2 The Wave Equation

The initial value problem for the wave equation on the whole number line is

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x), \quad -\infty < x < \infty, . \end{aligned} \tag{2.1}$$

It is a model for the small motions of many oscillating systems, e.g. the one dimensional displacement of a vibrating guitar string. In higher dimensions it is called the acoustic equation which models the propagation of sound waves through a medium. The sound velocity is c.

The wave equation in the one dimensional case can be derived in many different ways. We first give a simple derivation without to much detailed explanation. Then we present a more detailed discussion based on masses and springs. In class we will only discuss the first method and leave the lengthy discussion for you to read.

Derivation I: Newton's Law for Elastic String

In this derivation we consider a very short piece of a string which can only move in the vertical direction. We denote the vertical displacement at a point x and at time t by $u(x,t)$. We assume that the only forces acting on the string are the tensions at the left end and pulling to the left and the tension at the right end pulling to the right. We denote this force by T. The tension acts in a direction tangent to the string but we only consider vertical forces so we need to consider the vertical components of the string at each end. To do this we need to introduce the angle between the string and the horizontal line. If we call the vertical forces due to tension F then Newton's law states that $F = ma$ where m is the mass and a is the acceleration of the string. We assume that the string has length Δx and a constant density ρ so that the mass is $m = \rho \Delta x$. The acceleration is $u_{tt}(x,t)$ so we have

$$F = \rho \Delta x u_{tt}(x,t).$$

Our basic assumption is that the displacement is so small that the angles θ_1 and θ_2 are small enough that $\sin(\theta_1) \approx \tan(\theta_1)$ and $\sin(\theta_2) \approx \tan(\theta_2)$. Then using the fact from calculus that the derivative is the slope of the tangent line and

In order to compute F we consider the diagram below and use the basic assumption to obtain

$$\begin{aligned} \text{Tension } &= F = T\sin(\theta_2) - T\sin(\theta_1) \approx T[\tan(\theta_2) - \tan(\theta_1)] \\ &= T[u_x(x + \Delta x, t) - u_x(x,t)]. \end{aligned}$$

Therefore we have

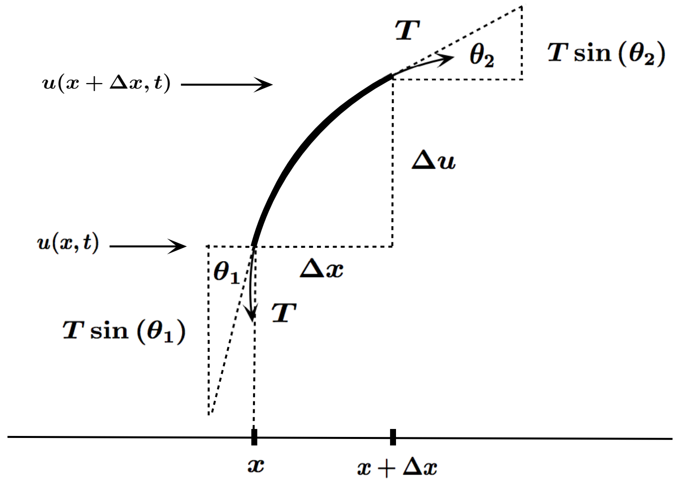
$$\rho \Delta x u_{tt}(x,t) \approx T[u_x(x + \Delta x, t) - u_x(x,t)]$$

so that

$$u_{tt}(x,t) \approx \frac{T}{\rho} \frac{[u_x(x + \Delta x, t) - u_x(x,t)]}{\Delta x}$$

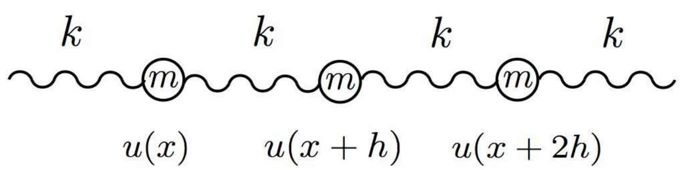
Finally we pass to the limit as Δx tends to zero to obtain

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad c^2 = \frac{T}{\rho}.$$



Mass and Springs

Derivation II: Newton's Law for Collection of Masses and Springs Imagine an array of little weights of mass m interconnected with springs of length h. Suppose also that the springs have a stiffness of k.



Mass and Springs

Here $u(x)$ measures the distance from the equilibrium of the mass at the spatial point x. The forces exerted on the mass m at the location $x+h$ are:

$$F_{\text{Newton}} = m\,a(t) = m\,\frac{\partial^2 u}{\partial t^2}(x+h,t)$$

and

$$\begin{aligned} F_{\text{Newton}} &= F_{x+2h} + F_x \\ &= k[u(x+2h,t) - u(x+h,t)] + k[u(x,t) - u(x+h,t)]. \end{aligned}$$

Here we have used Hooke's law for an elastic spring which states that $F_{Hooke} = -kx$ where x is the distance that the spring is stretched from equilibrium, F_{Hooke} is the restoring force exerted by the spring and k is the "spring constant." So, if we imagine the spring starting at position $u(x+2h,t)$ which is stretched to the point $u(x+h,t)$ then we have

$$F_{x+2h} = -k[u(x+h,t) - u(x+2h,t)] = k[u(x+2h,t) - u(x+h,t)].$$

Similarly the Hooke's force at x is

$$F_x = -k[u(x+h,t) - u(x,t)] = k[u(x,t) - u(x+h,t)].$$

Therefore, according to Newton's second law,

$$m\frac{\partial^2 u}{\partial t^2}(x+h,t) = k[u(x+2h,t) - 2u(x+h,t) + u(x)].$$

If the array of weights consists of N weights spaced evenly over the length $\ell = Nh$ of total mass $M = Nm$ and with the total stiffness of the array $K = k/N$ we can write this equation as:

$$\frac{\partial^2 u}{\partial t^2}(x+h,t) = \frac{K\ell^2}{M}\frac{[u(x+2h,t) - 2u(x+h,t) + u(x)]}{h^2}.$$

Here we are assuming that K, M and ℓ are constants which means that as N goes to infinity and h goes to zero, the values of m and k vary in such a way that the expression

$$\frac{K\ell^2}{M} = c^2$$

remains a constant. The constant c is the propagation speed in the particular media. Here

$$\lim_{h\to 0}\frac{[u(x+2h,t) - 2u(x+h,t) + u(x)]}{h^2} = \frac{\partial^2 u}{\partial t^2}(x,t).$$

Thus we end up with equation (2.1).

Let us explain why the above limit is correct. Suppose that f is a C^4 function near a point x. Then the Taylor formula from Calculus says that for h sufficiently small there exists a number $\xi_1 \in [x, x+h]$ so that

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f^{(iv)}(\xi_1)h^4.$$

On the other hand we can also say that for h sufficiently small there exists a number $\xi_2 \in [x-h, x]$ so that

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f^{(iv)}(\xi_2)h^4.$$

From these we can write

$$f''(x)\frac{h^2}{2} = f(x+h) - f(x) - f'(x)h - \frac{1}{6}f'''(x)h^3 - \frac{1}{24}f^{(iv)}(\xi_1)h^4$$

and

$$f''(x)\frac{h^2}{2} = f(x-h) - f(x) - f'(x)h + \frac{1}{6}f'''(x)h^3 - \frac{1}{24}f^{(iv)}(\xi_1)h^4.$$

Adding these expressions and dividing by h^2 we obtain

$$f''(x) = \frac{[f(x+h) - 2f(x) + f(x-h)]}{h^2} - \left[\frac{1}{24}f^{(iv)}(\xi_1) + \frac{1}{24}f^{(iv)}(\xi_2)\right]h^2.$$

Note that as h goes to zero ξ_1 and ξ_2 converge to x so the term

$$\left[\frac{1}{24} f^{(iv)}(\xi_1) + \frac{1}{24} f^{(iv)}(\xi_2)\right]$$

is bounded. Passing to the limit as h goes to zero we see that the last term goes to zero and we have

$$f''(x) = \lim_{h \to 0} \frac{[f(x+h) - 2f(x) + f(x-h)]}{h^2}.$$

2.1 The D'Alembert Formula

We consider the initial value problem for the wave equation on the whole number line

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x), \quad -\infty < x < \infty, . \end{aligned} \tag{2.2}$$

Notice that the PDE in (2.2) can be written as

$$\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right] u(x,t) = 0.$$

Actually the order in which we write the two terms is not relevant so that solutions of either

$$\left[\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right] u(x,t) = 0, \quad \text{or} \quad \left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right] u(x,t) = 0$$

are also solutions of (2.2). Each of these equations is first order linear and as we learned in the first chapter of material for the course, the general solution of these equations can be written respectively as

$$u(x,t) = F(x+ct), \quad \text{and} \quad u(x,t) = G(x-ct),$$

where F and G are arbitrary functions.

Since the equation (2.2) is linear it is easy to see that for any sufficiently smooth functions F and G the function

$$u(x,t) = F(x+ct) + G(x-ct) \tag{2.3}$$

solves the wave equation in (2.2). Namely, we see (by the chain rule)

$$u_t(x,t) = F'(x+ct)(c) + G'(x-ct)(-c)$$

$$\begin{aligned} u_{tt}(x,t) &= F''(x+ct)(c^2) + G''(x-ct)(c^2) \\ u_x(x,t) &= F'(x+ct) + G'(x-ct) \\ u_{xx}(x,t) &= F''(x+ct) + G''(x-ct) \end{aligned}$$

so

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= \big(F''(x+ct)(c^2) + G''(x-ct)(c^2)\big)\\ &\qquad - c^2\left(F''(x+ct) + G''(x-ct)\right) = 0.\end{aligned}$$

To find F and G so that (2.3), in addition, satisfies the initial conditions, we proceed as follows: Setting $t = 0$ in (2.3) we have

$$u(x,0) = F(x) + G(x) = f(x). \tag{2.4}$$

Taking the derivative with respect to t of (2.3) and again setting $t = 0$ we obtain

$$u_t(x,0) = cF'(x) - cG'(x) = g(x). \tag{2.5}$$

Carrying out the integration with respect to x from any x_0 to x in (2.6) and we obtain

$$F(x) - G(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + K$$

where K is an arbitrary constant.

Thus we obtain a system of two equations in two unknowns for F and G:

$$\begin{aligned}F(x) + G(x) &= f(x)\\ F(x) - G(x) &= \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + K\end{aligned}$$

We can solve this linear system of equations using Cramer's rule

$$F(x) = \frac{\begin{vmatrix} f(x) & 1 \\ \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + K & -1\end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1\end{vmatrix}} = \frac{1}{2}f(x) + \left(\frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{K}{2}\right)$$

and

$$G(x) = \frac{\begin{vmatrix} 1 & f(x) \\ 1 & \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + K\end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1\end{vmatrix}} = \frac{1}{2}f(x) - \left(\frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{K}{2}\right)$$

Combining these results with (2.3) we obtain

$$\begin{aligned}u(x,t) &= \frac{f(x+ct) + f(x-ct)}{2} + \left(\frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \frac{K}{2}\right)\\ &\qquad - \left(\frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds + \frac{K}{2}\right)\\ &= \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.\end{aligned}$$

We thus obtain the solution of the initial value problem (2.2) which is known as the D'Alembert formula.

Theorem 2.1. *If $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Then there is a unique C^2 solution of (2.2) given by the D'Alembert formula.*

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds. \tag{2.6}$$

Example 2.1. We consider the problem (2.1) with the initial conditions $f(x) = \sin(x)$ and $g(x) = 0$. In this case the solution (2.6) becomes

$$u(x,t) = \frac{1}{2} (\sin(x+ct) + \sin(x-ct)) = \sin(x)\cos(ct).$$

We can view the solution as dividing the initial shape $\sin(x)$ into two copies of $\dfrac{\sin(x)}{2}$. As time increases the single wave splits into two waves one of these travels to the left and one travels to the right at a constant speed c.

Remark 2.1. More generally we have the following interpretation of Eq. (2.6). If we plot the graph of $y = f(x-ct)$ for any fixed t, we see that it is exactly the same as the graph of $y = f(x)$ but it is shifted a distance ct units to the right. Notice a given height $f(x_0)$ at $x = x_0$ gets moved ct units to the right in time t. Thus we say that $f(x-ct)$ is a *wave* which travels with the velocity c in the positive x direction. The expression $f(x+ct)$, similarly, represents a wave moving to the left with a constant velocity c. If a disturbance occurs at a point x_0 (i.e. a value $f(x_0)$) then it it felt at the point x_1 after a time

$$t = \frac{(x_1 - x_0)}{c}.$$

Thus we see that for the wave equation data (disturbances) is propagated (or travels) at a finite rate of speed.

Example 2.2. We consider the problem (2.1) with $c = 1$ and the initial conditions $f(x) = x^2$ and $g(x) = 2x$. In this case the solution (2.6) becomes

$$u(x,t) = \frac{(x+t)^2 + (x-t)^2 + (x+t)^2 - (x-t)^2}{2} = (x+t)^2.$$

Example 2.3. We consider the problem (2.1) with the initial conditions

$$f(x) = [H(x+1) - H(x-1)], \quad g(x) = 0.$$

In this case the solution (2.6) becomes

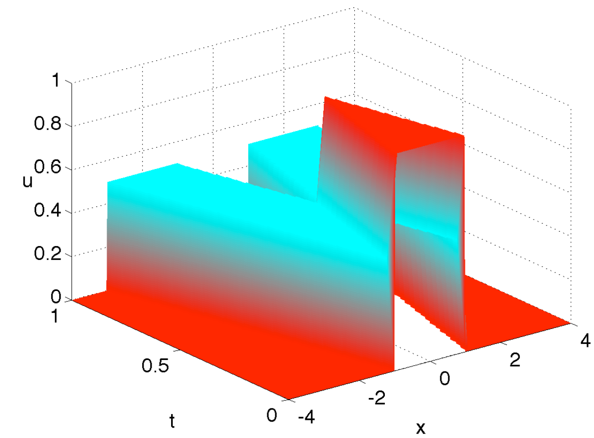
$$u(x,t) = \frac{1}{2}\Big(H(x-ct+1) - H(x-ct-1) + H(x+ct+1) - H(x+ct-1) \Big).$$

Here $H(x)$ is the Heaviside function

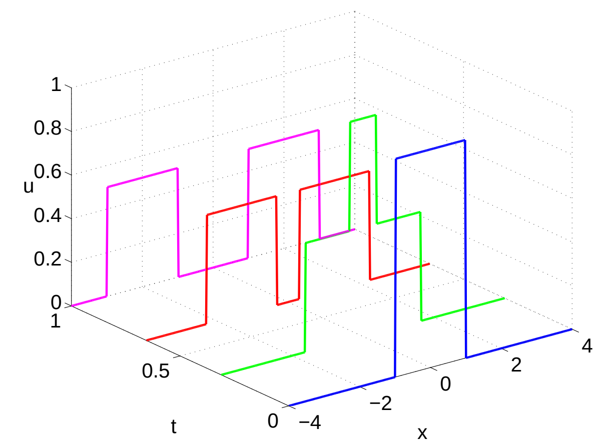
$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

so that $f(x)$ represents a box with value 1 for $-1 \leq x \leq 1$ and 0 elsewhere.

In our numerical example we have set $c = 2$.



Solution Surface



A Few Solution Curves

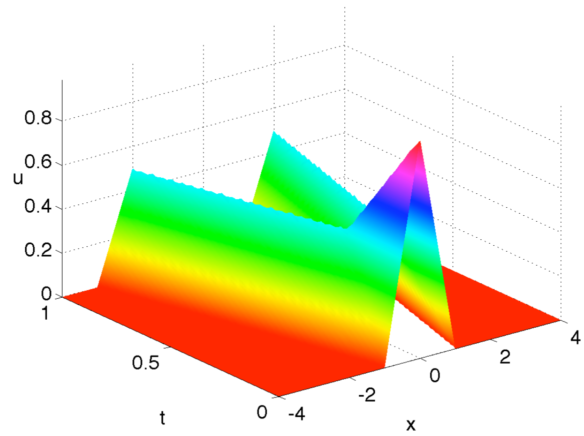
Example 2.4. Finally let us consider the problem (2.1) with the initial conditions

$$f(x) = \begin{cases} (x+1), & -1 < x \leq 0 \\ (1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \quad g(x) = 0.$$

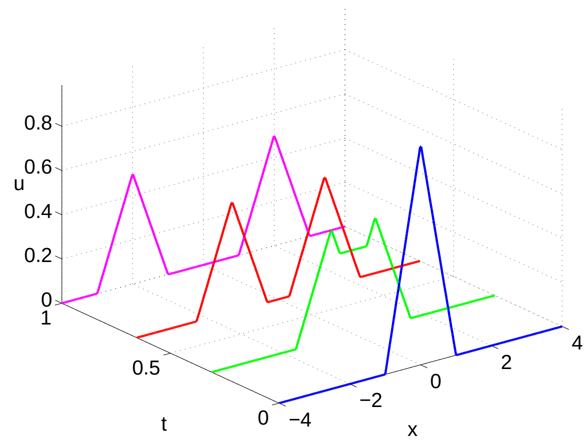
In this case the solution (2.6) becomes

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big).$$

In our numerical example we have set $c = 2$.



Solution Surface



A Few Solution Curves

Definition 2.1. For a function $u(x,t)$ we define the energy at time t as the integral

$$e(t) \equiv \frac{1}{2} \int_{\mathbb{R}} \left[u_t^2(x,t) + c^2 u_x^2(x,t) \right] \, dx \tag{2.7}$$

provided the integral is finite.

Theorem 2.2 (Conservation of Energy Wave Equation)**.** *Suppose that $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ with*

$$\int_{\mathbb{R}} f'(x)^2 \, dx < \infty, \qquad \textit{and} \qquad \int_{\mathbb{R}} g(x)^2 \, dx < \infty.$$

Let $u(x,t)$ be the unique solution to the problem (2.1)*. Then*

$$e(t) < \infty \qquad \textit{and} \quad \frac{de}{dt}(t) = 0 \quad \forall t \in \mathbb{R}$$

therefore $e(t)$ is constant. In particular,

$$e(t) = e(0) = \frac{1}{2} \int_{\mathbb{R}} \left[g(x)^2 + c^2 f'(x)^2 \right] \, dx.$$

If we knew that
$$\lim_{x\to\infty} u_x(x,t)u_t(x,t) = 0 \quad \text{for every} \quad t$$
then we could have a simple proof of 2.2 as follows:
$$\begin{aligned}
\frac{de(t)}{dt} =& \frac{1}{2}\int_{\mathbb{R}} \left(2u_t u_{tt} + c^2 2u_x u_{xt}\right)\, dx \\
=& \int_{\mathbb{R}} u_t u_{tt}\, dx + c^2 \int_{\mathbb{R}} u_x u_{xt}\, dx \\
=& \int_{\mathbb{R}} u_t u_{tt}\, dx + c^2 \left[u_x u_t \Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} u_{xx} u_t\, dx \right] \\
=& \int_{\mathbb{R}} u_t \left(u_{tt} - c^2 u_{xx}\right)\, dx = 0.
\end{aligned}$$
After some discussion involving domain of dependence and range of influence we could make this rigorous for initial data that vanish outside a bounded region. But we will take a different and somewhat more lengthy approach. We will show that $e(t) = e(0)$ directly using the D'Alembert formula. To this end we must compute u_t and u_x from the D'Alembert formula.
$$\begin{aligned}
u_t &= \frac{c}{2}\left(f'(x+ct) - f'(x-ct)\right) + \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) \\
u_x &= \frac{1}{2}\left(f'(x+ct) + f'(x-ct)\right) + \frac{1}{c2}\left(g(x+ct) - g(x-ct)\right),
\end{aligned}$$
In the following, in order to simplify notation we will use
$$f'_+ = f'(x+ct), \quad g_+ = g(x+ct), \quad f'_- = f'(x-ct), \quad g_- = g(x-ct).$$
Then we have
$$\begin{aligned}
e(t) =& \frac{1}{2}\int (u_t^2 + c^2 u_x^2)\, dx \\
=& \frac{1}{2}\int \left(\frac{c}{2}\left(f'(x+ct) - f'(x-ct)\right) + \frac{1}{2}\left(g(x+ct) + g(x-ct)\right)\right)^2\, dx \\
&+ \frac{c^2}{2}\int \left(\frac{1}{2}\left(f'(x+ct) + f'(x-ct)\right) + \frac{1}{2c}\left(g(x+ct) - g(x-ct)\right)\right)^2\, dx \\
=& \frac{1}{8}\int \left(c^2\left(f'_+ - f'_-\right)^2 + 2c\left(f'_+ - f'_-\right)(g_+ + g_-) + (g_+ + g_-)^2\right)\, dx \\
&+ \frac{1}{8}\int \left(c^2\left(f'_+ + f'_-\right)^2 + 2c\left(f'_+ + f'_-\right)(g_+ - g_-) + (g_+ - g_-)^2\right)\, dx \\
=& \frac{c^2}{8}\int \left(\left(f'_+ - f'_-\right)^2 + \left(f'_+ + f'_-\right)^2\right)\, dx \\
&+ \frac{2c}{8}\int \left(\left(f'_+ - f'_-\right)(g_+ + g_-) + \left(f'_+ + f'_-\right)(g_+ - g_-)\right)\, dx \\
&+ \frac{1}{8}\int \left((g_+ + g_-)^2 + (g_+ + g_-)^2\right)\, dx
\end{aligned}$$

$$
\begin{aligned}
=&\frac{c^2}{4}\int\left((f'(x+ct))^2+(f'(x-ct))^2\right)\,dx\\
&+\frac{c}{2}\int\left((f'(x+ct)g(x+ct))-(f'(x-ct)g(x-ct))\right)\,dx\\
&+\frac{1}{4}\int\left((g(x+ct))^2+(g(x-ct))^2\right)\,dx\\
=&\frac{1}{2}\int\left(g(x)^2+c^2f'(x)^2\right)\,dx\\
=&e(0)
\end{aligned}
$$

where on the last step we have used the following simple results obtained by change of variables in the integrals. The change of variables $s = x + ct$ gives

$$\int (f'(x+ct))^2\,dx=\int f'(s)^2\,ds,\quad \int (g(x+ct))^2\,dx=\int g(s)^2\,ds$$

and the change of variables $s = x - ct$ gives

$$\int (f'(x-ct))^2\,dx=\int f'(s)^2\,ds,\quad \int (g(x-ct))^2\,dx=\int g(s)^2\,ds.$$

If we would accept the general fact that energy is conserved then we can demonstrate an important result which states that the problem is well posed: (1) a solution exists, (2) the solution is unique, (3) the solution is a classical solution (it is smooth enough to satisfy the equation).

Theorem 2.3 (Well Posedness). *Suppose that $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ with*

$$\int_{\mathbb{R}} f'(x)^2\,dx<\infty,\qquad \text{and}\qquad \int_{\mathbb{R}} g(x)^2\,dx<\infty.$$

The the problem (2.1) *has a unique solution which is a classical solution.*

Proof. We know there is a solution given by the D'Alembert formula and assuming $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ it is clear that this solution is classical. We only need to show that the solution is unique. The way to do this is to use conservation of energy.

Let us assume that u and v are solutions of

$$
\begin{aligned}
u_{tt}(x,t)&=c^2u_{xx}(x,t),\quad -\infty<x<\infty,\quad t>0\\
u(x,0)&=f(x),\quad -\infty<x<\infty,\\
u_t(x,0)&=g(x),\quad -\infty<x<\infty,.
\end{aligned}
$$

Then $w = u - v$ is a solution of

$$
\begin{aligned}
w_{tt}(x,t)&=c^2w_{xx}(x,t),\quad -\infty<x<\infty,\quad t>0\\
w(x,0)&=0,\quad -\infty<x<\infty,\\
w_t(x,0)&=0,\quad -\infty<x<\infty,.
\end{aligned}
$$

Then due to conservation of energy we have

$$e(t) \equiv \frac{1}{2}\int_{\mathbb{R}} \left[w_t^2(x,t) + c^2 w_x^2(x,t)\right]\, dx = e(0) = \frac{1}{2}\int_{\mathbb{R}} \left[g(x)^2 + c^2 f'(x)^2\right]\, dx = 0.$$

But this implies

$$w_t(x,t) = 0, \qquad w_x(x,t) = 0 \qquad \text{for all} \quad x \in \mathbb{R},\ t > 0,$$

which means the total derivative $dw = w_t(x,t)\,dt + w_x(x,t)\,dx = 0$ and therefore $w(x,t)$ is a constant. But at $t = 0$ we have $w(x,0) = 0$ so $w(x,t) = 0$ for all x and all t. Finally this means $u(x,t) = v(x,t)$.

□

2.2 Wave Equation on a Half Line

In this section we consider the initial-boundary value problem for the wave equation on $0 < x < \infty$. This is our first encounter, in this class, with a new idea. Since we are solving the problem on the interval $0 < x < \infty$ it turns out we need to have some information about the semi-infinite string at the point $x = 0$. This information is called a boundary condition. For example, if the string is firmly fixed at height $u(x,0) = 0$ we have Dirichlet boundary condition. That is the case considered first here.

$$\begin{aligned}
u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < \infty, \quad t > 0 \\
u(x,0) &= f(x), \quad 0 < x < \infty, \\
u_t(x,0) &= g(x), \quad 0 < x < \infty, \\
u(0,t) &= 0.
\end{aligned} \tag{2.8}$$

Let us imagine that $f(x)$ and $g(x)$ are zero for $x < 0$. Then we can write the odd extensions of f and g denoted by $\widetilde{f}(x)$ and $\widetilde{g}(x)$ by

$$\widetilde{f}(x) = f(x) - f(-x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases},$$

$$\widetilde{g}(x) = g(x) - g(-x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0 \end{cases}.$$

Next we replace the (2.8) by the following initial value problem on the whole line

$$\begin{aligned}
\widetilde{u}_{tt}(x,t) &= c^2 \widetilde{u}_{xx}(x,t), \ \xi \in \mathbb{R}, \quad t > 0 \\
\widetilde{u}(x,0) &= \widetilde{f}(x), \quad x \in \mathbb{R}, \\
\widetilde{u}_t(x,0) &= \widetilde{g}(x), \quad x \in \mathbb{R}.
\end{aligned} \tag{2.9}$$

$$\tag{2.10}$$

Notice that the solution to this problem is automatically odd since the initial data are odd. To see this look at the D'Alembert solution.

$$\widetilde{u}(x,t) = \frac{\widetilde{f}(x+ct) + \widetilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(s)\, ds.$$

implies that

$$\widetilde{u}(-x,t) = \frac{\widetilde{f}(-x+ct) + \widetilde{f}(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \widetilde{g}(s)\, ds$$

In the first term use the fact that $\widetilde{f}(-p) = -\widetilde{f}(p)$, for all $p \in \mathbb{R}$ which gives

$$\frac{\widetilde{f}(-x+ct) + \widetilde{f}(-x-ct)}{2} = -\frac{\widetilde{f}(x-ct) + \widetilde{f}(x+ct)}{2}.$$

In the second term use the change of variables $s = -r$ (so that $ds = -dr$) and, for the limits, we will have $-r = s = -x - ct$ implies $r = x + ct$ and $-r = s = -x + ct$ implies $r = x - ct$ so we get

$$\frac{1}{2c}\int_{-x-ct}^{-x+ct} \widetilde{g}(s)\, ds = -\frac{1}{2c}\int_{x+ct}^{x-ct} \widetilde{g}(-r)\, dr = \frac{1}{2c}\int_{x+ct}^{x-ct} \widetilde{g}(r)\, dr = -\frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(r)\, dr.$$

We have shown that

$$\widetilde{u}(-x,t) = -\frac{\widetilde{f}(x-ct) + \widetilde{f}(x+ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{g}(r)\, dr = -\widetilde{u}(x,t).$$

Therefore we conclude that $\widetilde{u}(x,t) = -\widetilde{u}(-x,t)$ for all x and all t. So for $x = 0$ we have $\widetilde{u}(0,t) = -\widetilde{u}(0,t)$ which implies $\widetilde{u}(0,t) = 0$ for all t.

Now for $x > 0$ and $t > 0$ we have $(x + ct) > 0$ so that

$$\widetilde{f}(x+ct) = f(x+ct) - f(-(x+ct)) = f(x+ct),$$

$$\widetilde{f}(x-ct) = f(x-ct) - f(ct-x).$$

and

$$\widetilde{g}(s) = g(s) - g(-s).$$

So for $x > 0$ and $t > 0$ we have

$$u(x,t) = \widetilde{u}(x,t) = \frac{f(x+ct) + f(x-ct) - f(ct-x)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \left(g(s) - g(-s)\right)\, ds.$$

This formula can be simplified further by looking more carefully at the special cases $(x-ct) > 0$ and $(x - ct) < 0$ in the first quadrant of x-t space. Clearly for $(x - ct) > 0$ we have

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.$$

So we need only consider the case $(x - ct) < 0$.

In this case we have $f(x-ct)=0$ so

$$\frac{f(x+ct)+f(x-ct)-f(ct-x)}{2}=\frac{f(x+ct)-f(ct-x)}{2}.$$

We also have

$$\begin{aligned}
\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds &= \frac{1}{2c}\int_{0}^{x+ct} g(s)\,ds - \frac{1}{2c}\int_{x-ct}^{0} g(-s)\,ds \\
&= \frac{1}{2c}\int_{0}^{x+ct} g(s)\,ds - \frac{1}{2c}\int_{0}^{ct-x} g(s)\,ds \\
&= \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds.
\end{aligned}$$

Finally we can write the solution as

$$u(x,t)=\begin{cases} \dfrac{f(x+ct)+f(x-ct)}{2}+\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds, & x>ct \\[2ex] \dfrac{f(x+ct)-f(ct-x)}{2}+\dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct} g(s)\,ds, & x<ct \end{cases}. \tag{2.11}$$

Remark 2.2. 1. Equation (2.11) says that for $x>ct$ the solution is exactly the same as D'Alembert's solution for an infinite wave, while for $x-ct$, the solution is modified as a result of the wave reflecting from the boundary (notice also that the sign of the wave is reflected, i.e. it becomes negative) (see Example 2.5 below).

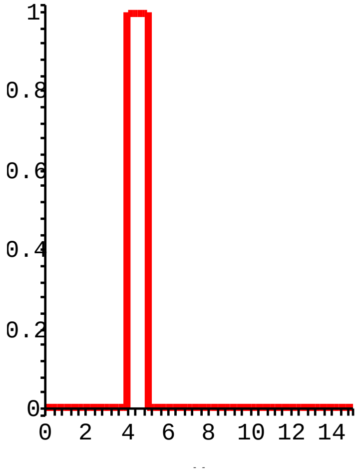
2. The solution would, of course, change if we changed the boundary condition at zero. For example we could impose a Neumann condition.

Example 2.5. We consider the problem (2.8) with the initial conditions

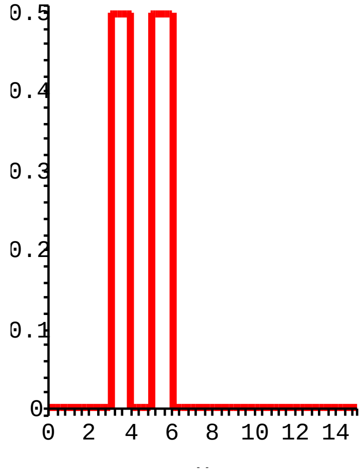
$$f(x)=\begin{cases} 1 & 4<x<5 \\ 0 & \text{otherwise} \end{cases}$$

and $g(x)=0$. In this case the solution (2.11) becomes

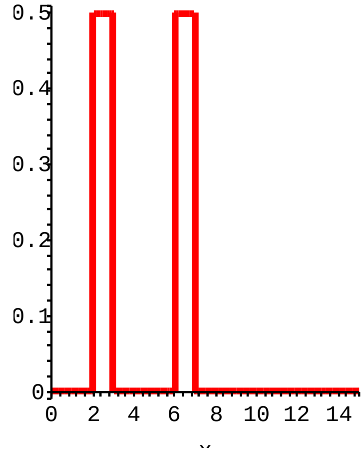
$$u(x,t)=\begin{cases} 1/2\begin{cases} 1 & 4<x+t<5 \\ 0 & \text{otherwise} \end{cases} + 1/2\begin{cases} 1 & 4<x-t<5 \\ 0 & \text{otherwise} \end{cases} & t<x \\[3ex] 1/2\begin{cases} 1 & 4<x+t<5 \\ 0 & \text{otherwise} \end{cases} - 1/2\begin{cases} 1 & 4<-x+t<5 \\ 0 & \text{otherwise} \end{cases} & x<t \end{cases}$$



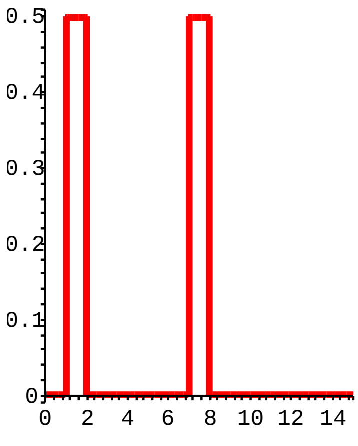
$t = 0$



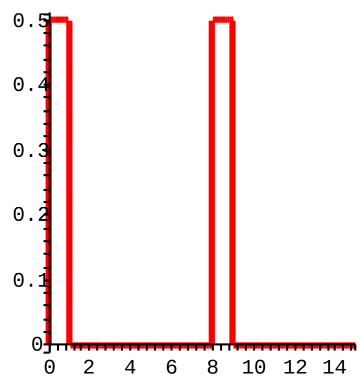
$t = 1$



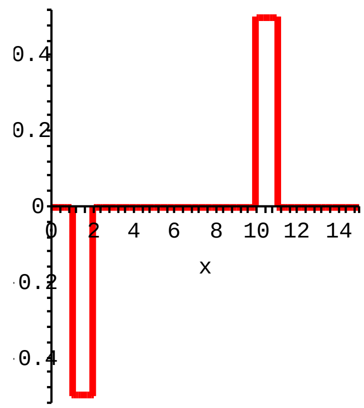
$t = 2$



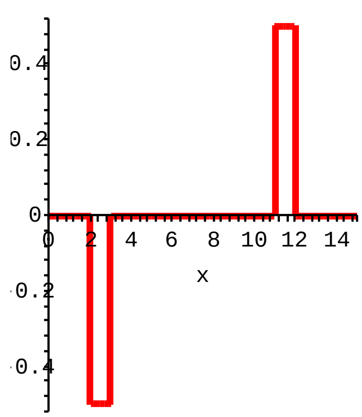
$t = 4$



$t = 5$



$t = 6$



$t = 7$

2.3 Assignment 2

1. Use D'alemberts formula to solve

$$u_{tt}(x,t) = 9u_{xx}(x,t), \quad -\infty < x < \infty \quad t > 0 \tag{2.12}$$
$$u(x,0) = 1, \quad u_t(x,0) = 0$$

2. Use D'alemberts formula to solve

$$u_{tt}(x,t) = 4u_{xx}(x,t), \quad -\infty < x < \infty \quad t > 0 \tag{2.13}$$
$$u(x,0) = 0, \quad u_t(x,0) = 1$$

3. Use D'alemberts formula to solve

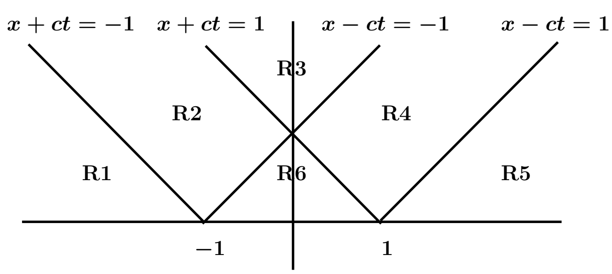
$$u_{tt}(x,t) = u_{xx}(x,t), \quad -\infty < x < \infty \quad t > 0 \tag{2.14}$$
$$u(x,0) = \sin(x), \quad u_t(x,0) = \cos(x)$$

4. Use D'alemberts formula to solve

$$u_{tt}(x,t) = u_{xx}(x,t), \quad -\infty < x < \infty \quad t > 0 \tag{2.15}$$
$$u(x,0) = e^{-x^2}, \quad u_t(x,0) = \frac{1}{1+x^2}$$

5. Use D'alemberts formula to solve

$$u_{tt}(x,t) = 4u_{xx}(x,t), \quad -\infty < x < \infty \quad t > 0 \tag{2.16}$$
$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$



Hint: To do Problem 5 simply pick a point (x,t) in each region and compute the integral

$$\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

For example, if (x,t) is in region **R1** then $x + ct < -1$ which means that the upper limit of integration is less than -1 so $g(s) = 0$ on $[x - ct, x + ct]$. Therefore

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds = 0, \quad \forall \;\; (x,t) \in R1.$$

6. Solve the semi-infinite string wave equation problem

$$
\begin{aligned}
u_{tt}(x,t) &= u_{xx}(x,t), \quad 0 < x < \infty \quad t > 0 \\
u(0,t) &= 0 \qquad t > 0 \\
u(x,0) &= xe^{-x^2}, \quad u_t(x,0) = 0.
\end{aligned}
$$

7. Consider the problem

$$
\begin{aligned}
u_{tt}(x,t) &= 4u_{xx}(x,t), \quad 0 < x < \infty \quad t > 0 \\
u(x,0) &= \begin{cases} x, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}, \\
u_t(x,0) &= \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}
\end{aligned}
$$

Find the energy $e(t)$ at time $t = 3$.