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda.$$

(Here $\dot{\psi}$ means the derivative of ψ with respect to t and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding nonzero functions φ and ψ.

We obtain

$$\varphi'' - \lambda\varphi = 0, \qquad \dot{\psi} - \lambda\psi = 0.$$

The solution of the second equation is

$$\psi(t) = Ce^{\lambda t} \tag{4.15}$$

where C is an arbitrary constant. Furthermore, the boundary conditions give

$$\varphi(0)\psi(t) = 0, \quad \varphi'(\ell)\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\varphi''(x) - \lambda\varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi'(\ell) = 0. \tag{4.16}$$

2. **Find Eigenvalues and Eignevectors** The next main step is to find the eigenvalues and eigenfunctions from (4.16). There are, in general, three cases:

   (a) If $\lambda = 0$ then $\varphi(x) = ax + b$ so applying the boundary conditions we get

   $$0 = \varphi(0) = b, \quad 0 = \varphi'(\ell) = a \quad \Rightarrow a = b = 0.$$

   Zero is not an eigenvalue.

   (b) If $\lambda = \mu^2 > 0$ then

   $$\varphi(x) = a\cosh(\mu x) + b\sinh(\mu x)$$

   and

   $$\varphi'(x) = a\mu\sinh(\mu x) + b\mu\cosh(\mu x).$$

   Applying the boundary conditions we have

   $$0 = \varphi'(0) = a\mu \Rightarrow a = 0 \quad 0 = \varphi'(\ell) = b\mu\cosh(\mu\ell) \quad \Rightarrow b = 0.$$

   Therefore, there are no positive eigenvalues.

   Consider the following alternative argument: If $\varphi''(x) = \lambda\varphi(x)$ then multiplying by φ we have $\varphi(x)\varphi''(x) = \lambda\varphi(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

   $$\lambda\int_0^\ell \varphi(x)^2\,dx = \int_0^\ell \varphi(x)\varphi''(x)\,dx = -\int_0^\ell \varphi'(x)^2\,dx + \varphi(x)\varphi'(x)\Big|_0^\ell.$$

   Since $\varphi(0) = \varphi'(\ell) = 0$ we conclude

   $$\lambda = -\frac{\int_0^\ell \varphi'(x)^2\,dx}{\int_0^\ell \varphi(x)^2\,dx}$$

   and we see that λ must be less than or equal to zero.

   (c) So, finally, consider $\lambda = -\mu^2$ so that

   $$\varphi(x) = a\cos(\mu x) + b\sin(\mu x)$$

   and

   $$\varphi'(x) = -a\mu\sin(\mu x) + b\mu\cos(\mu x).$$

   Applying the boundary conditions we have

   $$0 = \varphi(0) = a\mu \Rightarrow a = 0 \quad 0 = \varphi'(\ell) = b\mu\cos(\mu\ell).$$

From this we conclude $\cos(\mu\ell) = 0$ which implies

$$\mu = \frac{(2n-1)\pi}{2\ell}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{(2n-1)\pi}{2\ell}\right)^2, \quad \varphi_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots .. \qquad (4.17)$$

From (4.15) we also have the associated functions $\psi_n(t) = e^{\lambda_n t}$.

3. **Write Formal Infinite Sum** From the above considerations we can conclude that for any integer N and constants $\{b_n\}_{n=0}^N$

$$u_n(x,t) = \sum_{n=1}^N b_n \psi_n(t)\varphi_n(x) = \sum_{n=1}^N b_n e^{\lambda_n t} \sin\left(\frac{(2n-1)\pi x}{2\ell}\right).$$

satisfies the differential equation in (4.14) and the boundary conditions.

4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants b_n so that the initial condition $u(x,0) = \varphi(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for u as an infinite sum

$$u(x,t) = \sum_{n=1}^\infty b_n e^{\lambda_n t} \sin\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

and we try to find $\{b_n\}$ satisfying

$$\varphi(x) = u(x,0) = \sum_{n=1}^\infty b_n \sin\left(\frac{(2n-1)\pi x}{2\ell}\right).$$

But this nothing more than a Sine type expansion of the function φ on the interval $(0,\ell)$. Using

$$\varphi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

we have

$$\varphi(x) = \sum_{k=1}^\infty b_k \varphi_k(x).$$

We proceed as usual by multiplying both sides by $\varphi_n(x)$ and integrating from 0 to ℓ and using the orthogonality (described below in (4.19), (4.21)).

$$\int_0^\ell \varphi_n(x)\varphi(x)\,dx = \sum_{k=1}^\infty b_k \int_0^\ell \varphi_n(x)\varphi_k(x)\,dx$$

which implies

$$b_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\varphi_n(x)\,dx. \qquad (4.18)$$

Orthogonality:

$$\int_0^\ell \varphi_n(x)\varphi_k(x)\,dx = \begin{cases} \dfrac{\ell}{2}, & n = k \\ 0, & n \neq k \end{cases}.$$

To see this recall that $\varphi_j'' = \lambda_j \varphi_j$ and $\varphi_j(0) = 0$, $\varphi_j'(\ell) = 0$. First consider $n \neq k$ so $\lambda_n \neq \lambda_k$ and therefore

$$\lambda_n \int_0^\ell \varphi_n(x)\varphi_k(x)\,dx = \int_0^\ell \varphi_n''(x)\varphi_k(x)\,dx \tag{4.19}$$

$$= -\int_0^\ell \varphi_n'(x)\varphi_k'(x)\,dx + [\varphi_n'(x)\varphi_k(x)]\Big|_0^\ell$$

$$= \int_0^\ell \varphi_n(x)\varphi_k''(x)\,dx + [\varphi_n'(x)\varphi_k(x) - \varphi_n(x)\varphi_k'(x)]\Big|_0^\ell$$

$$= \lambda_k \int_0^\ell \varphi_n(x)\varphi_k(x)\,dx. \tag{4.20}$$

Therefore

$$(\lambda_n - \lambda_k)\int_0^\ell \varphi_n(x)\varphi_k(x)\,dx = 0 \;\Rightarrow\; \int_0^\ell \varphi_n(x)\varphi_k(x)\,dx = 0.$$

For $n = k$ we have

$$\int_0^\ell \varphi_n^2(x)\,dx = \int_0^\ell \sin^2\left(\frac{(2n-1)\pi x}{2\ell}\right)\,dx \tag{4.21}$$

$$= \frac{1}{2}\int_0^\ell \left(1 - \cos\left(\frac{(2n-1)\pi x}{\ell}\right)\right)\,dx$$

$$= \frac{\ell}{2} - \frac{1}{2}\left(\frac{\ell}{(2n-1)\pi}\right)\sin\left(\frac{(2n-1)\pi x}{\ell}\right)\Big|_0^\ell$$

$$= \frac{\ell}{2}. \tag{4.22}$$

As an explicit example for the initial condition consider $\varphi(x) = x$. Let us recall that $\mu_n = \left(\dfrac{(2n-1)\pi}{2\ell}\right)$

$$
\begin{aligned}
b_n &= \frac{2}{\ell}\int_0^\ell \varphi(x)\varphi_n(x)\,dx = \frac{2}{\ell}\int_0^\ell x\sin\left(\mu_n x\right)\,dx \\
&= \frac{2}{\ell}\int_0^\ell x\left(-\frac{\cos\left(\mu_n x\right)}{\mu_n}\right)'\,dx \\
&= \frac{2}{\ell}\left[-x\frac{\cos(\mu_n x)}{\mu_n}\bigg|_0^\ell + \int_0^\ell \frac{\cos(\mu_n x)}{\mu_n}\,dx\right] \\
&= \frac{2}{\ell}\left[-\ell\frac{\cos(\mu_n \ell)}{\mu_n} + \frac{\sin(\mu_n x)}{\mu_n^2}\bigg|_0^\ell\right] \\
&= \frac{2}{\ell}\left[-\ell\frac{\cos((2n-1)\pi/2)}{\mu_n} + \frac{\sin((2n-1)\pi/2)}{\mu_n^2}\right] \\
&= \frac{8\ell(-1)^{n+1}}{(2n-1)^2\pi^2}
\end{aligned}
$$

We arrive at the solution

$$
u(x,t) = \frac{8\ell}{\pi^2}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n-1)^2}e^{\lambda_n t}\sin\left(\mu_n x\right). \tag{4.23}
$$

4.6 Heat Equation General Boundary Conditions: Sturm–Liouville Theory

We consider the heat equation on the interval (a,b) with general boundary conditions. These notes use the material found in the book in Sections 2.7–2.9.

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) - q(x)u(x,t), \quad a<x<b, \quad t>0 && (4.24)\\
\alpha_1 u(a,t) - \alpha_2 u_x(a,t) &= 0, && (4.25)\\
\beta_1 u(b,t) + \beta_2 u_x(b,t) &= 0, && (4.26)\\
u(x,0) &= \varphi(x) && (4.27)
\end{aligned}
$$

where we assume that $\alpha_1^2+\alpha_2^2 \neq 0$ and $\beta_1^2+\beta_2^2 \neq 0$.

If we carry out our standard procedure of separation of variables we seek

$$
u(x,t) = \varphi(x)\psi(t).
$$

Substituting into (4.24) and dividing both sides by $\varphi(x)\psi(t)$ gives

$$
\frac{\dot\psi(t)}{\psi(t)} = \frac{\varphi''(x) - q(x)\varphi(x)}{\varphi(x)}
$$

Since the left side is independent of x and the right side is independent of t, it follows that the expression must be a constant:

$$\frac{\dot{\psi}(t)}{\psi(t)} = \frac{\varphi''(x) - q(x)\varphi(x)}{\varphi(x)} = \lambda.$$

(Here $\dot{\psi}$ means the derivative of ψ with respect to t and φ' means means the derivative of φ with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions φ and ψ.

We obtain

$$\varphi'' - q(x)\varphi(x) - \lambda\varphi = 0, \qquad \dot{\psi} - \lambda\psi = 0.$$

The solution of the second equation is

$$\psi(t) = Ce^{k\lambda t} \tag{4.28}$$

where C is an arbitrary constant. Furthermore, the boundary conditions give

$$(\alpha_1\varphi(a) - k\varphi(a))\psi(t) = 0\ , \quad (\beta_1\varphi(b) + \beta_2\varphi(b))\psi(t) = 0 \quad \text{for all } t.$$

Since $\psi(t)$ is not identically zero we obtain the desired eigenvalue problem

$$\begin{aligned} &\varphi''(x) - q(x)\varphi(x) - \lambda\varphi(x) = 0, \\ &\alpha_1\varphi(a) - \alpha_2\varphi(a) = 0, \\ &\beta_1\varphi(b) + \beta_2\varphi(b) = 0 \end{aligned} \tag{4.29}$$

This is an eigenvalue problem which is referred to as a Regular Sturm-Liouville problem.

Theorem 4.2. *The problem* (4.29) *has infinitely many eigenpairs* $\{\lambda_n, \varphi_n(x)\}$ *which satisfy the following properties:*

1. *The eigenvalues are all simple, i.e. they are eigenvalues of multiplicity one which means that* $\lambda_j \neq \lambda_k$ *for* $j \neq k$.

2. *The eigenvalues are all real and all but a finite number are negative. If* $\alpha_1, \alpha_2, \beta_1, \beta_2 \neq 0$ *then all the eigenvalues are less than or equal to zero.*

3. *If we order the eigenvalues in decreasing order by*

$$\lambda_n < \lambda_{n-1} < \cdots < \lambda_2 < \lambda_1$$

   then

$$\lambda_n \to -\infty \quad \text{as} \quad n \to \infty.$$

4. *The nth eigenfunction* $\varphi_n(x)$ *is real and has exactly* $(n-1)$ *zeros in the interval* (a, b)

5. *The eigenfunctions are orthogonal in the following sense*

$$\int_a^b \varphi_n(x)\varphi_m(x)\,dx = 0 \quad \text{for} \quad n \neq m.$$

6. If $\varphi(x)$ is piecewise smooth ($PC^{(1)}(a,b)$ in my notation in class), then

$$\frac{(\varphi(x+)+\varphi(x-))}{2} = \sum_{n=1}^{\infty} c_n\varphi_n(x), \quad a<x<b,$$

where

$$c_n = \frac{\int_a^b f(x)\varphi_n(x)\,dx}{\int_a^b \varphi_n^2(x)\,dx}.$$

At least for continuous φ, Theorem 4.2 allows us to conclude that the solution to (4.24)-(4.27) is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{\lambda_n t}\varphi_n(x).$$

4.7 Heat Equation with Conduction and Convection

We consider the heat equation on the interval $(0,1)$ with two extra terms that correspond to heat conduction and convection.

$$\begin{aligned} u_t(x,t) &= k\big(u_{xx}(x,t) - 2au(x,t)_x + bu(x,t)\big), \quad 0<x<\ell, \quad t>0 && (4.30)\\ u(0,t) &= 0, && (4.31)\\ u(\ell,t) &= 0, && (4.32)\\ u(x,0) &= \varphi(x). && (4.33) \end{aligned}$$

There are many different ways to approach this problem and one would be to apply separation of variable directly. The dissadvantange to this is that one gets a more complicated ode for $\varphi(x)$ and there is a more difficult analysis of the eigenvalues and eigenvectors.

We will take a different approach which allows us to use our earlier work after a change of dependent variables. So to this end let us define $v(x,t)$ via

$$u(x,t) = e^{ax+\beta t}v(x,t), \qquad \beta = k(b-a^2). \tag{4.34}$$

Thus we have

$$v(x,t) = e^{-(ax+\beta t)}u(x,t)$$

and we can compute

$$\begin{aligned} v_t - kv_{xx} &= e^{-(ax+\beta t)}\left(-\beta u + u_t\right) - k\left[e^{-(ax+\beta t)}\left(-au+u_x\right)\right]_x \\ &= e^{-(ax+\beta t)}\left\{(-\beta u + u_t) - k\left[-a(-au+u_x) + (-au_x + u_{xx})\right]\right\} \\ &= e^{-(ax+\beta t)}\left[u_t - k(u_{xx} - 2au_x + a^2u) + \beta u\right] \\ &= e^{-(ax+\beta t)}\left[u_t - k(u_{xx} - 2au_x + a^2u + (b-a^2)u)\right] \\ &= e^{-(ax+\beta t)}\left[u_t - k(u_{xx} - 2au_x + bu)\right] = 0. \end{aligned}$$

Furthermore
$$v(0,t) = e^{-\beta t}u(0,t) = 0, \qquad v(\ell,t) = e^{-(a\ell+\beta t)}u(\ell,t) = 0$$
and
$$v(x,0) = e^{-ax}u(x,0) = e^{-ax}\varphi(x).$$
Therefore, $v(x,t)$ is the solution of
$$\begin{gathered} v_t = kv_{xx} \\ v(0,t) = 0, \qquad v(\ell,t) = 0 \\ v(x,0) = e^{-ax}\varphi(x). \end{gathered}$$
We have eigenvalues and eigenfunctions
$$\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2, \quad \sin\left(\frac{n\pi}{\ell}x\right)$$
and we obtain the solution to this problem as
$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right) \quad \text{with} \;\; b_n = \frac{2}{\ell}\int_0^\ell e^{-ax}\varphi(x)\sin\left(\frac{n\pi}{\ell}x\right)\,dx.$$

Finally our solution to (4.30)-(4.33) can be written as
$$u(x,t) = e^{ax+\beta t}\sum_{n=1}^{\infty} b_n e^{\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right).$$

4.8 Assignment Eigenvalues and Heat Equation

1. Fourier Series Examples: The Fourier series for $f(x) = x^2$ on $-\pi \le x \le \pi$ gives
$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \quad -\pi \le x \le \pi$$
   Find values of x and give a justification for using them along with the above information to show that
   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$
   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n^2} = \frac{\pi^2}{12}$
2. Find the Fourier Cosine series expansion of $f(x) = \sin(x)$ on $0 \le x \le \pi$.
3. Find the Fourier series expansion of $f(x) = \cos^2(x)$ on $-\pi \le x \le \pi$. (THINK)
4. Determine all solutions

(a) $y'' - y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = -4.$

(b) $y'' + 4y = 0, \quad 0 < x < \pi, \quad y(0) = 0, \quad y'(\pi) = 0.$

(c) $y'' + y = 0, \quad 0 < x < 2\pi, \quad y(0) = 0, \quad y(2\pi) = 1.$

(d) $y'' - 2y' + y = 0, \quad -1 < x < 1, \quad y(-1) = 0, \quad y(1) = 2.$

5. Find all eigenvalues λ and eigenvectors y (i.e., y nonzero)

(a) $y'' + \lambda y = 0, \quad 0 < x < \pi, \quad y(0) = 0, \quad y(\pi) = 0.$

(b) $y'' + \lambda y = 0, \quad 0 < x < \pi, \quad y'(0) = 0, \quad y'(\pi) = 0.$

(c) $y'' + \lambda y = 0, \quad 0 < x < \pi, \quad y(0) = 0, \quad y'(\pi) = 0.$

(d) $y'' + \lambda y = 0, \quad 0 < x < 2\pi, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$

6. Solve the heat problem $u_t = u_{xx}$ with

(a) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $u(x,0) = \sin(x)$

(b) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $u(x,0) = x(\pi - x)$

(c) BC: $u(0,t) = 0,\ u(\pi,t) = 0$ IC: $u(x,0) = \sin(x) - 7\sin(3x)$

(d) BC: $u_x(0,t) = 0,\ u_x(\pi,t) = 0$, IC: $u(x,0) = 1 - \cos(x)$

(e) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $u(x,0) = \sin(x)\cos(x)$

(f) BC: $u_x(0,t) = 0,\ u_x(\pi,t) = 0$, IC: $u(x,0) = \cos^2(x)$ (Hint: half-angle formula)

(g) BC: $u(0,t) = 0,\ u(\pi,t) = 0$, IC: $u(x,0) = \begin{cases} -1, & 0 \le x < \pi/2 \\ 1, & \pi/2 \le x < \pi \end{cases}$

(h) BC: $u_x(0,t) = 0,\ u_x(\pi,t) = 0$, IC: $u(x,0) = \begin{cases} -1, & 0 \le x < \pi/2 \\ 1, & \pi/2 \le x < \pi \end{cases}$

7. Solve the initial boundary value problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0, \quad u(\pi,t) = 0 \\ &u(x,0) = x \end{aligned}$$

8. Solve the initial boundary value problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0 \\ &u_x(0,t) = 0, \quad u_x(\pi,t) = 0 \\ &u(x,0) = x \end{aligned}$$

9. Solve the initial boundary value problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0, \quad u_x(\pi,t) = 0 \\ &u(x,0) = x \end{aligned}$$

10. Solve the initial boundary value problem

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) - 2u_x(x,t), \quad 0 < x < 1, \quad t > 0 \\
u(0,t) &= 0, \quad u(1,t) = 0 \\
u(x,0) &= e^x \sin(\pi x)
\end{aligned}
$$

11. Solve the initial boundary value problem

$$
\begin{aligned}
u_t(x,t) &= 2(u_{xx}(x,t) - 3u(x,t)), \quad 0 < x < 1, \quad t > 0 \\
u(0,t) &= 0, \quad u(1,t) = 0 \\
u(x,0) &= 2\sin(\pi x) - 3\sin(2\pi x)
\end{aligned}
$$