Math 4354 Partial Differential Equations

Prerequisites for this course include three semesters of calculus, linear algebra and especially ODEs. If you cannot solve ODEs as studied in Math 3354 then you should not be in this class.

1 Partial Differential Equations

A *partial differential equation* PDE for a scalar function u is an equation involving more than one dependent variable (e.g. x, y, t, etc) and some number of partial derivatives of the function u with respect to these independent variables. So for example if the independent variables are x and y then the equation might look like

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0.$$

Note that we will use various notations for partial derivatives. For example

$$\frac{\partial f}{\partial x} = \partial_x f = f_x, \quad \frac{\partial^2 f}{\partial x^2} = \partial_{xx} f = f_{xx}.$$

In this case the equation is said to be of second order. In general, the order (call it m) of an equation is the highest order of the derivatives that appear in the equation. We say that u is a solution of the equation in a region $\Omega \subset \mathbb{R}^2$, if u has continuous derivatives up to and including order m and u satisfies the equation at each point $(x, t) \in \Omega$.

1. In applications variables like x, y, z are often space variables and a solution may be required in some region Ω of space. In this case there will be some conditions to be satisfied on the boundary of Ω (usually denoted $\partial\Omega$); these are called boundary conditions (BCs).

2. Also in applications, one of the independent variables can be time (t say), then there will be some initial conditions (ICs) to be satisfied (i.e., u is given at $t = 0$ everywhere in Ω).

3. Again in applications, systems of PDEs can arise involving the dependent variables $u_1, u_2, u_3, ..., u_m$, with $m > 1$ with some (at least) of the equations involving more than one u_i. That is, sometimes it happens that one must consider several such equations simultaneously, in which case we speak of a *system of PDEs*. For example as system of three equations in two dependent variables would have $\boldsymbol{u} = (u_1(x,t), u_2(x,t))$ and the differential equation becomes

$$\boldsymbol{F}(x, y, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \cdots) = \begin{pmatrix} F_1(x, y, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \cdots) \\ F_2(x, y, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \cdots) \\ F_3(x, y, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \cdots) \end{pmatrix} = 0.$$

In the study of differential equations there are three main points that must be addressed:

1. Existence
2. Uniqueness
3. Continuous Dependence on Data.

If these can be verified for a PDE then we say the problem is *Well-Posed.* If any of the three fails the problem is said to be ill-posed (or improperly posed).

1.1 First Order Linear Equations

A linear equation is one in which the equation and any boundary or initial conditions do not include any product of the dependent variables or their derivatives; an equation that is not linear is a nonlinear equation.

We will begin our study with the special case of *First Order Linear Equations in two variables.* The general form of such equations is

$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = g(x,t). \tag{1.1}$$

It is sometimes convenient to write this equation in a more compact form as $Lu = g$ where $Lu = au_x + bu_t + cu$.

Probably the most important aspect of such equations is that if $Lu_1 = g_1$ and $Lu_2 = g_2$ then $L(\alpha u_1 + \beta u_2) = (\alpha g_1 + \beta g_2)$. This is referred to as the *principle of superposition.*

1.2 Constant Coefficient Equations

In order to get started we begin by studying the simplest case of equations in the form

$$u_t(x,t) + cu_x(x,t) = 0 \tag{1.2}$$

where $c > 0$ is a constant. Note that we use x and t instead of x and y since in many applications the time derivatives physically refer to velocity, acceleration, etc.

Note that if we write the gradient in (x,t) as

$$\nabla u = \begin{bmatrix} u_x \\ u_t \end{bmatrix}$$

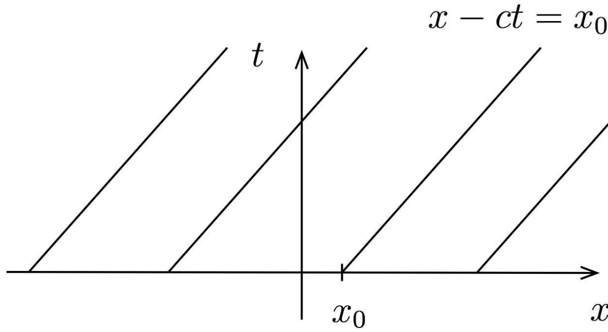
then the left hand side of equation (1.2) is actually a directional derivative in the direction $\nu = [c,\ 1]$, namely:

$$u_t(x,t) + cu_x(x,t) = \nu \cdot \nabla u.$$

In (x,t) space a line parallel to the vector ν can be written as

$$x - ct = x_0, \tag{1.3}$$

where x_0 is the point at which the line intersects the x axis when $t = 0$. Notice that the slope of this line is $1/c$, i.e., a change in c units in the x direction corresponds to a change of 1 unit in the t direction.



Characteristics are straight lines

Notice that as x_0 takes on every value in $\mathbb{R}$ the characteristic lines completely fill $\mathbb{R}^2$, i.e., for every $(x,t) \in \mathbb{R}^2$ there is a x_0 so that x_0 is the initial point for the line with slope $1/c$ passing through $(x_0, 0)$ and the point (x,t).

Let us restrict $u(x,t)$ to a line $x - ct = x_0$ and denote the resulting function by $v(t)$:

$$v(t) \equiv u(ct + x_0, t). \tag{1.4}$$

Now if u is a solution to (1.2), then $v(t)$ should be a constant. Namely we have

$$\begin{aligned}
\frac{d}{dt}v(t) &= \frac{d}{dt}u(ct + x_0, t) \\
&= u_x(ct + x_0, t)\frac{d(ct + x_0)}{dt} + u_t(ct + x_0, t) \\
&= u_x(ct + x_0, t)c + u_t(ct + x_0, t) = 0,
\end{aligned}$$

which means that $v(t)$ is a constant, which means that $v(t) = v(0)$. In terms of u this means that

$$u(x_0 + ct, t) = v(t) = v(0) = u(x_0, 0).$$

So on a characteristic line $x - ct = x_0$ we have

$$u(x,t) = u(x - ct, 0). \tag{1.5}$$

At this point we present the analog of the initial value problem for ODEs by defining the *Initial Value Problem.*

Definition 1.1. Let $f(x)$ be a continuously differentiable function on $\mathbb{R}$. The initial value problem is to find u satisfying

$$u_t(x,t) + cu_x = 0, \qquad u(x,0) = f(x). \tag{1.6}$$

Theorem 1.1. *The unique solution $u(x,t)$ to the initial value problem is given by*

$$u(x,t) = f(x - ct). \tag{1.7}$$

The proof of this result follows immediately from the formula (1.5). Concerning the question of well-posedness we only need to show continuous dependence on the initial data. Suppose that $f(x)$ ad $g(x)$ are two initial functions and let $u(x,t)$ and $v(x,t)$ be the corresponding solutions of (1.6). The because the problem is linear we have that $w = u - v$ is a solution of the PDE with initial condition $w(x,0) = f(x) - g(x)$. But furthermore we have

$$u(x,t) - v(x,t) = w(x,t) = f(x - ct) - g(x - ct)$$

so that

$$\sup_{x,t} |u(x,t) - v(x,t)| = \sup_{x,t} |w(x,t)| = \sup_{x} |f(x) - g(x)|$$

where, here we have noted that as (x,t) vary of $\mathbb{R}^2$ the quantity $(x - ct)$ varying over $\mathbb{R}$ so that

$$\sup_{x} |f(x) - g(x)| = \sup_{x,t} |f(x - ct) - g(x - ct)|.$$

So if $|f(x) - g(x)| < \delta$ for all x then we must also have $\sup_{x,t} |u(x,t) - v(x,t)| < \delta$. Therefore continuous dependence follows since a small difference in the initial data implies a small difference in the solution.

In these problems c is the *velocity of propagation* and the solution (1.7) represents a signal (or wave) propagating to the right (when $c > 0$) with velocity c. When $c < 0$ the signal propagates to the left.

Definition 1.2. The lines $x - ct =$ constant are called the *characteristic lines* for the equation because information (the value of u) is carried along them.

1.3 Equations with Spatially Dependent Velocity

In this section we consider the next simplest case of equations in the form

$$u_t(x,t) + c(x)u_x(x,t) = 0, \quad u(x,0) = f(x), \tag{1.8}$$

where c is no longer a constant but can depend on x.

Once again denoting the gradient in (x,t) as

$$\nabla u = \begin{bmatrix} u_x \\ u_t \end{bmatrix}$$

we have a situation similar to what happened above in that at a point (x,t) the the left hand side of equation (1.8) is a directional derivative in the direction $\nu(x) = [c(x),\, 1]$, namely:

$$u_t(x,t) + c(x)u_x(x,t) = \nu \cdot \nabla u.$$

Since, once again in this case, this directional derivative is zero we see that the vector $[c(x),\, 1]$ must be tangent to level curves of the solution. If the level curves are parameterized in (x,t) space by $(x(t),t)$ for $t \in \mathbb{R}$, then we must have

$$\frac{dx}{dt} = c(x) \tag{1.9}$$

which now is a possibly nonlinear ODE.

Let us assume that $c(x)$ is a continuously differentiable function so that the fundamental theorem of ODEs implies it has a smooth solution for all initial conditions $x_0 \in \mathbb{R}$.

Let us recall what we learned in the case when $c(x) = c$ is a constant. We learned that the solution u is constant along solutions of the equation (1.9), namely, along the lines $x = x_0 + ct$. The same situation occurs here and to see this let $v(t) = u(x(t),t)$ be the restriction of the solution u to such a curve. Then by the chain rule we must have

$$\frac{dv}{dt} = \frac{d}{dt}u(x(t),t) = u_x\frac{dx}{dt} + u_t\frac{dt}{dt} = c(x)u_x + u_t = 0.$$

Thus $v(t)$ is constant along a solution curve to the equation (1.9). Now these curves will not always be lines so we call the curves *Characteristic Curves*. Notice that on such a curve the PDE reduces to an ODE.

Now due to the fact that $v(t)$ is a constant we must have $v(t) = v(0)$ so if (x,t) is a point on the characteristic curve with initial point $(x_0, 0)$ then $u(x,t) = u(x_0,0)$. In the case when c was a constant we easily found $x - ct = x_0$ so we obtained a closed form solution to our problem as given in (1.5). In the more general case considered now we start at a point (x,t) and follow the characteristic curve back from (x,t) to $(x_0, 0)$. Here we need to assume that the curve actually reaches the x axis at a point $(x_0, 0)$. Now this process defines a function p that takes a point (x,t) and gives us a number x_0: $p(x,t) = x_0$. We can use this notation to write

$$u(x,t) = u(p(x,t),0) = f(p(x,t)). \tag{1.10}$$

Example 1.1. Consider the following initial value problem

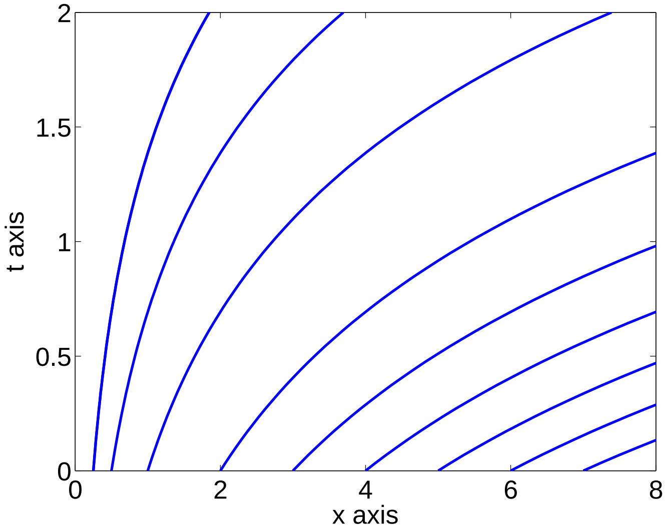
$$u_t(x,t) + xu_x(x,t) = 0, \quad u(x,0) = f(x), \tag{1.11}$$

In this example we have $c(x) = x$ so the equation for the characteristics is

$$\frac{dx}{dt}(t) = x(t), \quad x(0) = x_0,$$

with solution

$$x(t) = x_0 e^t.$$



Characteristics

The characteristics are depicted in the above figure.

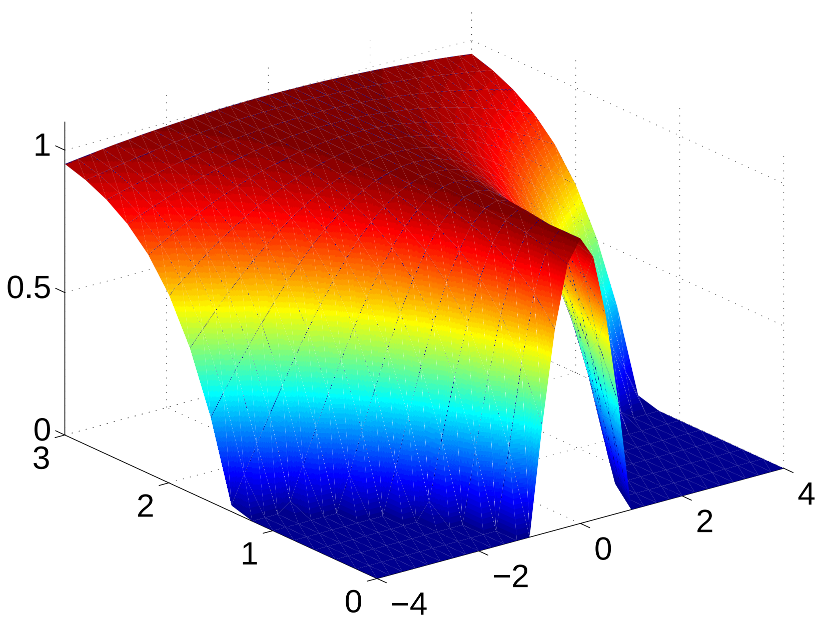
In this example we can solve for x_0 to obtain $x_0 = xe^{-t}$ and our solution is given by (1.7) (see also (1.5))

$$u(x,t) = f(xe^{-t}), \quad \forall \;\; x,t \in \mathbb{R}, \quad .$$

For the initial function

$$f(x) = \begin{cases} \cos\left(\dfrac{\pi x}{2}\right), & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

we have solution depicted in the following figure



Solution Surface

1.4 More general Method

In this section we look at the most general case of first oder linear PDEs that we will study in this class

$$a(x,t)u_x(x,t) + b(x,t))u_t(x,t) + c(x,t)u(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.12}$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}.$$

Step 1. Solve the characteristic equations

$$\frac{dx}{ds} = a(x,t), \quad \frac{dt}{ds} = b(x,t) \tag{1.13}$$
$$x(0) = \tau, \quad t(0) = 0.$$

This gives a transformation from (x,t) to (s,τ) space: namely

$$x = x(s,\tau), \quad t = t(s,\tau). \tag{1.14}$$

Step 2. Solve the IVP for the following ODE

$$\frac{du}{ds} + c\big(x(s,\tau), t(s,\tau)\big)u = 0, \quad 0 < s < \infty, \tag{1.15}$$
$$u(0) = f(\tau).$$

Step 3. After solving the ODE (and IVP) we obtain a solution $u(s,\tau)$. Next we need to solve the equations in (1.14) for s and τ in terms of x and t (sometimes this can be very hard). Once this is done we have $s = s(x,t)$ and $\tau = \tau(x,t)$ which we substitute into our solution u are we are done.

Example 1.2. Consider the following initial value problem

$$xu_x(x,t) + u_t(x,t) + tu(x,t) = 0, \quad u(x,0) = \sin(x), \tag{1.16}$$

Step 1. Solve the characteristic equations

$$\frac{dx}{ds} = x, \quad \Rightarrow \quad x(s) = c_1 e^s \tag{1.17}$$
$$\frac{dt}{ds} = 1, \quad \Rightarrow \quad t(s) = s + c_2.$$

With $x(0) = \tau$ and $t(0) = 0$ we have $c_1 = \tau$ and $c_2 = 0$ and we have the transformation from (x,t) to (s,τ) space: namely

$$x = \tau e^s, \quad t = s. \tag{1.18}$$

Step 2. Solve the IVP for the following ODE

$$\frac{du}{ds} + su = 0, \quad 0 < s < \infty, \tag{1.19}$$
$$u(0) = \sin(\tau).$$

The solution is given by

$$u(s, \tau) = \sin(\tau) e^{-s^2/2}. \tag{1.20}$$

Step 3. At this point we have a solution $u(s, \tau)$. We need to solve the equations in (1.18) for s and τ in terms of x and t. We have

$$s = t, \quad \tau = xe^{-t}.$$

Therefore we finally obtain

$$u(x, t) = \sin\left(xe^{-t}\right) e^{-t^2/2}. \tag{1.21}$$

1.5 Assignment 1

1. For each of the problems the characteristics are $x = x_0 + ct$. Let $v(t) = u(x_0 + ct, t)$ be the restriction of the solution u to the characteristic line. Find the ODE solved by v in each case and solve it. Then use this to obtain the solution $u(x, t)$.

   (a) $u_t + cu_x + u = 0$ with $u(x, 0) = f(x)$.

   (b) $u_t + cu_x = xt$ with $u(x, 0) = f(x)$.

   (c) $u_t + cu_x = u^2$ with $u(x, 0) = f(x)$.

2. Solve the initial value problem $xu_x + tu_t + 2u = 0$ for $x \in \mathbb{R}$ and $t > 1$ with $u(x, 1) = \sin(x)$ for $x \in \mathbb{R}$.

3. Solve the initial value problem $u_x + xu_t = u^2$ with $u(x, 0) = 1$ for $x > 0$.

4. Solve the initial value problem $u_t + 2txu_x = u$ with $u(x, 0) = x^2$.

5. Solve the initial value problem $u_x + u_t + tu = 0$ with $u(x, 0) = f(x)$ for $x \in \mathbb{R}$.