First Order Equation Techniques

I. (RHS only contains x) $\boxed{y' = f(x)}$ $\Rightarrow$ $\boxed{y = \int f(x)\,dx}$

II. (Separable) $\boxed{f(y)y' = g(x)}$ $\Rightarrow$ $\boxed{\int f(y)\,dy - \int g(x)\,dx = C}$

Implicit and explicit:

1. If we can write the answer as $y = \varphi(x)$ then we have an explicit answer.
2. If we leave the answer in the form $F(x, y) = C$ we have an implicit solution.

III. (First Order Linear) $\boxed{y' + P(x)y = Q(x)}$ Integrating Factor $\boxed{\mu = e^{\int P dx}}$ and

Gen. Sol. $\boxed{y = 1/\mu(x)\left[\int^x \mu(t)Q(t)\,dt + C\right]}$

IV. (RHS Linear in x and y) $\boxed{y' = f(ax + by + c)}$ $\boxed{v = ax + by + c \quad \Rightarrow \quad v' = a + f(v)}$ is separable.

V. (RHS only contains y/x) $\boxed{y' = f(y/x)}$ $\boxed{v = y/x \quad \Rightarrow \quad xv' + v = f(v)}$ is separable.

VI. (Bernoulli) $\boxed{y' + P(x)y = Q(x)y^n, n \neq 0, 1}$ $\boxed{v = y^{1-n} \quad \Rightarrow \quad v' + (1 - n)P(x)v = (1 - n)Q(x)}$ is First Order Linear.

VII. (Exact) $\boxed{M(x, y)\,dx + N(x, y)\,dy = 0}$ is exact if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$. If exact then there exists $F(x, y)$ so that $\boxed{M = \dfrac{\partial F}{\partial x}, \quad N = \dfrac{\partial F}{\partial y}}$. Use these to find solution $\boxed{F(x, y) = C}$.

VIII. (Reduce to First Order) $\boxed{F(x, y, y', y'') = 0}$ can sometimes be reduced to first order:

(a) *y missing:* For $\boxed{F(x, y', y'') = 0}$ use $\boxed{p = y', p' = y'' \Rightarrow F(x, p, p') = 0}$

(b) *x missing:* For $\boxed{F(y, y', y'') = 0}$ use $\boxed{p = \dfrac{dy}{dx}, y'' = \dfrac{dp}{dx} = \dfrac{dp}{dy}\dfrac{dy}{dx} = p\dfrac{dp}{dy} \Rightarrow F\left(y, p, p\dfrac{dp}{dy}\right) = 0}$