$$
\begin{aligned}
&u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\
&u(0,t) = 0, \quad u(\ell,t) = 0 \\
&u(x,0) = f(x), \quad u_t(x,0) = g(x) \\
&\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(\mu_n\, ct) + b_n \sin(\mu_n\, ct)\right) \sin(\mu_n x) \\
&a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin(\mu_n\, x)\; dx, \quad b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\sin(\mu_n\, x)\; dx.
\end{aligned}
$$

$$
\begin{aligned}
&u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\
&u_x(0,t) = 0, \quad u_x(\ell,t) = 0 \\
&u(x,0) = f(x), \quad u_t(x,0) = g(x) \\
&\Rightarrow \quad u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(c\mu_n\, t) + b_n \sin(c\mu_n\, t)\right) \cos(\mu_n\, x) \\
&a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos(\mu_n x)\; dx, \quad a_0 = \frac{2}{\ell}\int_0^\ell f(x)\, dx, \\
&b_n = \frac{2}{n\pi c}\int_0^\ell g(x)\cos(\mu_n\, x)\; dx, \quad b_0 = \frac{2}{\ell}\int_0^\ell g(x)\, dx.
\end{aligned}
$$

$$
\begin{aligned}
&u_t(x,t) = k u_{xx}(x,t) + F(x,t), \\
&u(0,t) = 0, \quad u(\ell,t) = 0 \\
&u(x,0) = \varphi(x)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
&u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x) + \sum_{n=1}^{\infty}\left(\int_0^t e^{k\lambda_n(t-\tau)}F_n(\tau)\, d\tau\right)\varphi_n(x) \\
&b_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\varphi_n(x)\, dx, \quad F_n(t) = \frac{2}{\ell}\int_0^\ell F(x,t)\varphi_n(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&u_t(x,t) = k u_{xx}(x,t) + f(x), \quad 0 < x < \ell, \\
&u(0,t) = 0, \quad u(\ell,t) = 0 \\
&u(x,0) = \varphi(x)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
u(x,t) \quad &= \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x) + \sum_{n=1}^{\infty} f_n\left(\frac{e^{k\lambda_n t} - 1}{k\lambda_n}\right)\varphi_n(x), \\
b_n \quad &= \frac{2}{\ell}\int_0^\ell \varphi(x)\varphi_n(x)\, dx, \quad f_n = \frac{2}{\ell}\int_0^\ell f(x)\varphi_n(x)\, dx.
\end{aligned}
$$

Steady State $\psi''(x) = -\frac{1}{k}f(x), \quad \psi(0) = 0, \quad \psi(\ell) = 0 \;\; 0 < x < \ell, .$

$$
\begin{aligned}
&u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < \ell, \\
&u(0,t) = \gamma_0(t), \quad u(\ell,t) = \gamma_1(t) \\
&u(x,0) = \varphi(x) \\
&\text{set} \\
&h(x,t) = \gamma_0(t) + \frac{x}{\ell}(\gamma_1(t) - \gamma_0(t)) \\
&v(x,t) = u(x,t) - h(x,t)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
&v_t(x,t) = k v_{xx}(x,t) - h_t(x,t), \quad 0 < x < \ell, \\
&v(0,t) = 0, \quad v(\ell,t) = 0 \\
&v(x,0) = v_0(x) = v_0(x)\left(\gamma_0(0) + \frac{x}{\ell}(\gamma_1(0) - \gamma_0(0))\right) \\
&v(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x) - \sum_{n=1}^{\infty}\left(\int_0^t e^{k\lambda_n(t-\tau)}\frac{dh_n}{d\tau}(\tau)\, d\tau\right)\varphi_n(x) \\
&b_n = \frac{2}{\ell}\int_0^\ell \varphi_n(x)v_0(x)\, dx, \qquad h_n(t) = \frac{2}{\ell}\int_0^\ell h(x,t)\varphi_n(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < \ell, \\
&u(0,t) = \alpha, \quad u(\ell,t) = \beta \\
&u(x,0) = \varphi(x) \\
&\text{set} \;\; h(x,t) = \alpha + (\beta - \alpha)\frac{x}{\ell} \equiv U(x) \\
&v_0(x) = \varphi(x) - h(x,0) = \varphi(x) - U(x) \\
&v(x,t) = u(x,t) - h(x,t)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
&v_t(x,t) = k v_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\
&v(0,t) = 0, \quad v(\ell,t) = 0 \\
&v(x,0) = v_0(x) \\
&v(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x), \quad b_n = \frac{2}{\ell}\int_0^\ell v_0(x)\varphi_n(x)\, dx \\
&u(x,t) = U(x) + \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x)
\end{aligned}
$$